Recursion for Integral Coefficients

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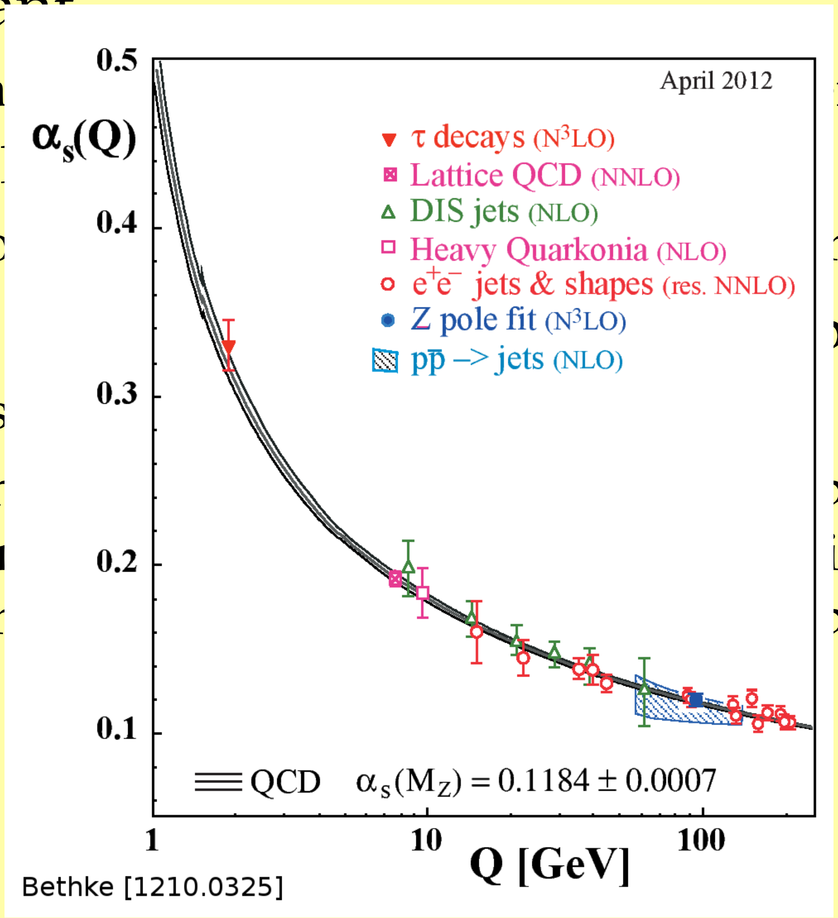
Precision Calculations for the LHC

- Test our understanding of the Standard Model in detail
  - ensure that experimenters understand their detectors
  - and that theorists understand the theory
- Uncertainties in Standard Model measurements
  - precision measurements of M_W
- Backgrounds to Frontier Physics
  - precision measurements of Higgs branchings and couplings
- Backgrounds to New Physics
  - cascade decays (e.g. SUSY searches)
  - candidate resonances

The Challenge

- Strong coupling is not small: $\alpha_s(M_Z) \approx 0.12$ and running is importa
  - $\Rightarrow$ events h ... rs (jets)
  - $\Rightarrow$ each jet ...
  - $\Rightarrow$ higher-c ... nportant
- Processes ... $p_T(W)$ & M_W
  - $\Rightarrow$ need res
- Confinemer ... ping partons to hadrons, bu ... infrared-safe) avoiding sm ... power corrections)

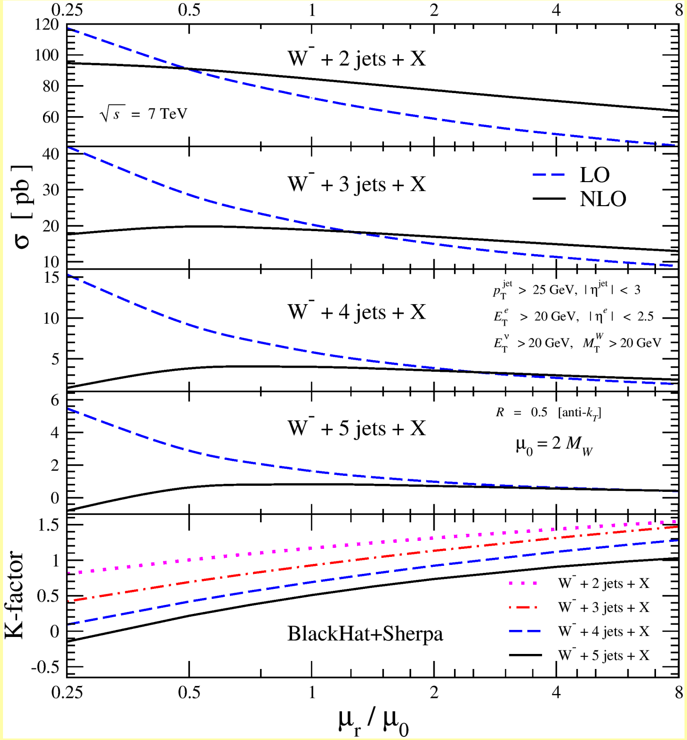


What's the Right Scale?

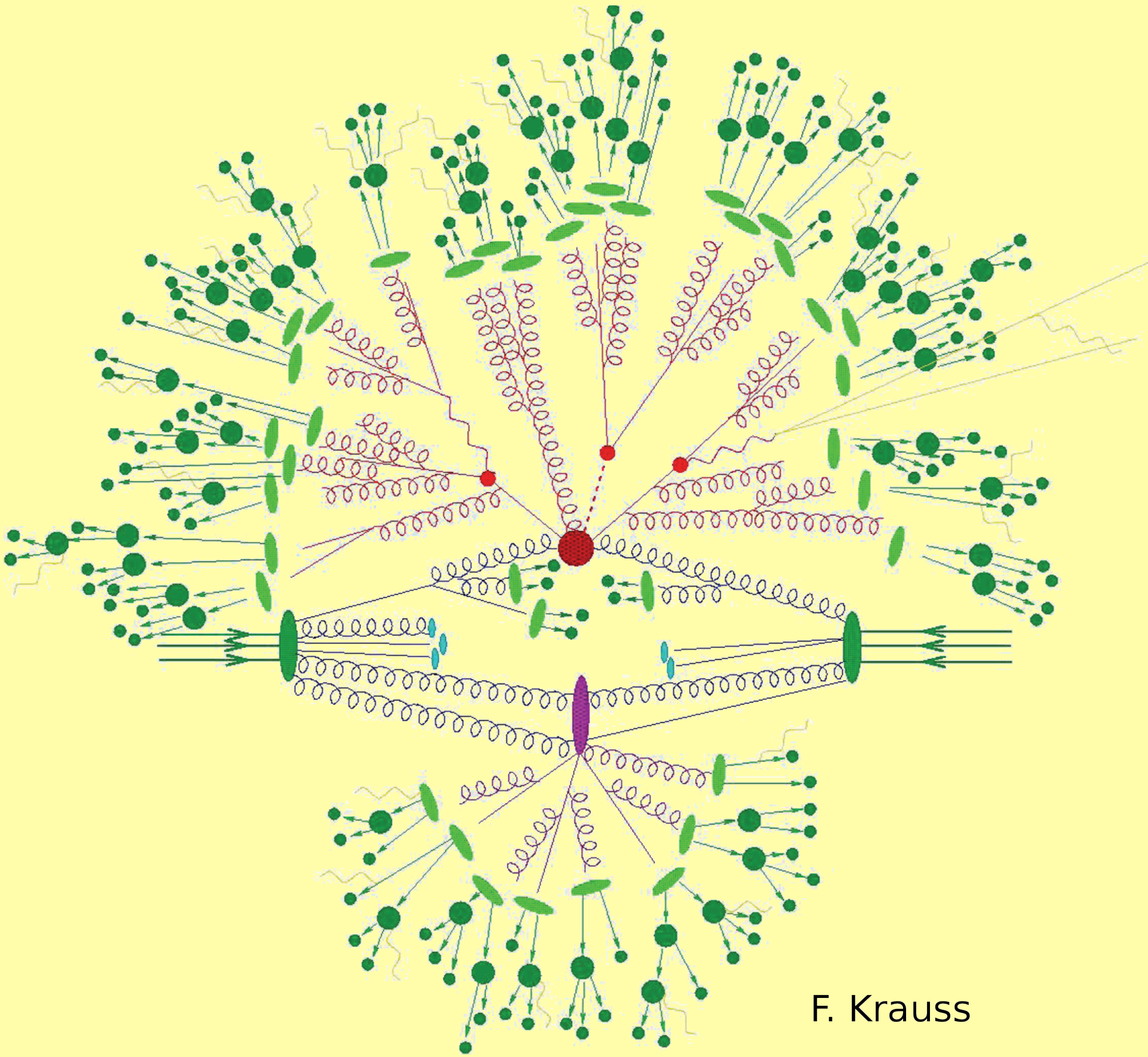
- Need to introduce renormalization scale to define α_s, and a factorization scale to separate long-distance physics
- Physical observables should be independent of these unphysical scales
- But truncated perturbation theory isn't: the dependence is typically of **O**(first omitted order)
- Leading Order (LO —"tree level") will have unacceptably large dependence
- Next-to-Leading Order (NLO) reduces this dependence

NLO Calculations in QCD

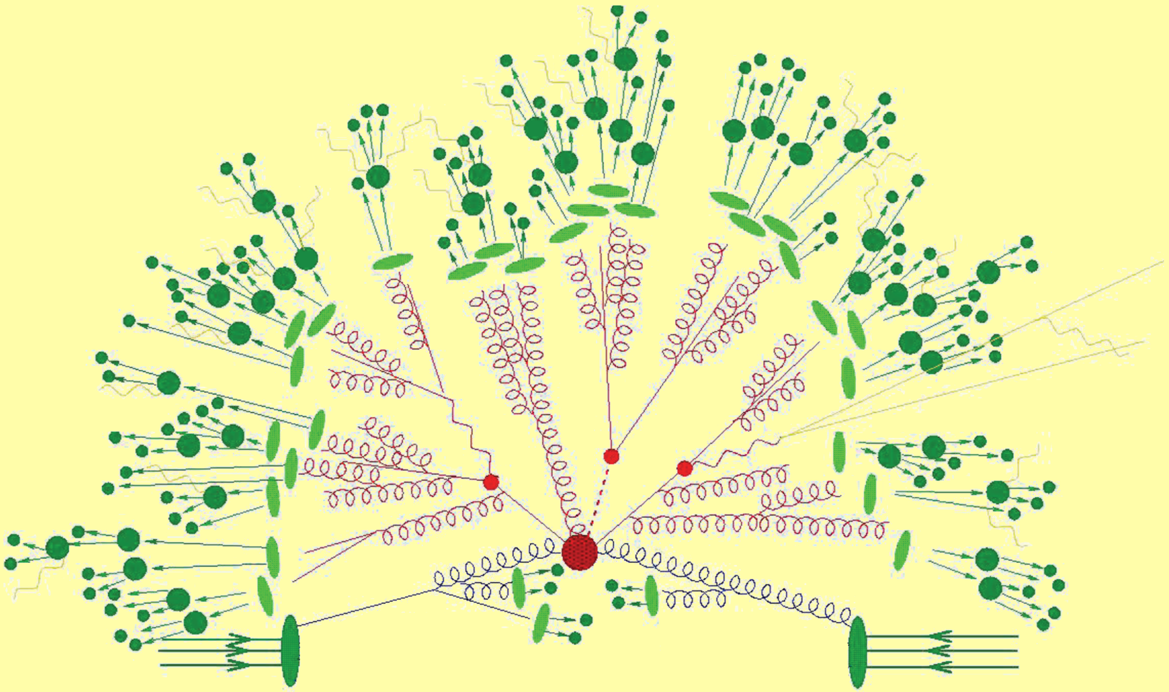
- Dramatic advances in NLO calculations over the last decade
- A standard tool
- Software libraries for one-loop amplitudes for a large classes of processes, also with many jets



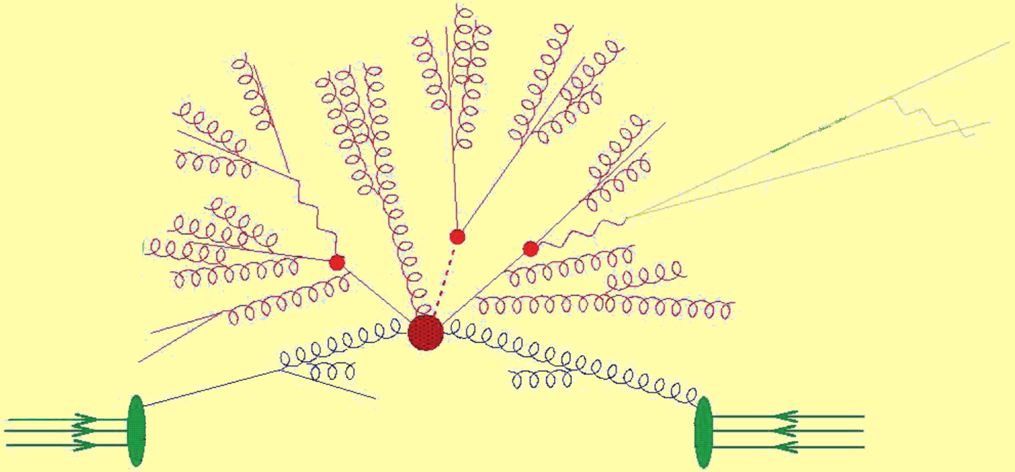
- NNLO will place even heavier demands on one-loop amplitudes
- Is there still room for improvement in methods?



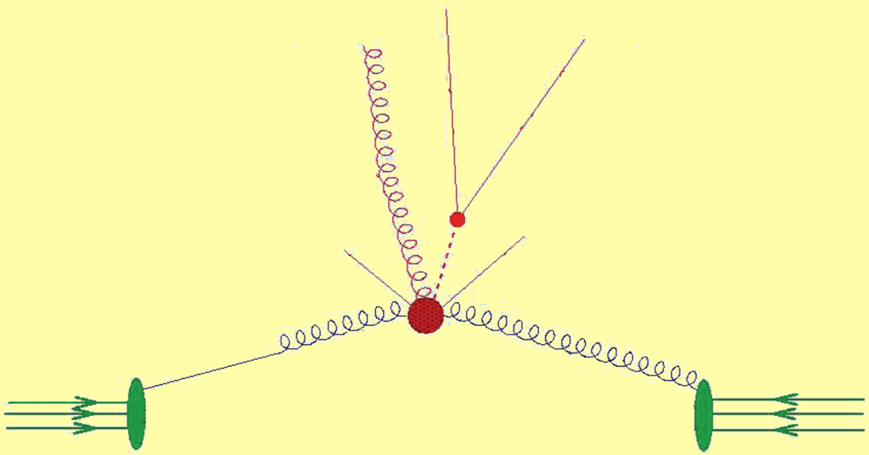
F. Krauss



F. Krauss

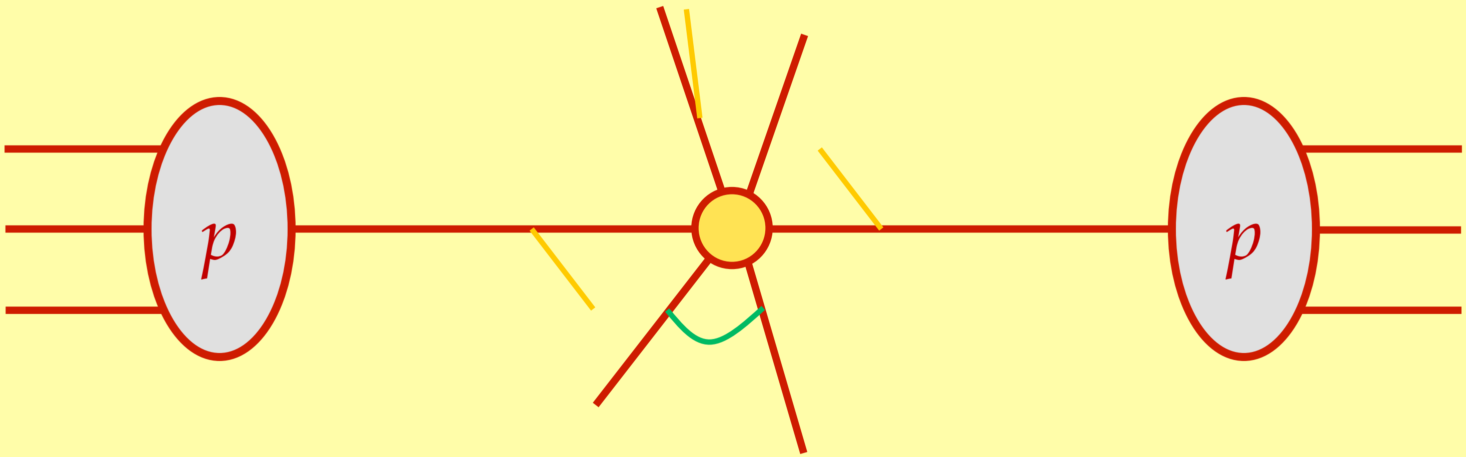


F. Krauss



F. Krauss

QCD-Improved Parton Model



$$\frac{d\sigma}{dv} = \sum_{a,b} \underbrace{\int dx_a dx_b\, f_a f_b}_{\text{long-distance}} \underbrace{\int d\text{Phase}_{ab}\ \hat{\sigma}_{ab} \delta(v - \text{Observable})}_{\text{short-distance}}$$

Parton-hadron duality

Computing Amplitudes

- Master formula (all loops)

$$\text{Ampl} = \sum_{j\in\text{Basis}} c_j(\epsilon)\,\text{Int}_j$$

Process-dependent
Rational function of spinors

Process-independent

- Master integrals determined for all *L*-loop processes, or restricted to set for given process, using IBP

- Coefficients to be computed using generalized unitarity

Implementation

- Each amplitude will be evaluated 10^5–10^7 times
- Want a numerically efficient & stable implementation
- Analytics are nice for low-point amplitudes
- Not necessarily feasible for higher-point amplitudes
  - And may not even be fastest
- Hybrid: analytics for Int_j, purely numerical evaluation for $c_j(\epsilon)$

Generalized Unitarity: Contour Integration

- Cut all propagators in the target integral topology
- Reduces corresponding amplitude to product of trees
- Cut is implemented by contour integration
- Real Integration → Complex Integration → Contour Integration
- Feynman Integrals → Their Coefficients
- Contours: Multidimensional tori that encircle *global poles*: common singularities of all cut propagators

- In one dimension, contour integration corresponds to performing a Laurent expansion, and taking the coefficient of the simple pole,

$$\cdots + \frac{c_2}{z^2} + \frac{c_1}{z} + c_0 + c_{-1} z + \cdots$$

- The analogous statement holds in higher complex dimensions when the integrand factorizes

Generalized Unitarity

- Coefficients

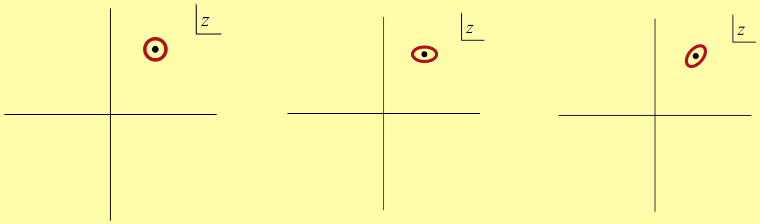
$$c_j = \sum_{\text{contours}\,\Gamma} a_\Gamma \oint_\Gamma \prod_t A_t^{(0)}$$

- Coefficients a_Γ determined by requirement that total derivatives give no contribution

- Many examples known, no general formula yet
  - limited to D=4 cuts
  - integrals with leading singularities only

Fun with Multivariate Contour Integration

- One-dimensional contour integrals are independent of the contour's shape



- Unique contour (yielding non-vanishing residue) for each pole

Fun with Multivariate Contour Integration

- Not true in higher dimensions!
- Consider $$\frac{z_1 dz_1 \wedge dz_2}{z_2(a_1 z_1 + a_2 z_2)(b_1 z_1 + b_2 z_2)}$$
- Requiring any two denominators to vanish forces $z_1 = z_2 = 0$
- "Degenerate" residue: denominator can vanish on some contours encircling the global pole
- Nonhomologous contours: more than one for each global pole
- Corresponds to different groupings of denominators in algebraic approach (Griffiths & Harris; Cattani & Dickenstein)

Nonhomologous Contours

- Arise for double boxes

- Performing integrations non-democratically (heptacut + z contour) isolates one of the two topologies:
  - Horizontal double box
  - Vertical double box
- "Bad" sharing vanquished

- Within each topology, different masters (different numerators) isolated by different linear combinations of heptacut solutions + z contour: these share some contours

A Generic Coefficient

- Perform maximal cut, then a multivariate contour integral over remaining degrees of freedom

$$c_j = \sum_\Gamma a_\Gamma \oint_\Gamma \prod_i dz_i \ \mathrm{Jacobian}^{-1} \prod_d A_d^{\mathrm{tree}}(z_i)$$

- Restrict attention to coefficients for which we can isolate contours cleanly
- Change variables to factorize poles

$$\frac{1}{(z_1 - z_1^0)(z_2 - z_2^0)\cdots}$$

- Assume that all factors have only "simple" poles

$$\text{Laurent}\, F = \sum_{j=-1}^{\infty} F^{[-j]} z^j$$

or appropriate multivariate generalization

$$\text{Laurent}\, F = \sum_{j_i=-1}^{\infty} F^{[-\vec{\jmath}]} \prod_i z_i^{j_i}$$

- Peer into the Laurent expansions to get a more explicit formula for the global residue

$$\underset{\{z\}=\{z^0\}}{\text{Global Residue}} \prod_d F_d(z) =$$

$$\sum_d F_d^{[1]} \prod_{r\neq d} F_d^{[0]}$$

$$+ \sum_{d_1,d_2,d_3} F_{d_1}^{[1]} F_{d_2}^{[1]} F_{d_3}^{[-1]} \prod_{r\neq d_i} F_d^{[0]}$$

$$+ \sum_{d_i} F_{d_1}^{[1]} F_{d_2}^{[1]} F_{d_3}^{[1]} F_{d_4}^{[-1]} F_{d_5}^{[-1]} \prod_{r\neq d_i} F_d^{[0]}$$

$$+ \sum_{d_i} F_{d_1}^{[1]} F_{d_2}^{[1]} F_{d_3}^{[1]} F_{d_4}^{[-2]} \prod_{r\neq d_i} F_d^{[0]} + \cdots$$

Calculating Laurent[Amplitude]

- How can we calculate $A^{[j]}$?

a) Calculate the A analytically, then extract the Laurent expansion analytically
   - suitable for small number of legs

But:
   - not efficient at larger n (exponential vs polynomial complexity)
   - not amenable to direct numerical calculation
   - doesn't allow combining subexpressions across coefficients

b) Seek recursive approach

$$A_n = f(A_i, A_j, \ldots) \qquad i, j, \ldots \leq n$$

$$\text{Laurent } A_n = f(\text{Laurent } A_i, \text{Laurent } A_j, \ldots) \qquad i, j, \ldots \leq n$$

- Apply this to recursion relation
  - BCFW not so suitable
  - Use Berends–Giele instead

- Recursion for off-shell current $J\downarrow\mu$

- Schematically (in amputated form)

$$J^\mu(1,\ldots,n) = \\ -\sum_{j=1}^{n-1}{}' V_3^{\mu\nu'\rho'} \frac{d_{\nu'\nu}}{K_{1,j}^2} \frac{d_{\rho'\rho}}{K_{j+1,n}^2} J^\nu(1,\ldots,j) J^\rho(j+1,\ldots,n) \\ -\sum_{j_1=1}^{n-2}{}' \sum_{j_2=j_1+1}^{n-1}{}' V_4^{\mu\nu'\tau'\rho'} \frac{d_{\nu'\nu}}{K_{1,j_1}^2} \frac{d_{\tau'\tau}}{K_{j_1+1,j_2}^2} \frac{d_{\rho'\rho}}{K_{j_2+1,n}^2} \\ \times J^\nu(1,\ldots,j_1) J^\tau(j_1+1,\ldots,j_2) J^\rho(j_2+1,\ldots,n)$$

Continuing schematically

$$\text{Laurent}\, J^{\mu}(1,\ldots,n) =$$

$$-\sum_{j=1}^{n-1}{}' \text{Laurent}\ V_3^{\mu\nu'\rho'}\ \text{Laurent}\ \frac{d_{\nu'\nu}}{K_{1,j}^2}\ \text{Laurent}\ \frac{d_{\rho'\rho}}{K_{j+1,n}^2}$$

$$\times\ \text{Laurent}\ J^{\nu}(1,\ldots,j)\,\text{Laurent}\ J^{\rho}(j+1,\ldots,n)$$

$$-\cdots$$

We obtain a system of recursions for the expansion coefficients

$$J^{[r]\mu}(1,\ldots,n) =$$

$$-\sum_{j=1}^{n-1}{}' \sum_{r_i=r_{\min}(r)<0}^{1} \delta_{r_1+r_2+r_3+r_4+r_5} V_3^{[r+r_1]\mu\nu'\rho'} \left(\frac{d_{\nu'\nu}}{K_{1,j}^2}\right)^{[r_2]} \left(\frac{d_{\rho'\rho}}{K_{j+1,n}^2}\right)^{[r_3]}$$

$$\times\ J^{[r_4]\nu}(1,\ldots,j) J^{[r_5]\rho}(j+1,\ldots,n)$$

$$-\cdots$$

Example: One-Loop Triangle

- Master formula for one-loop amplitudes

Process-dependent ϵ-free
Rational function of spinors

Process-independent

$$\mathrm{Ampl} = \sum_{j \in \mathrm{Basis}} c_j \, \mathrm{Int}_j + \mathrm{Rational}$$

D=4 Unitarity

D-dimensional unitarity: coefficient of μ↑2 integrals

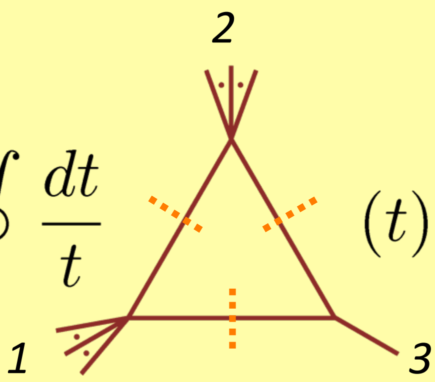
- Basis consists of boxes, triangles, and bubbles

Example: Triangle Coefficients

Four-fold contour integral translates into maximal cuts, plus one additional degree of freedom for triangles.

Coefficients given by residues at ∞ — Forde (2007)

$$\mathrm{coeff} = \frac{1}{2\pi i} \oint \frac{dt}{t}$$



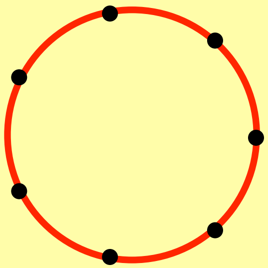
$$(t)$$

Can also write this as $-\mathrm{Inf}_t \left(A_1(t) A_2(t) A_3(t)\right)\big|_{t=0}$

where Inf is the expansion as $t \to \infty$

Example: Triangle Coefficients

In BLACKHAT, contour integral is evaluated numerically using a discrete Fourier projection (exact!)

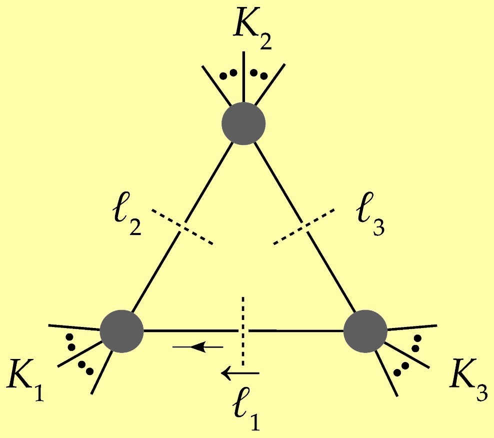


Known box integrand first subtracted for numerical stability as in the OPP procedure

Ossola–Papadopoulos–Pittau

$$A_1(t)A_2(t)A_3(t) - \sum_j \frac{\mathrm{Res}_{t=t_j}\, A_1A_2A_3}{t-t_j}$$

Example: Triangle Coefficient



Build massless momenta

$$K_1^{\flat,\mu} = K_1^\mu - \frac{K_1^2}{2K_1\cdot K_2^\flat}K_2^{\flat,\mu}$$

$$K_2^{\flat,\mu} = K_2^\mu - \frac{K_2^2}{2K_2\cdot K_1^\flat}K_1^{\flat,\mu}$$

- Parametrization

$$|\ell_i\rangle = t|K_2^\flat\rangle + a_{i1}|K_1^\flat\rangle$$

$$|\ell_i] = |K_2^\flat] + \frac{a_{i2}}{t}|K_1^\flat]$$

- t is the remaining degree of freedom; Jacobian is t

Large-t Behavior

Behavior depends only on helicities of cut legs

$$A = A^{[1]} t + A^{[0]} + A^{[-1]} t^{-1} + A^{[-2]} t^{-2} + \cdots$$

(except for (+,+) helicities, where $A \sim t^{-3}$)

- Side remark about gravity
  - Naively $\sim t^{n-1}$
  - Because of KLT, $\sim t^{2}$

Global Residue

- Want global residue at $t=\infty$
- Extract coefficient of t^0 in $A_1\, A_2\, A_3$

- Formula for triangle coefficient

$$
\begin{aligned}
-\big[&A_1^{[1]}A_2^{[1]}A_3^{[-2]} + A_1^{[-2]}A_2^{[1]}A_3^{[1]} + A_1^{[1]}A_2^{[-2]}A_3^{[1]} \\
&+A_1^{[1]}A_2^{[0]}A_3^{[-1]} + A_1^{[-1]}A_2^{[1]}A_3^{[0]} + A_1^{[0]}A_2^{[-1]}A_3^{[1]} \\
&+A_1^{[0]}A_2^{[1]}A_3^{[-1]} + A_1^{[-1]}A_2^{[0]}A_3^{[1]} + A_1^{[1]}A_2^{[-1]}A_3^{[0]} \\
&+A_1^{[0]}A_2^{[0]}A_3^{[0]}\big]
\end{aligned}
$$

- Want direct recursive equations for the $A_i^{[j]}$

- BCFW is awkward:
  - If shift legs aren't ℓ_1, ℓ_2, $\Rightarrow$ in general some contributions have a *t*-dependent internal line
  - the shift induces *t* dependence on other lines in the lower-point amplitudes
  - We then have to track new kinds of amplitudes, with *t* expansions on many legs
  - Shift legs ℓ_1, ℓ_2 require additional factors to adjust the *t* power-counting, and the rules for assigning them are cumbersome
- Instead, use Berends–Giele recursion
  - Also has better large-*n* scaling

Interchange with Recursion

$$J_n^\mu = \sum V_{3,\mu\nu\rho} \frac{1}{K_{1\cdots i}^2} J_i^\nu \frac{1}{K_{i+1\cdots n}^2} J_{n-i}^\rho + V_{4,\ldots} \cdots$$ Berends–Giele (1988)

$$\text{Laurent } J_n^\mu =$$

$$\sum \text{Laurent}\left[V_{3,\mu\nu\rho}\frac{1}{K_{1\cdots i}^2}\frac{1}{K_{i+1\cdots n}^2}\right]\text{Laurent } J_i^\nu \text{ Laurent } J_{n-i}^\rho + \cdots$$

explicitly

recursively

- First, rewrite it to make it purely cubic (still amputating the propagator)

Duhr, Höche, Maltoni; Gleisberg, Höche

$$J^\mu(1,\ldots,n) = \sum_{j=1}^{n-1}{}' \Bigg\{ - V_3^{\mu\nu'\rho'} \frac{d_{\nu'\nu}(K_{1,j})}{K_{1,j}^2} \frac{d_{\rho'\rho}(K_{j+1,n})}{K_{j+1,n}^2} J^\nu(1,\ldots,j) J^\rho(j+1,\ldots,n)$$

$$- V_{\mathrm{T}}^{\mu\nu'\tau'\rho'} \frac{d_{\nu'\nu}(K_{1,j})}{K_{1,j}^2} d_{\rho'\tau'\rho\tau}(K_{j+1,n}) J^\nu(1,\ldots,j) J^{\rho\tau}(j+1,\ldots,n)$$

$$- V_{\mathrm{T}}^{\mu\nu'\rho'\tau'} d_{\rho'\tau'\rho\tau}(K_{1,j}) \frac{d_{\nu'\nu}(K_{j+1,n})}{K_{j+1,n}^2} J^{\rho\tau}(1,\ldots,j) J^\nu(j+1,\ldots,n) \Bigg\}$$

$J^{\mu\nu}$ is a fictitious-tensor current, which also has a cubic recursion

$$J^{\mu\nu}(1,\ldots,n) = V_{\mathrm{T}}^{\mu\nu\rho'\tau'} \sum_{j=1}^{n-1}{}' \frac{d_{\rho'\rho}(K_{1,j})}{K_{1,j}^2} \frac{d_{\tau'\tau}(K_{j+1})}{K_{j,+1}^2} J^\rho(1,\ldots,j) J^\tau(j+1,\ldots,n)$$

(more recent alternative by Mafra & Schlotterer)

- Next, switch to a helicity form
- Switch to light-cone gauge $d^{\rm LC}_{\mu\nu}(K) = -g_{\mu\nu} + \frac{q^\mu K^\nu + K^\mu q^\nu}{q\cdot K}\,,$
- Using massless momentum $K^\flat = K - \frac{K^2}{2q\cdot K}q$
- Rewrite

$$d^{\rm LC}_{\mu\nu}(K) = \sum_{\sigma=\pm} \varepsilon^{(\sigma)}_\mu(K^\flat,q)\varepsilon^{(\sigma)*}_\nu(K^\flat,q) + \frac{K^2 q^\mu q^\nu}{(q\cdot K^\flat)^2}$$

- This gives us three terms in a sum (±,0) instead of summing over four momentum components, and definite-helicity currents
  - many terms in sum drop out
  - adjust phases to make *t* power-counting cleaner
  - non-zero vertices are $V_3\,(--+)$, $V_3\,(++-)$, $V_3\,(0-+)$ & cyclic, $V_T\,(-+[-+])$, $V_T\,(+-[-+])$
  - absorb additional factors into $V_3\,(0-+)$ vertex

- With $\rho_\pm = 1/K^2$, $\rho_0 = 1/q\cdot K$, the recursion is

$$J^{(\lambda)}(1,\ldots,n) =$$

$$\sum_{j=1}^{n-1}{}' \Bigg\{ -\sum_{\lambda_{1,2}=\pm,0} V_3(\lambda,-\lambda_1,-\lambda_2)\rho_{\lambda_1}\rho_{\lambda_2} J^{(\lambda_1)}(1,\ldots,j) J^{(\lambda_2)}(j+1,\ldots,n)$$

$$-\sum_{\lambda_{1,2}=\pm} V_{\mathrm{T}}(\lambda,-\lambda_1,(-\lambda_2)(\lambda_2))\rho_{\lambda_1} J^{(\lambda_1)}(1,\ldots,j) J^{(\lambda_2,-\lambda_2)}(j+1,\ldots,n)$$

$$+\sum_{\lambda_{1,2}=\pm} V_{\mathrm{T}}(\lambda,-\lambda_2,(-\lambda_1)(\lambda_1))\rho_{\lambda_2} J^{(\lambda_1,-\lambda_1)}(1,\ldots,j) J^{(\lambda_2)}(j+1,\ldots,n)\Bigg\}$$

Example: Simple forms

- Shorthand $\langle\langle 1\cdots n\rangle\rangle \equiv \langle 12\rangle\langle 23\rangle\cdots\langle (n-1)n\rangle$

$$J^{(+)}(1^+,\ldots,n^+) = 0$$

$$J^{(-)}(1^+,\ldots,n^+) = 2i\frac{q\cdot K_{1,n}K_{1,n}^2}{\langle q\,1\rangle\,\langle q\,n\rangle\,\langle\langle 1\cdots n\rangle\rangle}$$

$$J^{(+)}(1^+,\ldots,j^-,(j+1)^+,\ldots,n^+) = i\frac{K_{1,n}^2}{2q\cdot K_{1,n}\,\langle q\,1\rangle\,\langle q\,n\rangle\,\langle\langle 1\cdots n\rangle\rangle}$$

$$J^{(+-)}(1^-,2^+,\ldots,n^+) = -i\frac{\langle q\,1\rangle^2\,\langle 1\,n\rangle}{2\,\langle\langle 1\cdots n\rangle\rangle\,\langle q\,n\rangle^2}$$

$$-\,i\frac{\langle q\,1\rangle^3\;q\cdot K_{1,n}}{2\,\langle\langle 1\cdots n\rangle\rangle\,\langle q\,n\rangle}\sum_{j=1}^{n-1}\frac{\langle j\,j{+}1\rangle}{q\cdot K_{1,j}\,\langle q\,j\rangle\,\langle q\,j{+}1\rangle}$$

Recursions

- Take off-shell to be $-\ell_1$, on-shell legs to be ℓ_2, 1, 2, …
- Only first current ($\ell_2\, 1\cdots j$) is t-dependent; second ($j+1\cdots n$) is just the usual tree current
- $V_3 \sim O(t)$, $1/(\ell_2 + K_{1\cdots j})^2 \sim O(t^{-1})$
- Schematically

$$J^{[1]} = \sum_j \left(V_3 \frac{1}{(\ell_2 + K_{1\ldots j})^2}\right)^{[0]} J^{[1]}(\ell_2, 1, \ldots j) J^{\text{tree}}(j+1, \ldots, n) + \cdots$$

$$J^{[0]} = \sum_j \sum_{r=-1}^{0} \left(V_3 \frac{1}{(\ell_2 + K_{1\ldots j})^2}\right)^{[r]} J^{[-r]}(\ell_2, 1, \ldots j) J^{\text{tree}}(j+1, \ldots, n) + \cdots$$

$$J^{[-1]} = \sum_j \sum_{r=-2}^{0} \left(V_3 \frac{1}{(\ell_2 + K_{1\ldots j})^2}\right)^{[r]} J^{[-1-r]}(\ell_2, 1, \ldots j) J^{\text{tree}}(j+1, \ldots, n) + \cdots$$

Tower of Recursions

- $J\uparrow[1]$ is a function of lower-point $J\uparrow[1]$ and $J\uparrow$tree
- $J\uparrow[0]$ is a function of lower-point $J\uparrow[1]$, $J\uparrow[0]$, and $J\uparrow$tree
- $J\uparrow[-1]$ is a function of lower-point $J\uparrow[1]$, $J\uparrow[0]$, $J\uparrow[-1]$, and $J\uparrow$tree

Example

- Simplest configuration

$$J^{[1]\,(-)}(\ell_2^+, 1^+, \ldots, n^+) = 0$$

$$J^{[0]\,(-)}(\ell_2^+, 1^+, \ldots, n^+) = -i\,\frac{\langle K_1^\flat | K_{1\ldots n} | K_2^\flat]\,\left\langle q\, K_2^\flat \right\rangle}{\left\langle 1\, K_1^\flat \right\rangle \langle q\, n\rangle\, \langle\!\langle 1 \cdots n \rangle\!\rangle}$$

$$\vdots$$

Next: Purely Rational Terms

- Can be recast as contour integrals for $I_4[\mu^4]$, $I_3[\mu^2]$, $I_2[\mu^2]$ where μ^2 are (-2ϵ)-dimensional components

Badger

  - Single-variable expansion in μ^2 for the box
  - Two-variable expansion in μ^2 and t for the triangle
  - Bubbles will need some additional tricks

Summary

- Recursive approach to integral coefficients
- Exploit general structure of integral coefficients as global residues, and interchange Laurent expansion and recursion
- Compatible with purely numerical evaluation
- Opens possibilities for more common sub-expression evaluation