Five-point two-loop master integrals in QCD

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Based on work with with Thomas Gehrmann and Johannes Henn

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and work in preparation

Outline

Introduction

1-loop NLO established in the last decade as the new standard for high-multiplicity processes. BlackHat, Gosam, OpenLoops, NJet ...

2-loop NNLO is the current frontier

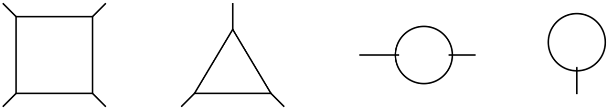
- $2 \to 2$ processes calculated recently ($\gamma\gamma$, ZZ, $Z\gamma$, $W\gamma$, WW, $t\bar{t}$, Hj, Wj, jj)
  [Catani, Cieri, de Florian, Ferrera, Grazzini, Gehrmann, G.-De Ridder, Glover, Czakon, Fiedler, Mitov, Kallweit, Maierhöfer, Rathlev, Chen, Jaquier, Melnikov, Caola, Schulze, Tejeda-Yeomans, Huss, Morgan, Boughezal, Campbell, Ellis, Focke, Giele, Liu, Petriello, Williams ...]
  Many recent results on $pp \to HH$ [Munich+OpenLoops (DeFlorian et al.); Mass effects (NLO): Gosam+SecDec (Borowka et al.), DeGrassi,Giardino,Gröber]

- **$2 \to 3$ computations are still an open field**
  E.g.: $pp \to 3j$ is an important process for precise α_s determination.

Introduction

Among NNLO bottle-necks:
two-loop scattering amplitudes $\longrightarrow$ **purely virtual contribution**.

At one-loop Feynman diagrams can be decomposed into a small set of **master integrals** (MIs), all of which are known.



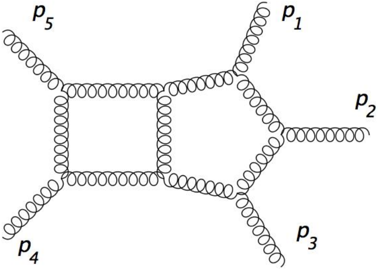
At two-loop much larger set of MIs $\rightarrow$ extends to higher multiplicities.

2-loop five-point planar integrals

I present the computation of the full set of planar master integrals at 5-pt

$$G_{\{a_1,\ldots,a_{11}\}} = \int \frac{d^D k_1\, d^D k_2}{(i\pi^{D/2})^2} \frac{D_9^{-a_9} D_{10}^{-a_{10}} D_{11}^{-a_{11}}}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5} D_6^{a_6} D_7^{a_7} D_8^{a_8}}$$

$$\begin{aligned}
D_1 &= -k_1^2\,, \\
D_2 &= -(k_1+p_1)^2\,, \\
D_3 &= -(k_1+p_1+p_2)^2\,, \\
D_4 &= -(k_1+p_1+p_2+p_3)^2\,, \\
D_5 &= -k_2^2\,, \\
D_6 &= -(k_2+p_1+p_2+p_3)^2\,, \\
D_7 &= -(k_2+p_1+p_2+p_3+p_4)^2\,, \\
D_8 &= -(k_1-k_2)^2\,, \\
D_9 &= -(k_1+p_1+p_2+p_3+p_4)^2\,, \\
D_{10} &= -(k_2+p_1)^2\,, \\
D_{11} &= -(k_2+p_1+p_2)^2
\end{aligned}$$



$$s_{ij} = (p_i + p_j)^2$$

$$\vec{x} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{51}\}$$

Also calculated by [Papadopoulos, Tommasini, Wever]

Integration-by-part identities and differential equations

Given a Feynman integral

$$G(a_1,a_2,\ldots,a_n) = \int \prod_{j=1}^{l} \frac{d^D k_j}{i\pi^{D/2}} \frac{1}{D_1^{a_1}\ldots D_n^{a_n}} \quad , \quad \text{where } D_i = (k_j - p_i, - \ldots)^2$$

Integration by part identities

$$\int \prod_{j=1}^{l} \frac{d^D k_j}{i\pi^{D/2}} \left(\frac{\partial}{\partial k_j^\mu} v^\mu \frac{1}{D_1^{a_1}\ldots D_n^{a_n}} \right) = 0$$

(v^μ is appropriately chosen vector, e.g. $k_j^\mu - p_1^\mu$)

$\rightarrow$ terms with same denominators D_i , but different indices $a_1, a_2, \ldots$

relate different integrals $\Longrightarrow$ we can reduce them to MIs.[Laporta alg.]

AIR[Anastasiou, Lazopoulos], Fire [Smirnov], Reduze [Studerus, Manteuffel], LiteRed [Lee]

Integration-by-part identities and differential equations

Derivaties w.r.t external kinematic invariants, e.g. $s_{12} = (p_1 + p_2)^2$

$$\frac{\partial}{\partial s_{12}} \int \prod_{j=1}^{l} \frac{d^D k_j}{i\pi^{D/2}} \frac{1}{D_1^{a_1} \ldots D_n^{a_n}} = \int \prod_{j=1}^{l} \frac{d^D k_j}{i\pi^{D/2}} \frac{1}{2 s_{12}} \left((p_1+p_2)^\mu \frac{\partial}{\partial (p_1+p_2)^\mu} \right) \frac{1}{D_1^{a_1} \ldots D_n^{a_n}}$$

$$\longrightarrow \sum c_{s_{12};b_1\ldots,b_n} \int \prod_{j=1}^{l} \frac{d^D k_j}{i\pi^{D/2}} \frac{1}{D_1^{b_1} \ldots D_n^{b_n}}$$

on the R.H.S.:
same D_i s , but different indices: same topology + its subtopologies appear

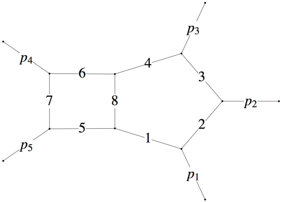
reduced to master integrals using IBP relations

$\Rightarrow$ differential equations for MIs. [Gehrmann, Remiddi]

Codes used: Fire [Smirnov], Reduze [von Manteuffel]

2-loop five-point planar integrals

61 MIs



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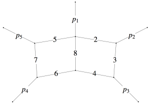

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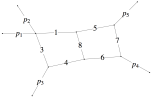

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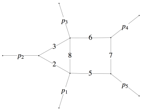

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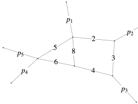

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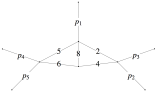

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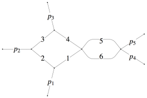

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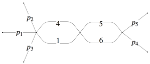

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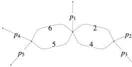

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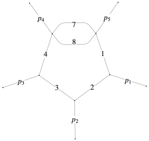

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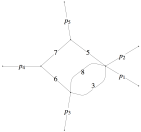

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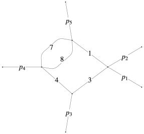

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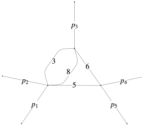

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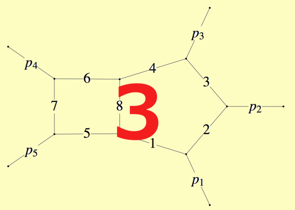
2-loop five-point planar integrals

61 MIs , 10 new



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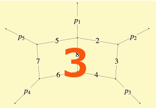
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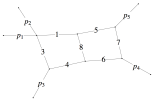
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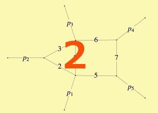
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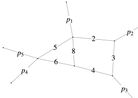
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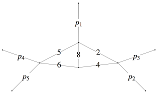
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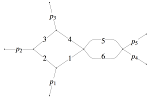
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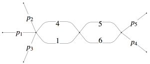
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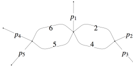
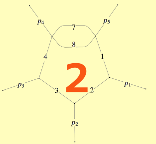

{11, G[1, {0, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0}], 1

≤ 4 point MIs known

[Gehrmann, Remiddi (2001)]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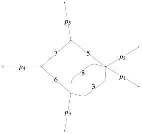
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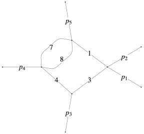
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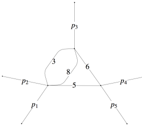
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Integration-by-part identities and differential equations

MIs basis is not unique. Suitable choice considerably simplifies diff. eqs.:

$\partial_{x_i}\vec{f} = A_i(x,\varepsilon)\vec{f} \longrightarrow \partial_{x_i}\vec{f} = \varepsilon\tilde{A}_i(x)\vec{f}$ can be integrated order by order in ε.

[J. Henn]

$$\vec{f}(x,\varepsilon) = \vec{f}_0(x) + \varepsilon\vec{f}_1(x) + \varepsilon^2\vec{f}_2(x) + \ldots$$

$$\begin{aligned} \partial_{x_i}\vec{f}_0(x) &= 0 \\ \partial_{x_i}\vec{f}_1(x) &= \tilde{A}_i(x)\vec{f}_0 \\ \partial_{x_i}\vec{f}_2(x) &= \tilde{A}_i(x)\vec{f}_1(x) \\ &\ldots \end{aligned} \quad\Longrightarrow\quad \begin{aligned} \vec{f}_0(x) &= \vec{f}_0 \\ \vec{f}_1(x) &= \int dx A(x)\vec{f}_0 \\ \vec{f}_2(x) &= \int dx A(x)\vec{f}_1(x) \\ &\ldots \end{aligned}$$

Transcendental weight = number of successive integrations

Starting from $\vec{f}_0(x) = \vec{f}_0$ $\rightarrow$ weight-0 constant

$\Rightarrow$ each order in ε has **uniform transcendental weight** .

The alphabet

Further simplification:

$$d\vec{f}(\vec{x},\varepsilon) = \varepsilon\, d\left[\sum_k a_k \log \alpha_k(\vec{x})\right]\vec{f}(\vec{x},\varepsilon)$$

a_k constant matrices. The list of functions $\{\alpha_1, \ldots \alpha_n\}$ is the **alphabet**.

Alphabet of 26 letter

$$\left\{ v_1 \;,\; v_3+v_4 \;,\; v_1-v_4 \;,\; v_1+v_2-v_4 \;,\; \Delta \;,\; \frac{a-\sqrt{\Delta}}{a+\sqrt{\Delta}} \right\} + \text{cyclic}$$

with $a = v_1v_2 - v_2v_3 + v_3v_4 - v_1v_5 - v_4v_5 = \mathrm{tr}[\not{p}_4\not{p}_5\not{p}_1\not{p}_2] \quad (v_i \equiv s_{i,i+1})$

Gram determinant $\Delta = |2p_i \cdot p_j| = (\mathrm{tr}_5)^2$ with $\mathrm{tr}_5 = \mathrm{tr}[\gamma_5\not{p}_4\not{p}_5\not{p}_1\not{p}_2]$.

Chen-Iterated integrals

At each order in ε the solution of the differential equation

$$d\vec{f}(\vec{x},\varepsilon) = \varepsilon\, d\left[\sum_k a_k \log \alpha_k(\vec{x})\right] \vec{f}(\vec{x},\varepsilon)$$

is expressed in terms of Chen Iterated Integrals [Chen ('77)]

$$\begin{aligned}
CII[\alpha_i] &= \int_\gamma d\log\alpha_i \longrightarrow \log|\alpha_i| \\
CII[\alpha_i,\alpha_j] &= \int d\log\alpha_j \int d\log\alpha_i \\
CII[\alpha_i,\alpha_j,\alpha_k] &= \int d\log\alpha_k \int d\log\alpha_j \int d\log\alpha_i \\
&\longrightarrow \int_{0<t_1<\cdots<t_n<1} dt_1 \ldots dt_n \frac{d}{dt_1}\log\alpha_i(t_1)\ldots\frac{d}{dt_n}\log\alpha_j(t_n)
\end{aligned}$$

Chen-Iterated integrals

Given their definition via differential of logarithms, each entry of the C.I.I. satisfies

$$CII[\ldots, \frac{\alpha_i}{\alpha_j}, \ldots] = CII[\ldots, \alpha_i, \ldots] - CII[\ldots, \alpha_j, \ldots]$$

and form a **shuffle algebra**

$$CII[\,\vec{\alpha}\,]\, CII[\,\vec{\beta}\,] = \sum_{\gamma \in \alpha \sqcup\!\sqcup \beta} CII[\,\vec{\gamma}\,]\,.$$

E.g.:

$$\begin{aligned} CII[\alpha_i] &= \log|\alpha_i| \\ CII[\alpha_p, \alpha_q] + CII[\alpha_q, \alpha_p] &= CII[\alpha_p]\, CII[\alpha_q] = \log|\alpha_p| \log|\alpha_q| \end{aligned}$$

Representation as of C.I.I & Polylogarithms: weights 1 & 2

At **weight 1** : only logarithms of the Mandelstam invariant can appear

$$f_{1,1}^{(i)} = CII[v_i] = \log(-v_i)\,, \quad i = 1\ldots 5$$

as other letters would give rise to un-physical singularities.

At **weight 2** :

$$f_{2,1}^{(i)} = CII\left[\frac{v_i}{v_{i+2}}, \frac{v_{i+2}-v_i}{v_{i+2}}\right] \to \mathrm{Li}_2\left(1-\frac{v_i}{v_{i+2}}\right)\,, \quad i = 1\ldots 5$$

are the only weight-2 functions appearing.

$$\mathrm{Li}_n(z) = \int_0^z \frac{dt_n}{t_n}\int_0^{t_n}\frac{dt_{n-1}}{t_{n-1}}\ldots\int_0^{t_2}\frac{dt_1}{1-t_1} = \sum_{k\geq 1}\frac{z^k}{k^n}$$

Representation as of C.I.I & Polylogarithms: 3-pt @ w3

At **weight 3** :
the new independent functions appearing are

$$
\begin{aligned}
f_{3,1} &= CII\,[x_1, 1+x_1\,, 1+x_1] \;\longrightarrow\; -\mathrm{Li}_3\,(1+x_1)\\
f_{3,2} &= CII\,[x_2, 1+x_2\,, 1+x_2] \;\longrightarrow\; -\mathrm{Li}_3\,(1+x_2)\\
f_{3,3} &= CII\,[x_1, 1+1/x_1\,, 1+1/x_1] \;\longrightarrow\; -\mathrm{Li}_3\,(1+1/x_1)\\
f_{3,4} &= CII\,[x_1, 1+1/x_2\,, 1+1/x_2] \;\longrightarrow\; -\mathrm{Li}_3\,(1+1/x_2)
\end{aligned}
$$

It is useful to define the ratios:

$$[\, x_1 = -v_1\,/\,v_4 \;, \quad x_2 = -v_2\,/\,v_4\,] \quad \text{and cyclic}$$

E.g.:

$$-\frac{\alpha_6}{\alpha_4} = -\frac{v_1 - v_4}{v_4} \to 1 + x_1 \qquad \frac{\alpha_9}{\alpha_4} = \frac{v_4 - v_2}{v_4} \to 1 + x_2$$

$$\frac{\alpha_6}{\alpha_1} = \frac{v_1 - v_4}{v_4} \to 1 + 1/x_1 \qquad -\frac{\alpha_9}{\alpha_2} = -\frac{v_4 - v_2}{v_2} \to 1 + 1/x_2$$

$$-\frac{\alpha_{11}}{\alpha_4} = -\frac{v_1 + v_2 - v_4}{v_4} \to 1 + x_1 + x_2 \qquad -\frac{\alpha_{16}}{\alpha_4} = -\frac{v_1 + v_2}{v_4} \to x_1 + x_2$$

Representation as of C.I.I & Polylogarithms: 4-pt @ w3

At **weight 3** the 4-pt integrals require the letter $1+x_1+x_2$

$$
\begin{aligned}
f_{3,5} = {} & CII[x_1, 1+x_1, x_2] + CII[x_2, 1+x_2, x_1] \\
& - CII[x_1, 1+x_1, 1+x_1+x_2] - CII[x_2, 1+x_2, 1+x_1+x_2] \\
& + CII[x_1, x_2, 1+x_1+x_2] + CII[x_2, x_1, 1+x_1+x_2] \\
& - \zeta_2\, CII[1+x_1+x_2]
\end{aligned}
$$

$$
\begin{aligned}
\to & - \mathrm{Li}_3(-x_1) - \mathrm{Li}_3(-x_2) - \mathrm{Li}_3\left(\frac{1+x_1+x_2}{x_1}\right) - \mathrm{Li}_3\left(\frac{1+x_1+x_2}{x_2}\right) \\
& + \log(-x_2)\,\mathrm{Li}_2\left(\frac{1+x_1+x_2}{x_1}\right) + \log(-x_1)\,\mathrm{Li}_2\left(\frac{1+x_1+x_2}{x_2}\right) + 3\zeta_3
\end{aligned}
$$

together with their cyclic permutations.
These are the only 4-pt functions appearing at weight-3.

Representation as of C.I.I & Polylogarithms: 5-pt @ w3

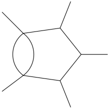
At weight 3 there is only **one genuine 5-point function** appearing

$$f_{3,6} \equiv \Phi_5 = \frac{2}{3} d_{37,3} + \Big\{ CII[\alpha_1/\alpha_3\,,\,\alpha_4/\alpha_8\,,\,\alpha_{21}] - CII[\alpha_4/\alpha_1\,,\,\alpha_3/\alpha_6\,,\,\alpha_{21}]$$
$$+ \zeta_2\, CII[\alpha_{21}] + \text{cycl.} \Big\} \qquad \left[\alpha_{21} = \frac{a_1 + \sqrt{\Delta}}{a_1 - \sqrt{\Delta}} \right]$$

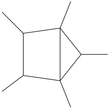
The fist two integrations can be easily performed analytically

$$\rightarrow \qquad \frac{2}{3} d_{37,3} + \sum_{\text{cyclic}} \int_0^1 dt\, \partial_t \left(\log \frac{\tilde{a}_1 + \sqrt{\tilde{\Delta}}}{\tilde{a}_1 - \sqrt{\tilde{\Delta}}} \right) \Big[\log(-\tilde{v}_1) \log(-\tilde{v}_4)$$
$$- \log(-\tilde{v}_4) \log(-\tilde{v}_3) + \frac{1}{2} \log^2(-\tilde{v}_3) - \frac{1}{2} \log^2(-\tilde{v}_1) + \zeta_2$$
$$+ \mathrm{Li}_2(1 - \tilde{v}_1/\tilde{v}_3) - \mathrm{Li}_2(1 - \tilde{v}_4/\tilde{v}_1) \Big] , \qquad \tilde{v}_i = -1 - t(v_i - 1)$$

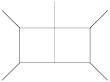
Representation as of C.I.I & Polylogarithms: 5-pt @ w3



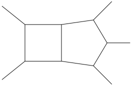
$$I_{37}^{(4)} = 3\,\Phi_5/2$$



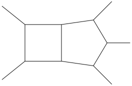
$$I_{41,45,47}^{(4)} = 3\,\Phi_5$$



$$I_{51,58}^{(4)} = -\Phi_5$$



$$I_{59}^{(4)} = \Phi_5/2$$



$$I_{61}^{(4)} = -\Phi_5/2$$

all other five-point integrals can be written using four-point functions.

Representation as of C.I.I & Polylogarithms: 3-pt @ w4

Besides the products of lower weight functions and the Li_4 functions

$$
\begin{aligned}
f_{4,1} &= CII\left[x_1, 1+x_1, 1+x_1, 1+x_1\right] \longrightarrow -\mathrm{Li}_4\left(1+x_1\right) \\
f_{4,2} &= CII\left[x_2, 1+x_2, 1+x_2, 1+x_2\right] \longrightarrow -\mathrm{Li}_4\left(1+x_2\right) \\
f_{4,3} &= CII\left[x_1, 1+1/x_1, 1+1/x_1, 1+1/x_1\right] \longrightarrow -\mathrm{Li}_4\left(1+1/x_1\right) \\
f_{4,4} &= CII\left[x_1, 1+1/x_2, 1+1/x_2, 1+1/x_2\right] \longrightarrow -\mathrm{Li}_4\left(1+1/x_2\right)
\end{aligned}
$$

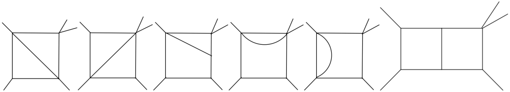
The additional functions appears

$$
f_{4,5} = CII\left[x_1, x_1, x_1, x_1\right] - 2\,CII\left[x_1, x_1, x_1, 1+x_1\right]
$$

$$
\begin{aligned}
\rightarrow\ & \frac{1}{6}\log^3(-x1)\Big[-\log(1+1/x1)-\log(1+x1)\Big] \\
& -\frac{1}{2}\log^2(-x1)\Big[-\mathrm{Li}_2(-1/x1)+\mathrm{Li}_2(-x1)\Big] \\
& +\log(-x1)\Big[\mathrm{Li}_3(-1/x_1)+\mathrm{Li}_3(-x_1)\Big]+\mathrm{Li}_4(-1/x_1)-\mathrm{Li}_4(-x_1)
\end{aligned}
$$

Representation as of C.I.I & Polylogarithms: 4-pt @ w4

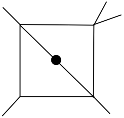
For 4-point topologies $\mathrm{Li}_{2,2}$ functions are needed



$$\rightarrow \mathrm{Li}_{2,2}(z_1, z_2) = \sum_{k_1 > k_2 \geq 1} \frac{z_1^{k_1}}{k_1^2} \frac{z_2^{k_2}}{k_2^2}$$

The letter $v_1 + v_2$ makes its appearence only at weight 4 and only in $\longrightarrow$

which also contains $\mathrm{Li}_{2,2}$ function



A useful representation for the four-points integrals is

$$\int_0^1 dt\, \partial_t \log \alpha_i(t)\, [\text{analytic w3 function}](t)$$

Representation as of C.I.I & Polylogarithms: 5-pt @ w4

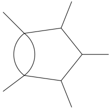
In some 5-point integrals Δ does not appear as a letter.
They all however depend on the 21$^{\text{th}}$ through 25$^{\text{th}}$ letters of the alphabet

$$r_i = \frac{a_i - \sqrt{\Delta}}{a_i + \sqrt{\Delta}} \qquad i = 1, \ldots, 5 \qquad \text{and}$$
$$a_i = \text{Cyclic}_i \Big[v_1 v_2 - v_2 v_3 + v_3 v_4 - v_1 v_5 - v_4 v_5 \Big]$$

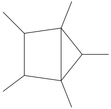
can be written in terms of the functions

$$f_{4,8}^{(i)} = \int d \log r_i \, \Phi_5 \; , \quad i = 1, \ldots, 5$$

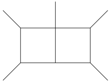
Representation as of C.I.I & Polylogarithms: 5-pt @ w4



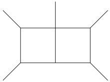
$$I^{(4)}_{38} = \frac{3}{2} f^{(4)}_{4,8} + \ldots$$



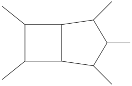
$$I^{(4)}_{40,44,46} = \frac{3}{2}\left(f^{(p)}_{4,8} - f^{(p+2)}_{4,8}\right) + \ldots \; [p = 5, 1, 3]$$



$$I^{(4)}_{49,56} = -f^{(p)}_{4,8} - f^{(p+1)}_{4,8} + \ldots \; [p = 5, 2]$$



$$I^{(4)}_{50,57} = 2\left(f^{(p)}_{4,8} + f^{(p+1)}_{4,8} - f^{(p+3)}_{4,8}\right) + \ldots \; [p = 5, 2]$$

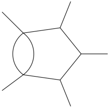


$$I^{(4)}_{60} = 2 f^{(4)}_{4,8} - f^{(1)}_{4,8} - f^{(2)}_{4,8} + \ldots$$

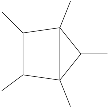
Representation as of C.I.I & Polylogarithms: 5-pt @ w4

The remaining 5-point integrals depend on the 26th letter of our alphabet Δ through

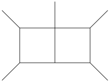
$$f_{4,9} = \int d\log\Delta\,\Phi_5$$



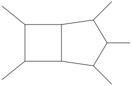
$$I^{(4)}_{37} = \frac{3}{2} f_{4,9} + \ldots$$



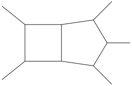
$$I^{(4)}_{41,45,47} = 3 f_{4,9} + \ldots$$



$$I^{(4)}_{51,58} = \frac{1}{2} f_{4,9} + \ldots$$



$$I^{(4)}_{59} = -f_{4,9} + \ldots$$



$$I^{(4)}_{61} = -\frac{1}{2} f_{4,9} + \ldots$$

Representation as of C.I.I & Polylogarithms: 5-pt @ w4

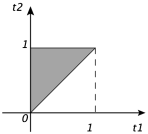
Due to the 1-fold integral representation of Φ_5 the functions

$$\int d\log r_i\, \Phi_5 \;, \quad i=1,\ldots,5 \qquad \text{and} \qquad \int d\log \Delta\, \Phi_5 \;,$$

come about as 2-fold integrals

$$\int_0^1 dt_2\, \partial_{t_2} \log \alpha_i(t_2) \int_0^{t_2} dt_1\, \partial_{t_1} \log \alpha_j(t_1)\, [\text{analytic w2 function}](t_1)$$

by exchanging order of integration it easily becomes a 1-fold integral [Henn,Caron-Huot]



$$\to \int_0^1 dt_1\, \partial_{t_1} \log \alpha_j(t_1)\, [\text{analytic w2 function}](t_1) \left[\int_{t_1}^1 dt_2\, \partial_{t_2} \log \alpha_i(t_2)\right]$$

Representation as Goncharov Polylogarithms

Goncharov Polylogarithms: very well suited for numerical evaluation
[Goncharov, implemented in C++ within GiNaC by Vollinga, Weinzierl]

$$G(a_1, a_2, \dots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \dots, a_n; t) ,$$

$$\text{with } G(x) = 1 , \quad G(0) = 0 \quad \text{and} \quad G(\vec{0}_n; x) = \frac{1}{n!} \log^n x .$$

Simple examples:

$$\begin{aligned} G(\vec{a}_n; x) &= \frac{1}{n!} \log^n \left(1 - \frac{x}{a}\right) \\ G(\vec{0}_{n-1}, a; x) &= -\mathrm{Li}_n\left(\frac{x}{a}\right) \\ G(\vec{0}_{n-1}, \vec{a}_p; x) &= (-1)^p S_{n,p}\left(\frac{x}{a}\right) \\ G(0,1,0,1/y; x) &= \mathrm{Li}_{2,2}(x,y) \end{aligned}$$

Representation as Goncharov Polylogarithms

With a suitable parametrization (Momentum Twistor variables) [Hodges (2009)]

$$\begin{aligned} v_1 &= x_1 \,, \\ v_2 &= x_1 x_2 x_4 \,, \\ v_3 &= (x_1/x_2)\left[x_3(x_4-1) + x_2 x_4 + x_2 z_3 (x_4 - x_5)\right] , \\ v_4 &= x_1 x_2 (x_4 - x_5) \,, \\ v_5 &= x_1 x_3 (1 - x_5) \end{aligned}$$

the Gram determinant becomes a perfect square

$$\sqrt{\Delta} = -x_1^2\Big[x_2 x_4(x_5-1) + x_3\Big(1 + x_2 x_5 + x_4(-2 - x_2 + x_5)\Big)\Big]$$

After partial-fractioning

$$d\vec{f}(\vec{x},\varepsilon) = \varepsilon\, d\left[\sum_k a_k \log \alpha_k(\vec{x})\right]\vec{f}(\vec{x},\varepsilon) \quad \longrightarrow \quad \partial_{x_i}\vec{f} = \varepsilon \sum_k \frac{\tilde{a}_k}{x_i - x_{i,k}}\vec{f}$$

the solution can be expressed in terms of Goncharov Polylogs.

Representation as Goncharov Polylogarithms

Integration path: polygonal chain along x_i axes $\longrightarrow$ $\{x_2, x_5, x_3, x_4, x_1\}$.

- x_2 is the first variable to be integrate to avoid the appearance of additional square roots from partial fractioning.

$$\partial_{x_2} I'^{n}_{W}(x_2; \{x_5, x_3, x_4, x_1\}_{\mathrm{B.}}) = \sum_{j,k} \frac{a_{2,j,k}}{x_2 - x_{2,k}(\{x_5, x_3, x_4, x_1\}_{\mathrm{B.}})} I'^{j}_{W-1}(x_2; \{x_5, x_3, x_4, x_1\}_{\mathrm{B.}})$$

$$\partial_{x_5} I''^{n}_{W}(x_2, x_5; \{x_3, x_4, x_1\}_{\mathrm{B.}}) = \sum_{j,k} \frac{a_{5,j,k}}{x_5 - x_{5,k}(x_2; \{x_3, x_4, x_1\}_{\mathrm{B.}})} I''^{j}_{W-1}(x_2, x_5; \{x_3, x_4, x_1\}_{\mathrm{B.}})$$

- we integrate x_1 last as the diff. eq. takes a simple diagonal form

$$\partial_{x_1} I^{n}_{W}(x_2, x_5, x_3, x_4, x_1) =$$

$$\sum_{j,k} \frac{a_{1,j,k}}{x_1 - x_{1,k}(x_2, x_5, x_3, x_4)} I^{j}_{W-1}(x_2, x_5, x_3, x_4, x_1) = -\frac{2}{x_1} I^{n}_{W-1}(x_2, x_5, x_3, x_4, x_1)$$

Representation as Goncharov Polylogarithms

The integration leads to expressions in terms of Goncharov polylogarithms

$$\int_{x_{i,\mathrm{B.}}}^{x_i} \frac{dx_i'}{x_i' - x_{i,\mathrm{pole}}} G(\ldots;x_i') = G(x_{i,\mathrm{pole}},\ldots;x_i) - G(x_{i,\mathrm{pole}},\ldots;x_{i,\mathrm{B.}})$$

Since parts of the integration path fall outside of the Euclidean region, a sign prescription to cancel imaginary parts is necessary:

$$\begin{aligned}
&G(\ldots;x_1)\ ,\ G(\ldots;-1) &&\longrightarrow - \\
&G(\ldots;x_2)\ ,\ G(\ldots;\frac{1+\sqrt{5}}{2}) &&\longrightarrow - \\
&G(\ldots;x_3)\ ,\ G(\ldots;1) &&\longrightarrow + \\
&G(\ldots;x_4)\ ,\ G(\ldots;\frac{-1+\sqrt{5}}{2}) &&\longrightarrow - \\
&G(\ldots;x_5)\ ,\ G(\ldots;0) &&\longrightarrow +
\end{aligned}$$

Boundary conditions

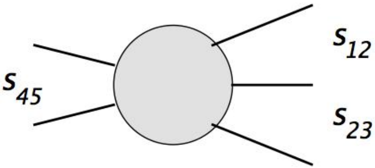
Boundary values can be obtained from physical conditions,
in kinematic limits with **singular diff. eq.** but **regular integrals**.

No singularities in the Euclidean region $s_{i,i+1} < 0$.

Un-physical singularities
appear in the limit

$$s_{45} \to s_{12} + s_{23}$$

and they need to cancel.



$\longrightarrow$ no need to compute any additional integrals.

Boundary conditions

$\Delta = 0$ defines hypersurface where divergencies need to cancel.

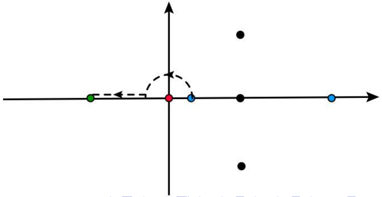
The symmetric point $\vec{x}_{sym} = \{-1,-1,-1,-1,-1\}$

is connected to the $\Delta = 0$ surface by

$$\vec{f}(\vec{x},\varepsilon) = P\exp\left[\varepsilon \int_\gamma dA\right] \vec{f}(\vec{x}_0,\varepsilon)$$

path $\gamma = \{-\frac{y}{(1-y)^2},-1,-1,-1,-1\} \longrightarrow$ reduced alphabet .

$$\text{Sym. pt} \rightarrow y = \frac{3\pm\sqrt{5}}{2}$$
$$\Delta = 0 \rightarrow y = -1$$



Applications: All-plus amplitude

We have applied our integrals to the **all-plus amplitude** (leading-colour).
[Badger, Frellesvig, Zhang (2013)]

$$\mathcal{A}_5\,(1^+,2^+,3^+,4^+,5^+)|_{\text{leading colour}} =$$

$$g_s^7 N_c^2\, c_\Gamma^2 \sum_{\sigma\in S_5} \mathrm{tr}(T^{a_{\sigma(1)}}\, T^{a_{\sigma(2)}}\, T^{a_{\sigma(3)}}\, T^{a_{\sigma(4)}}\, T^{a_{\sigma(5)}}) \sum_{\text{cycl}} A_5^{(2)}\,(\sigma(1)^+,\sigma(2)^+,\sigma(3)^+,\sigma(4)^+,\sigma(5)^+)$$

At one loop we have [Bern, Dixon, Dunbar, Kosower]

$$A^{(1)}(1^+2^+3^+4^+5^+) = \frac{-i\varepsilon(1-\varepsilon)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\Bigg(2\,(2-\varepsilon)\;\mathrm{tr}_5\; I^{[10-2\varepsilon]}_{[5;12345]}[1]$$
$$+\; s_{12}s_{23}\; I^{[8-2\varepsilon]}_{4;1234}[1] \;+\; s_{23}s_{34}\; I^{[8-2\varepsilon]}_{4;2345}[1] \;+\; s_{34}s_{45}\; I^{[8-2\varepsilon]}_{4;3451}[1] \;+\; s_{45}s_{51}\; I^{[8-2\varepsilon]}_{4;5123}[1]\Bigg)$$

$$\varepsilon^0 \;\to\; \frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\left(-\frac{1}{6}\,F_5^{(1)}\right) \quad , \quad F_5^{(1)} = v_1v_2+v_2v_3+v_3v_4+v_4v_5+v_5v_1 \;+\; \mathrm{tr}_5$$

Applications: All-plus amplitude

The infrared and ultraviolet structure is described by the one-loop amplitude

$$A^{(2)}_{5\,\text{plus}} = A^{(1)}_{5\,\text{plus}} \left[-\sum_{i=1}^{5} \frac{1}{\varepsilon^2} \left(\frac{\mu^2}{-v_i} \right)^{\varepsilon} \right] + \frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \left(-\frac{1}{6} F^{(2)}_5 \right) + O(\varepsilon) \,,$$

For the finite remainder we find: $(v_i = s_{i,i+1})$ [arXiv:1511.05409 [hep-ph]]

$$F^{(2)}_5 = \frac{5\pi^2}{12} F^{(1)}_5 + \sum_{i=0}^{4} \sigma^i \left\{ \frac{v_5 \mathrm{tr}\left[(1-\gamma_5) \not{p}_4 \not{p}_5 \not{p}_1 \not{p}_2 \right]}{(v_2 + v_3 - v_5)} I_{23,5} + \frac{1}{6} \frac{\mathrm{tr}\left[(1+\gamma_5) \not{p}_4 \not{p}_5 \not{p}_1 \not{p}_2 \right]^2}{v_1 v_4} + \frac{10}{3} v_1 v_2 + \frac{2}{3} v_1 v_3 \right\} .$$

with $I_{23,5}$ one-loop two-mass easy box function in six dimensions.

$$I_{23,5} = \zeta_2 - \mathrm{Li}_2\left(\frac{v_5 - v_3}{v_2} \right) - \mathrm{Li}_2\left(\frac{v_5 - v_2}{v_3} \right) + \mathrm{Li}_2\left(\frac{(v_5 - v_2)(v_5 - v_3)}{v_2 v_3} \right)$$

Checks

All master integrals checked against FIESTA in the Euclidean region.

All four point sub-topologies checked against [Gehrmann, Remiddi]

Amplitude:

Checks

All master integrals checked against FIESTA in the Euclidean region.

All four point sub-topologies checked against [Gehrmann, Remiddi]

Amplitude:

Checked against numerical results of [Badger, Frellesvig, Zhang (2013)]

Double and single pole cancellation provides non-trivial check.

Factorization properties of scattering amplitudes in the soft and **collinear limit** allow to connect provides a way to check them. When $p_4||p_5$, we have

$$
\begin{aligned}
A_5^{(2)}(1^+,2^+,3^+,4^+,5^+) \to & \mathrm{Split}^{P\to 45\,(1)}(P^-,4^+,5^+)A_4^{(1)}(1^+,2^+,3^+,P^+) \\
& + \mathrm{Split}^{P\to 45\,(1)}(P^+,4^+,5^+)A_4^{(1)}(1^+,2^+,3^+,P^-) \\
& + \mathrm{Split}^{P\to 45\,(0)}(P^-,4^+,5^+)A_4^{(2)}(1^+,2^+,3^+,P^+)\,.
\end{aligned}
$$

Ingredients: Bern,Dixon,Kosower (2014), Badger,Glover (2014), Bern,DeFreitas,Dixon (2002)

Recomputed by [Dunbar, Perkins (2016)] using on-shell recursions

Summary and Outlook

- Five-point two-loop MIs (planar) obtained using the Differential-Equation method, with MIs basis that makes the diff. eq. system canonical.
- Boundary conditions obtained by requiring the cancellation of spurious singularities in diff. eqs. $\rightarrow$ No further integration required.
- We have derived an analytic formula for the leading-color contribution of the all-plus 5-gluon amplitude.

Summary and Outlook

- Five-point two-loop MIs (planar) obtained using the Differential-Equation method, with MIs basis that makes the diff. eq. system canonical.
- Boundary conditions obtained by requiring the cancellation of spurious singularities in diff. eqs. $\rightarrow$ No further integration required.
- We have derived an analytic formula for the leading-color contribution of the all-plus 5-gluon amplitude.

- Analytic continuation outside Euclidean region ($\rightarrow$ physical region).
- Non-planar integrals: in progress.
- Application to other amplitudes.