Resurgence and Hydrodynamics in Gauss-Bonnet Holography

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Hydrodynamics in 3+1 Dimensions

The equation of motion for Hydrodynamics is the conservation equation

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1}$$

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$$T^{\mu\nu} = T^{\mu\nu}_{\textit{ideal}} + c_1\partial^\mu u^\nu + c_2\partial^\nu u^\mu + c_3\eta^{\mu\nu}\partial_\alpha u^\alpha + c_4 u^\mu u^\nu \partial_\alpha u^\alpha + \ldots \tag{3}$$

Generally symmetries of the theory can be used to constrain these co-efficients c_i.

Hydrodynamics in 3+1 Dimensions

For example, in a conformal theory all these co-efficients (at order $\partial^\mu u^\nu$) are contrained to η the shear viscousity

$$T^{\mu\nu} = T^{\mu\nu}_{ideal} - \eta\sigma^{\mu\nu} + \ldots \tag{4}$$

where $\sigma^{\mu\nu} = (\partial_\mu u^\nu + \partial^\nu u^\mu - \frac{2}{3}(u^\mu u^\nu + \eta^{\mu\nu})\partial_\alpha u^\alpha)$.

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In general we can include all derivatives of u^μ

$$T^{\mu\nu} = T^{\mu\nu}_{\textit{ideal}} + O(\sim \partial^\mu u^\nu) + O(\sim (\partial^\mu u^\nu)^2) + \dots \tag{5}$$

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- This series is known as the Gradient Expansion and orders itself in $\partial^\mu u^\nu << 1$ when u^μ is slowly varying.
- The co-efficients c_i are known as transport co-efficients and uniquely specify our theory.

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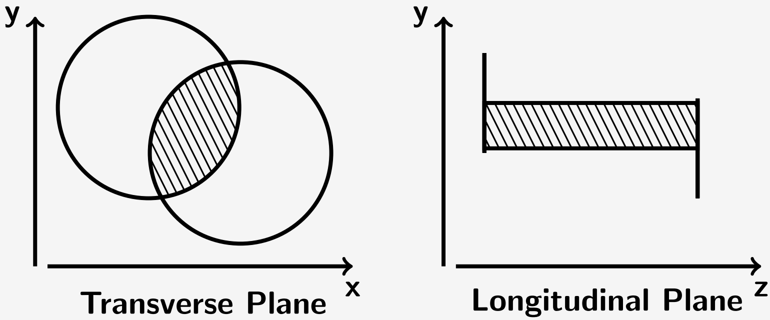

Figure: Head-on and Side profiles for a Lead-Lead collision. The overlapping region results in an energy density that evolves longitudinally according to hydrodynamics.

Bjorken Flow

- This energy density $T_{00} = \epsilon$ is a function of only the proper time, and the form is known to all orders:

$$\epsilon(\tau) = \tau^{-4/3}(\epsilon_0 + \epsilon_1 \tau^{-2/3} + \epsilon_2 \tau^{-4/3} + ...) \qquad (6)$$

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- To gain some understanding of this evolving system analytically, we need a way to calculate the energy co-efficients for a QCD-like theory at Strong Coupling.
- $N = 4$ SYM (a QCD-like theory) can be re-written at infinite coupling as a gravitational theory.

The Fluid-Gravity correspondence

We can perform classical gravity calculations to find strongly coupled QFT results.

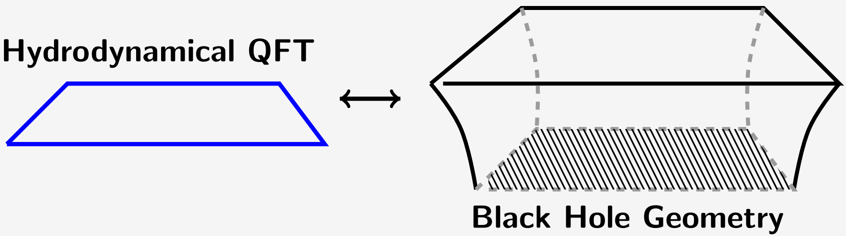

Figure: Some Gauge theories and Gravity theories are conjectured to be the same theory under a field redefinition.

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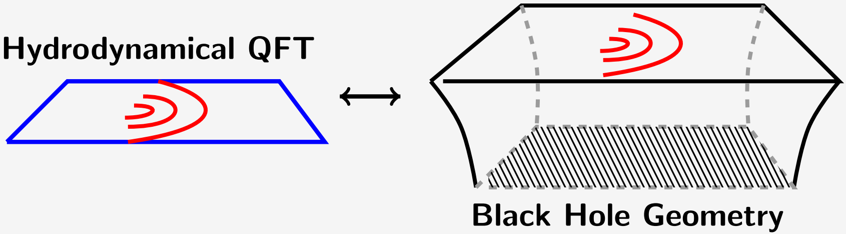

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The Fluid-Gravity Correspondence

The geometry that is dual to Bjorken Flow Hydrodynamics in $N = 4$ SYM at infinite coupling is given by

$$ds^2 = -r^2 A(r,\tau) d\tau^2 + 2 d\tau dr + (r\tau + 1)^2 e^{B(r,\tau)} dy^2 + r^2 e^{C(r,\tau)} dx_\perp^2 \tag{7}$$

where r is the radial distance towards the Black Hole, and τ is the proper time.

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where r is the radial distance towards the Black Hole, and τ is the proper time. A, B and C are defined by:

$$\begin{aligned} A(\tau, r) &= \sum_{i=0} \tau^{-\frac{2}{3}i} A_i(r^{-1}\tau^{-1/3}), \quad & A_0 &= 1 - \left(\tfrac{1}{r\tau^{1/3}}\right)^4 \\ B(\tau, r) &= \sum_{i=0} \tau^{-\frac{2}{3}i} B_i(r^{-1}\tau^{-1/3}), & B_0 &= 0 \\ C(\tau, r) &= \sum_{i=0} \tau^{-\frac{2}{3}i} C_i(r^{-1}\tau^{-1/3}), & C_0 &= 0. \end{aligned}$$

(Kinoshita, Mukohyama & Nakamura [arXiv:0807.3797v2])

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where r is the radial distance towards the Black Hole, and τ is the proper time. This looks a little like a space with a blackhole a horizon sinking into the radial direction.

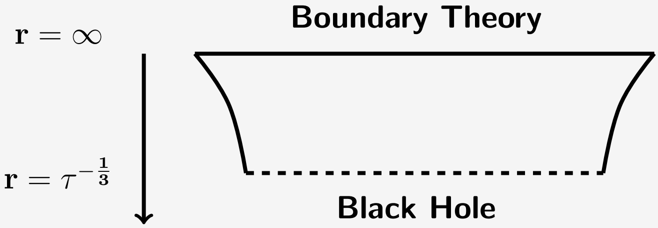

Figure: Schematic cartoon of the Geometry.

Resurgence

We can calculate ϵ_i to large orders from this solution.

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But after some finite order, the co-efficients start to contribute more and more!

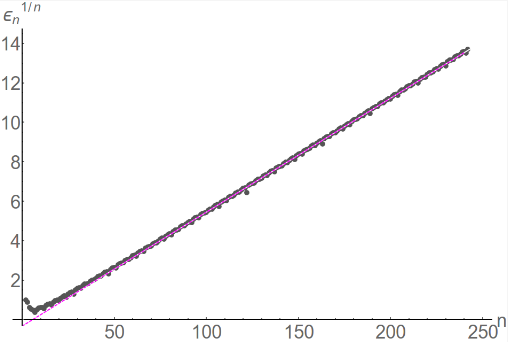

Figure: Energy density co-efficients $\epsilon_n^{1/n}$ as a function of order n. Note that $(n!)^{1/n} \sim n$ for large n. [arXiv:1302.0697v2]

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Using the identity:

$$\boxed{n!\left(\tau^{-\frac{2}{3}}\right)^{n}=\int_{0}^{\infty}du\,\left(\frac{e^{-u\tau^{2/3}}}{\tau^{2/3}}\right)u^{n}} \tag{10}$$

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as an integral of a converging series

$$\epsilon(\tau) = \int_0^\infty du \left(\frac{e^{-u\tau^{2/3}}}{\tau^{2/3}}\right)\left(\frac{\epsilon_0}{2!}u^2 + \frac{\epsilon_1}{3!}u^3 + \frac{\epsilon_2}{4!}u^4 + ...\right) \tag{12}$$

Resurgence

This convergent series is called the Borel Sum

$$\zeta(u) = \frac{\epsilon_0}{2!}u^2 + \frac{\epsilon_1}{3!}u^3 + \frac{\epsilon_2}{4!}u^4 + ... \tag{13}$$

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If we plot $\zeta(u)$ in the complex plane we can examine the pole structure.

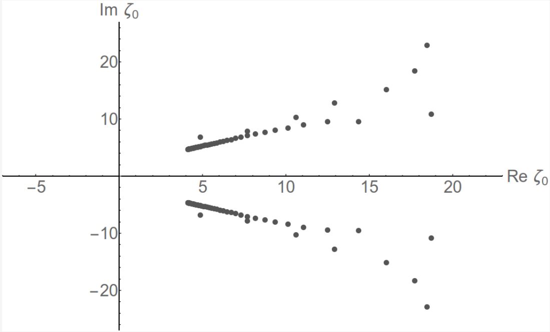

Figure: Poles of the $\zeta(u)$ series containing non-perturbative information. [arXiv:1302.0697v2]

Resurgence

To evaluate

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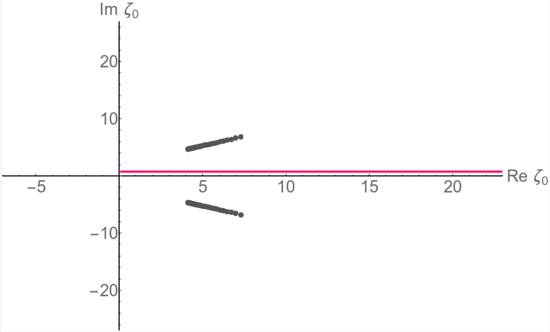

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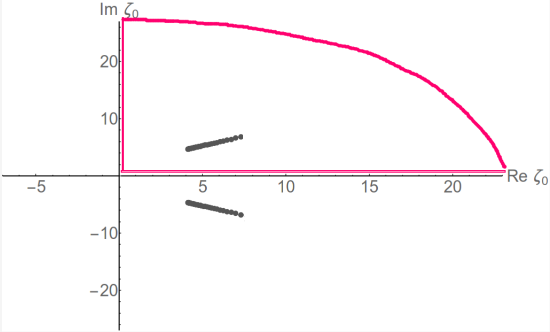

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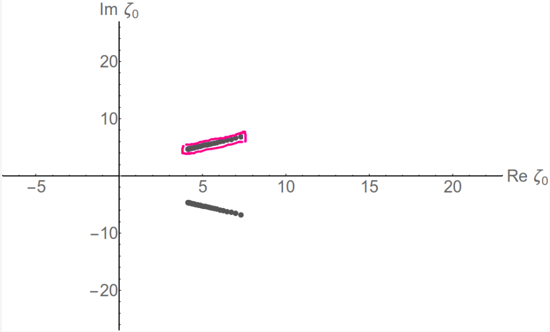

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Using the residue theorem one can find the leading non-perturbative mode

$$\int_C du \left(\frac{e^{-u\tau^{2/3}}}{\tau^{2/3}} \right) \zeta(u) \sim \tau^{\alpha} \mathsf{exp} \left(-i\frac{3}{2}\omega\tau^{2/3} \right) \tag{17}$$

with $\alpha = -1.5426 + 0.5192i$ and $\omega = 3.11 - 2.7471i$. (Heller, Janik and Witaszczyk [arXiv:1302.0697v2])

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The take home message is that non-perturbative behaviour is contained in the pole structure of our Borel Sum $\zeta(u)$

Strong (but finite) Coupling

The transport co-efficients have been found for infinitely coupled $N = 4$ SYM wth classical gravity:

$$S = \int d^5x \sqrt{-g} \, (R + 12) \tag{18}$$

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This corresponds to the well-known result of $\frac{\eta}{s} = \frac{1-4\lambda}{4\pi}$, a non-trivial consistency check!

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- We have suceeded in finding the first two co-efficients for strong (but finitely) coupled Bjorken Flow.
- Moving forward, we will attempt a high order computation to determine the singularity structure of the Borel plane, and extract non-perturbative information.