Isospin Breaking Corrections on the Lattice

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Outline

Motivation

Including Isospin Breaking effects on the lattice

Isospin Breaking corrections to hadron masses

QED corrections to the HVP

QED corrections to pion and kaon decay rates

Conclusions and Outlook

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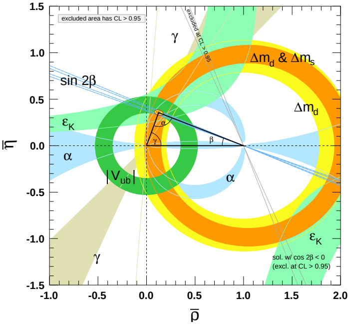
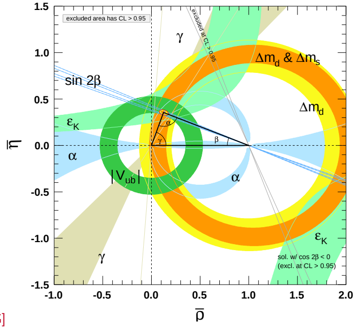
CKM Matrix elements



[PDG]

$$V_{\text{CKM}} = \begin{pmatrix} \mathbf{V_{ud}} & \mathbf{V_{us}} & \mathbf{V_{ub}} \\ \mathbf{V_{cd}} & \mathbf{V_{cs}} & \mathbf{V_{cb}} \\ \mathbf{V_{td}} & \mathbf{V_{ts}} & \mathbf{V_{tb}} \end{pmatrix}$$

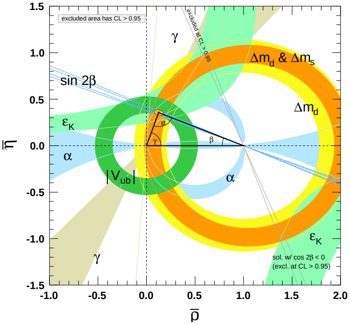
CKM Matrix elements



[PDG]

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & \color{red}{V_{us}} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

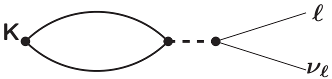
CKM Matrix elements



[PDG]

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

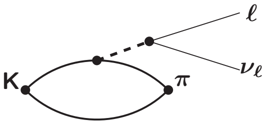
Leptonic Kaon Decay



$$\left| \frac{V_{us}}{V_{ud}} \frac{f_{K^\pm}}{f_{\pi^\pm}} \right| = 0.2760(4)$$

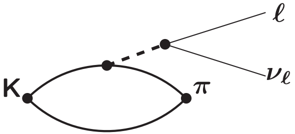
[M. Moulson, arXiv:1411.5252]

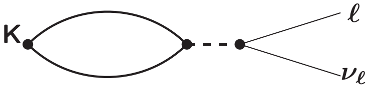
Semi-Leptonic Kaon Decay

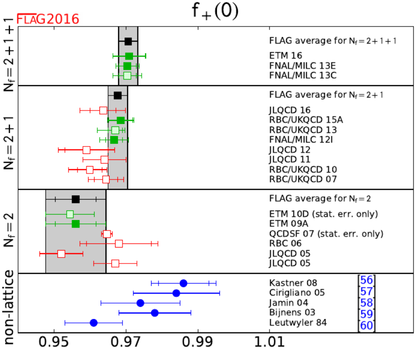


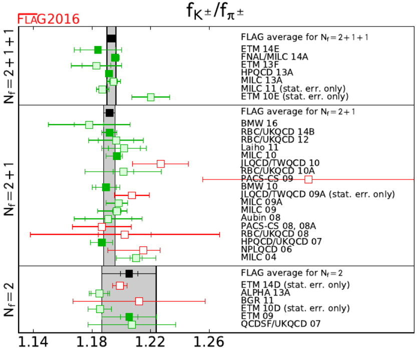
$$|V_{us}|\, f_+(0) = 0.2165(4)$$

Leptonic and Semi-Leptonic Kaon decays









FLAG average: $\mathbf{f_+(0) = 0.9706(27)}$ $\mathbf{f_{K^\pm}/f_{\pi^\pm} = 1.193(3)}$

Motivation

- Calculations usually done in isospin symmetric limit (treat up and down as equal)
- lattice calculation aiming at **1%** precision requires to include isospin breaking
- two sources of isospin breaking effects
  - different masses for up- and down quark (of $\mathcal{O}((\mathbf{m_d} - \mathbf{m_u})/\mathbf{\Lambda}_{\mathrm{QCD}})$)
  - Quarks have electrical charge (of $\mathcal{O}(\alpha)$)

- Calculations including QED on the lattice
  - Mainly focused on QED corrections to hadron masses

    [e.g. S. Borsanyi *et al.* Science **347** (2015) 1452; R. Horsley *et al.* JHEP **04** (2016) 093;D. Giusti *et al.* Phys.Rev. **D95** (2017) 114504]
  - First calculations of QED corrections to hadronic vacuum polarization

    [V.G. *et al.* arXiv:1706.05293; D. Giusti *et al.* arXiv:1707.03019]

- Methodology for calculating QED corrections to pion/kaon decay rates developed [N. Carrasco *et al.* Phys. Rev. **D91** (2015) 074506; V. Lubicz *et al.* Phys. Rev. **D95** (2017) 034504]

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QED on the lattice

- Euclidean path integral including QED

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[\Psi, \overline{\Psi}] \mathcal{D}[U] \mathcal{D}[A] \; O \; e^{-S_F[\Psi, \overline{\Psi}, U, A]} \, e^{-S_G[U]} \, e^{-S_\gamma[A]}$$

- non-compact photon action

$$S_\gamma [A] = \frac{a^4}{4} \sum_x \sum_{\mu,\nu} \left(\partial_\mu A_\nu (x) - \partial_\nu A_\mu (x) \right)^2$$

- two approaches for including QED
  - stochastic QED using **U(1)** gauge configurations
    [A. Duncan, E. Eichten, H. Thacker, Phys.Rev.Lett. **76**, 3894 (1996)]
  - perturbative QED by expanding the path integral in α
    [RM123 Collaboration, Phys.Rev. **D87**, 114505 (2013)]

$$\langle O \rangle = \langle O \rangle_0 + \frac{1}{2} e^2 \frac{\partial^2}{\partial e^2} \langle O \rangle \Big|_{e=0} + \mathcal{O}(\alpha^2)$$

- QED in a box $\rightarrow$ finite volume corrections

stochastic method

- Feynman gauge

$$\mathbf{S}_{\gamma}^{\mathrm{Feyn}}[\mathbf{A}] = \mathbf{S}_{\gamma}[\mathbf{A}] + \frac{1}{2}\sum_{\mathbf{x}}\left(\sum_{\mu}\partial_{\mu}\mathbf{A}_{\mu}(\mathbf{x})\right)^{2} = -\frac{1}{2}\sum_{\mathbf{x}}\sum_{\mu}\mathbf{A}_{\mu}\partial^{2}\mathbf{A}_{\mu}(\mathbf{x})$$

- in momentum space

$$\mathbf{S}_{\gamma}^{\mathrm{Feyn}}[\mathbf{A}] = \frac{1}{2\mathbf{V}}\sum_{\mathbf{k},\,\vec{\mathbf{k}}\neq 0}\hat{\mathbf{k}}^{2}\sum_{\mu}\left|\tilde{\mathbf{A}}_{\mu}(\mathbf{k})\right|^{2} \qquad \hat{\mathbf{k}}_{\mu} = \frac{2}{\mathbf{a}}\sin\left(\frac{\mathbf{a}\mathbf{k}_{\mu}}{2}\right)$$

- draw $\tilde{\mathbf{A}}_{\mu}(\mathbf{k})$ from Gaussian distribution with variance $\mathbf{2V}/\hat{\mathbf{k}}^{2}$
- electro quenched approximation
  $\rightarrow$ sea quarks electrically neutral
  $\rightarrow$ QED configurations generated independently of QCD configurations
- multiply **SU(3)** gauge links with **U(1)** photon fields

$$\mathbf{U}_{\mu}(\mathbf{x}) \rightarrow \mathbf{e}^{\mathbf{ieA}_{\mu}(\mathbf{x})}\mathbf{U}_{\mu}(\mathbf{x})$$

- unquenched calculation $\rightarrow$ generate combined QED+QCD configurations
- QED correction to all orders in α

The zero-mode of the photon field

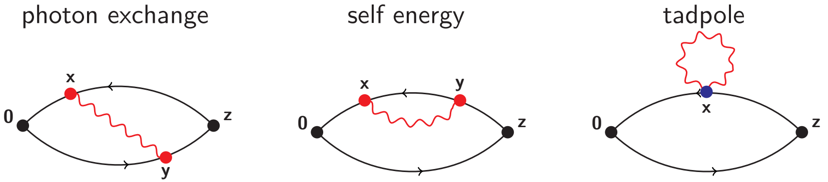
- zero-mode of the photon field
  shift symmetry of the of the photon action $\mathbf{A}_\mu(\mathbf{x}) \to \mathbf{A}_\mu(\mathbf{x}) + \mathbf{c}_\mu$
  $\to$ cannot be constrained by gauge fixing

- different prescriptions of QED:
- $\text{QED}_{\mathbf{TL}}$: remove the zero-mode of the photon field, i.e. $\tilde{\mathbf{A}}_\mu(\mathbf{k}=0)=\mathbf{0}$
  [A. Duncan, E. Eichten, H. Thacker, Phys.Rev.Lett. **76**, 3894 (1996)]
- $\text{QED}_{\mathbf{L}}$: remove all the spatial zero-modes, i.e. $\tilde{\mathbf{A}}_\mu(\mathbf{k_0}, \vec{\mathbf{k}}=0)=\mathbf{0}$
  [S. Uno and M. Hayakawa, Prog. Theor. Phys. **120**, 413 (2008)]
- $\text{QED}_{\mathbf{m}}$: use a massive photon and take $\mathbf{m}_\gamma \to \mathbf{0}$
  [M. Endres *et al.*,Phys. Rev. Lett. **117** (2016) 072002]
- $\text{QED}_{\mathbf{C}}$: $\mathbf{C}^*$ boundary conditions in spatial direction, i.e. fields are periodic up to charge conjugation [B. Luchini *et al.* JHEP **02** (2016) 076]

- for detailed discussion on different prescriptions of QED see e.g. [A. Patella 1702.03857]

Perturbative method

- calculate contributions at $\mathcal{O}(\alpha)$
- three different types of (quark-connected) diagrams



- e.g. photon exchange diagram for a charged Kaon

$$\mathbf{C}(\mathbf{z_0}) = \sum_{\vec{z}} \sum_{x,y} \mathrm{Tr}\left[\mathbf{S^s}(\mathbf{z},\mathbf{x})\, \mathbf{\Gamma}^c_\nu\, \mathbf{S^s}(\mathbf{x},\mathbf{0})\, \gamma_5\, \mathbf{S^u}(\mathbf{0},\mathbf{y})\, \mathbf{\Gamma}^c_\mu\, \mathbf{S^u}(\mathbf{y},\mathbf{z})\, \gamma_5 \right] \mathbf{\Delta}_{\mu\nu}(\mathbf{x}-\mathbf{y})$$

- photon propagator in Feynman gauge

$$\mathbf{\Delta}_{\mu\nu}(\mathbf{x}-\mathbf{y}) = \frac{1}{\mathbf{V}} \sum_{k,\vec{k}\neq 0} \frac{e^{ik\cdot(x-y)}}{\hat{k}^2}$$

- electro-quenched approximation: sea quarks are neutral

Stochastic vs Perturbative method

- stochastic method

- perturbative method

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  - QED corrections at fixed order in $\mathcal{O}(\alpha)$

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  - once the stochastic $\mathbf{U(1)}$ fields are generated, calculation proceeds without QED $\longrightarrow$ computationally cheaper than perturbative method
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- direct comparison of results and statistical errors for QED corrections to meson masses and hadronic vacuum polarization [V.G. *et al.* arXiv:1706.05293]
  $\rightarrow$ for our setup: stochastic method gives $\approx$ **1.5** times smaller statistical error for same numerical cost

strong Isospin Breaking

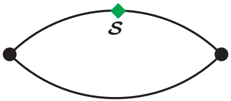
- different bare quark masses for up- and down quark
- expansion in $\mathbf{\Delta m = (m_u - m_d)}$ [G.M. de Divitiis *et al*, JHEP 1204 (2012) 124]

$$\langle \mathbf{O} \rangle_{\mathbf{m_u \neq m_d}} = \langle \mathbf{O} \rangle_{\mathbf{m_u = m_d}} + \mathbf{\Delta m} \frac{\partial}{\partial \mathbf{m}} \langle \mathbf{O} \rangle \Big|_{\mathbf{m_u = m_d}} + \mathcal{O}\left(\mathbf{\Delta m}^2\right)$$

with

$$\frac{\partial}{\partial \mathbf{m}} \langle \mathbf{O} \rangle \Big|_{\mathbf{m_u = m_d}} = \langle \mathbf{O}\, \mathcal{S} \rangle_{\mathbf{m_u = m_d}}$$

- scalar current $\mathcal{S} = \sum_{\mathbf{x}} \overline{\psi}(\mathbf{x})\, \psi(\mathbf{x})$



- quark mass tuning at the physical point, e.g. by fixing masses of charged pion, charged and neutral kaon

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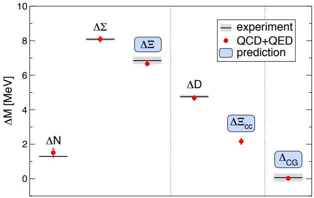
Isopin Breaking corrections to hadron masses

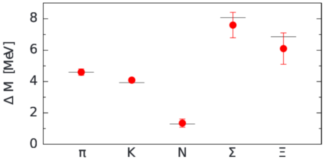
- Calculations of isospin breaking corrections to hadron masses
  [e.g. S. Borsanyi *et al.* Phys. Rev. Lett. **111** (2013) 252001; G. M. de Divitiis *et al.* Phys. Rev. **D87** (2013) 114505; S. Borsanyi *et al.* Science **347** (2015) 1452; R. Horsley *et al.* J. Phys. **G43** (2016) 10LT02; R. Horsley *et al.* JHEP **04** (2016) 093; S. Basek *et al.* PoS LATTICE2015 (2016) 259; Z. Fodor *et al.* Phys. Rev. Lett. **117** (2016) 082001; D. Giusti *et al.* Phys.Rev. **D95** (2017) 114504; V.G. *et al.* arXiv:1706.05293]

- examples

[S. Borsanyi *et al.* Science **347** (2015) 1452]



[R. Horsley *et al.* J. Phys. **G43** (2016) 10LT02]



- finite volume corrections, e.g. mesons in $\mathrm{QED_L}$ [BMW Collaboration, Science **347** (2015) 1452–1455]

$$\mathbf{m}^2(\mathbf{L}) \sim \mathbf{m}^2 \left\{ 1 - \mathbf{q}^2\alpha \left[\frac{\kappa}{\mathbf{mL}} \left(1 + \frac{2}{\mathbf{mL}} \right) \right] \right\} + \mathcal{O}\left(\frac{1}{\mathbf{L}^3} \right) \qquad \kappa = 2.837297$$

- universal up to $\mathcal{O}\left(\frac{1}{\mathbf{L}^2}\right)$
- [Z. Fodor *et al.* Phys. Rev. Lett. **117** (2016) 082001] $\mathcal{O}\left(\frac{1}{\mathbf{L}^3}\right)$ negligible within errors

up and down quark masses

- up and down quark masses $\overline{\textbf{MS}}$ at **2** GeV [FLAG 2016]

Collaboration	Ref.	pu	chi	co	fin	rel	ru	m_u	m_d	m_u/m_d
MILC 14	[32]	C	★	★	★	–	–			$0.4482(48)(^{+21}_{-115})(1)(165)$
ETM 14	[9]	A	★	★	★	★	*b*	2.36(24)	5.03(26)	0.470(56)
QCDSF/UKQCD 15$^{\ominus}$	[92]	P	○	■	○	–	–			0.52(5)
PACS-CS 12*	[67]	A	★	■	■	★	*a*	2.57(26)(7)	3.68(29)(10)	0.698(51)
Laiho 11	[68]	C	○	★	★	○	–	1.90(8)(21)(10)	4.73(9)(27)(24)	0.401(13)(45)
HPQCD 10‡	[66]	A	○	★	★	★	–	2.01(14)	4.77(15)	
BMW 10A, 10B^{+}	[6, 7]	A	★	★	★	★	*b*	2.15(03)(10)	4.79(07)(12)	0.448(06)(29)
Blum 10†	[20]	A	○	■	○	★	–	2.24(10)(34)	4.65(15)(32)	0.4818(96)(860)
MILC 09A	[26]	C	○	★	★	○	–	1.96(0)(6)(10)(12)	4.53(1)(8)(23)(12)	0.432(1)(9)(0)(39)
MILC 09	[25]	A	○	★	★	○	–	1.9(0)(1)(1)(1)	4.6(0)(2)(2)(1)	0.42(0)(1)(0)(4)
MILC 04, HPQCD/ MILC/UKQCD 04	[24] [75]	A	○	○	○	■	–	1.7(0)(1)(2)(2)	3.9(0)(1)(4)(2)	0.43(0)(1)(0)(8)
RM123 13	[37]	A	○	★	○	★	*c*	2.40(15)(17)	4.80 (15)(17)	0.50(2)(3)
RM123 11$^{\oplus}$	[93]	A	○	★	○	★	*c*	*2.43(11)(23)*	*4.78(11)(23)*	*0.51(2)(4)*
Dürr 11*	[55]	A	○	★	○	–	–	2.18(6)(11)	4.87(14)(16)	
RBC 07†	[22]	A	■	■	★	★	–	3.02(27)(19)	5.49(20)(34)	0.550(31)

- recent results
- [Z. Fodor *et al.* Phys. Rev. Lett. **117** (2016) 082001]
  $\mathbf{m_u/m_d = 0.485(11)(8)(14)}$
- [D. Giusti *et al.* Phys.Rev. **D95** (2017) 114504]
  $\mathbf{m_u/m_d = 0.512(30)}$

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Muon $\mathbf{a}_\mu$ and the hadronic vacuum polarisation (HVP)

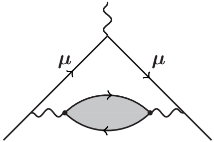
- experiment: polarized muons in a magnetic field [Bennet et al., Phys.Rev. **D73**, 072003 (2006)]

$$\mathbf{a}_\mu = \mathbf{11659208.9(5.4)(3.3)} \times \mathbf{10^{-10}}$$

- Standard Model [PDG]

$$\mathbf{a}_\mu = \mathbf{11659180.3(0.1)(4.2)(2.6)} \times \mathbf{10^{-10}}$$

- Comparison of theory and experiment: $\mathbf{3.6\sigma}$ deviation
- largest error on SM estimate from HVP



- current best estimate from $\mathbf{e^+e^-} \to$ hadrons [Davier et al., Eur.Phys.J. **C71**, 1515 (2011)]

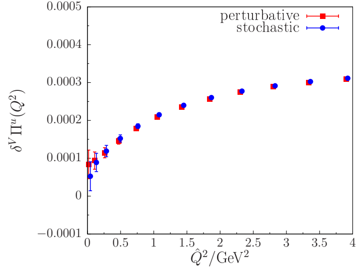
$$\mathbf{(692.3 \pm 4.2 \pm 0.3)} \times \mathbf{10^{-10}}$$

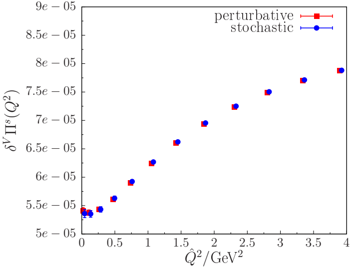
- lattice calculation at $\lesssim$ **1%** requires inclusion of isopin breaking effects

Results QED corrections to HVP

- first results on QED corrections to HVP [V.G. *et al.* arXiv:1706.05293; D. Giusti *et al.* arXiv:1707.03019]

- QED correction to HVP from [V.G. *et al.* arXiv:1707.03019] (at unphysical quark masses)



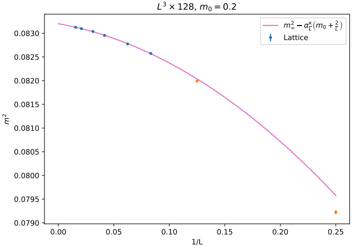


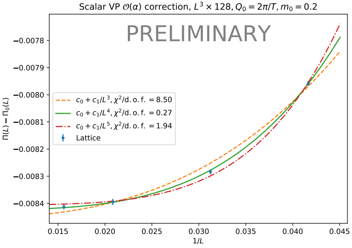
QED correction to $\mathbf{a}_\mu < \mathbf{1\%}$ for up quark, $\approx -\mathbf{0.1\%}$ for strange

- [D. Giusti *et al.* arXiv:1707.03019]: QED correction for strange quark $-\mathbf{0.032(21)\%}$ extrapolated to physical point

Finite volume corrections for the QED corrections to HVP

- 2-loop analytical calculation $\longrightarrow$ not done yet
- scalar QED as effective theory
- our approach: lattice scalar QED as quick numerical method for obtaining FV effects
- for HVP leading term scalar bubble diagram (two pion contribution)





[Plots by J. Harrison]

- for QED correction to masses we obtain results consistent with analytic formula from [BMW Collaboration, Science **347** (2015) 1452–1455]
- for HVP our data suggest that finite volume effects are of $\boldsymbol{\mathcal{O}(1/L^4)}$

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$K \to \ell\nu_\ell$ with QED [N. Carrasco *et al.* Phys.Rev. **D91** (2015) no.7, 074506]

- formulated in [N. Carrasco *et al.* Phys.Rev. **D91** (2015) no.7, 074506]
- contributions from photon emitted from hadron and absorbed by charged lepton $\to$ hadronic and leptonic part can no longer be factorised
- infrared (IR) divergences
  $\to$ canceled when combining contributions from virtual and real photons
  [F. Bloch and A. Nordsieck, Phys.Rev. **52** (1937) 54]
  $\to$ perturbative method for QED

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- $\mathbf{\Gamma_0}$: decay rate for $\mathbf{K}^+ \to \ell^+\nu_\ell$ including QED
- $\mathbf{\Gamma_1(\Delta E)}$: decay rate for $\mathbf{K}^+ \to \ell^+\nu_\ell\gamma$ with a photon of energy $\leq \mathbf{\Delta E}$ in final state
- sum $\mathbf{\Gamma(\Delta E) = \Gamma_0 + \Gamma_1(\Delta E)}$ free from IR divergences
  $\to \mathbf{\Gamma(\Delta E)}$ can be measured in experiment

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- $\mathbf{\Gamma_0}$: decay rate for $\mathbf{K}^+ \to \ell^+\nu_\ell$ including QED
- $\mathbf{\Gamma_1(\Delta E)}$: decay rate for $\mathbf{K}^+ \to \ell^+\nu_\ell\gamma$ with a photon of energy $\leq \mathbf{\Delta E}$ in final state
- sum $\mathbf{\Gamma(\Delta E) = \Gamma_0 + \Gamma_1(\Delta E)}$ free from IR divergences
  $\to$ $\mathbf{\Gamma(\Delta E)}$ can be measured in experiment

- choose $\mathbf{\Delta E}$ small, such that structure of hadron is not resolved
  $\to$ $\mathbf{\Gamma_1(\Delta E)}$ can be calculated in perturbation theory

$\mathsf{K} \to \ell\nu_\ell$ with QED [N. Carrasco *et al.* Phys.Rev. **D91** (2015) no.7, 074506]

$$\Gamma(\Delta E) = \Gamma_0 + \Gamma_1(\Delta E)$$

- split calculation in "lattice" part and "perturbative" part
  $\rightarrow$ such that both parts are IR finite

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- split calculation in "lattice" part and "perturbative" part
  $\rightarrow$ such that both parts are IR finite
- rewrite the decay rate as

$$\Gamma(\Delta E) = \lim_{V\to\infty}\left(\Gamma_0 - \Gamma_0^{\mathrm{pt}}\right) + \lim_{V\to\infty}\left(\Gamma_0^{\mathrm{pt}} + \Gamma_1(\Delta E)\right)$$

- Γ_0^{pt}: decay rate for $K^+ \to \ell^+\nu_\ell$ including QED with K^+ pointlike particle
- contribution from small momenta are the same for Γ_0 and Γ_0^{pt}
  $\rightarrow \Gamma_0 - \Gamma_0^{\mathrm{pt}}$ IR finite
  $\rightarrow$ then, also $\Gamma_0^{\mathrm{pt}} + \Gamma_1(\Delta E)$ must be IR finite, since $\Gamma(\Delta E)$ is IR finite

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- Γ_0^{pt}: decay rate for $\mathbf{K}^+ \to \ell^+\nu_\ell$ including QED with $\mathbf{K}^+$ pointlike particle
- contribution from small momenta are the same for Γ_0 and Γ_0^{pt}
  $\to \Gamma_0 - \Gamma_0^{\mathrm{pt}}$ IR finite
  $\to$ then, also $\Gamma_0^{\mathrm{pt}} + \Gamma_1(\Delta E)$ must be IR finite, since $\Gamma(\Delta E)$ is IR finite
- calculate $\Gamma^{\mathrm{pt}}(\Delta E)$ in perturbation theory

$K \to \ell\nu_\ell$ with QED [N. Carrasco *et al.* Phys.Rev. **D91** (2015) no.7, 074506]

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$$\Gamma(\Delta E) = \lim_{V\to\infty} \underbrace{\left(\Gamma_0 - \Gamma_0^{pt}\right)}_{=\Delta\Gamma_0(V)} + \underbrace{\lim_{V\to\infty} \left(\Gamma_0^{pt} + \Gamma_1(\Delta E)\right)}_{=\Gamma^{pt}(\Delta E)}$$

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- contribution from small momenta are the same for Γ_0 and Γ_0^{pt}
  $\to \Gamma_0 - \Gamma_0^{pt}$ IR finite
  $\to$ then, also $\Gamma_0^{pt} + \Gamma_1(\Delta E)$ must be IR finite, since $\Gamma(\Delta E)$ is IR finite
- calculate $\Gamma^{pt}(\Delta E)$ in perturbation theory
- calculate $\Delta\Gamma_0(V)$ on the lattice

$K \to \ell\nu_\ell$ with QED [N. Carrasco *et al.* Phys.Rev. **D91** (2015) no.7, 074506]

$$\Gamma(\Delta E) = \Gamma_0 + \Gamma_1(\Delta E)$$

- split calculation in "lattice" part and "perturbative" part
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- rewrite the decay rate as

$$\Gamma(\Delta E) = \underbrace{\lim_{V\to\infty}\left(\Gamma_0 - \Gamma_0^{pt}\right)}_{=\Delta\Gamma_0(V)} + \underbrace{\lim_{V\to\infty}\left(\Gamma_0^{pt} + \Gamma_1(\Delta E)\right)}_{=\Gamma^{pt}(\Delta E)}$$

- Γ_0^{pt}: decay rate for $K^+ \to \ell^+\nu_\ell$ including QED with K^+ pointlike particle
- contribution from small momenta are the same for Γ_0 and Γ_0^{pt}
  $\to \Gamma_0 - \Gamma_0^{pt}$ IR finite
  $\to$ then, also $\Gamma_0^{pt} + \Gamma_1(\Delta E)$ must be IR finite, since $\Gamma(\Delta E)$ is IR finite
- calculate $\Gamma^{pt}(\Delta E)$ in perturbation theory
- calculate $\Delta\Gamma_0(V)$ on the lattice
- FV correction to leptonic decay rate [V. Lubicz *et al.* Phys. Rev. **D95** (2017) 034504]

$$\Gamma_0(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell)\log(m_K L) + \frac{C_1(r_\ell)}{m_K L} + \mathcal{O}\left(\frac{1}{L^2}\right) \qquad r_\ell = \frac{m_\ell}{m_K}$$

Outline

Motivation

Including Isospin Breaking effects on the lattice

Isospin Breaking corrections to hadron masses

QED corrections to the HVP

QED corrections to pion and kaon decay rates

Conclusions and Outlook

Summary

- Lattice calculations with precision of $\lesssim \mathbf{1\%}$ require inclusion of isospin breaking and QED effects
- challenges for including QED on the lattice
  - photon zero mode
  - large finite volume corrections
  - IR divergences for some quantities like kaon/pion decay rate
- Isospin Breaking corrections to hadron masses
- First calculations for QED corrections to HVP
- method to calculate QED corrections to leptonic pion/kaon decay developed in [N. Carrasco *et al.* Phys.Rev. **D91** (2015) no.7, 074506]

Outlook

- Calculate the QED corrections to leptonic pion/kaon decay
- QED corrections to semileptonic $\mathbf{K_{l3}}$ decay

Outline

Motivation

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Conclusions and Outlook

Backup

a_μ: Experiment vs. Theory

- $\mathbf{a_\mu = (g_\mu - 2)/2}$
- measured and calculated very precisely $\longrightarrow$ test of the Standard Model
- experiment: polarized muons in a magnetic field [Bennet et al., Phys.Rev. **D73**, 072003 (2006)]

$$\mathbf{a_\mu = 11659208.9(5.4)(3.3) \times 10^{-10}}$$

- Standard Model

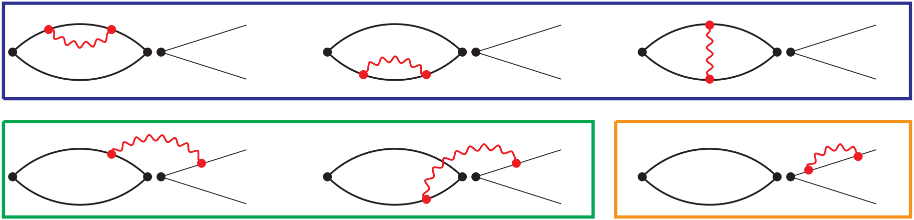
em	$\mathbf{(11658471.895 \pm 0.008) \times 10^{-10}}$	[Kinoshita et al., Phys.Rev.Lett. **109**, 111808 (2012)]
weak	$\mathbf{(15.36 \pm 0.10) \times 10^{-10}}$	[Gnendinger et al., Phys.Rev. **D88**, 053005 (2013)]
HVP	$\mathbf{(692.3 \pm 4.2 \pm 0.3) \times 10^{-10}}$	[Davier et al., Eur.Phys.J. **C71**, 1515 (2011)]
HVP(α^3)	$\mathbf{(-9.84 \pm 0.06) \times 10^{-10}}$	[Hagiwara et al., J.Phys. **G38**, 085003 (2011)]
LbL	$\mathbf{(10.5 \pm 2.6) \times 10^{-10}}$	[Prades et al.,Adv.Ser.Direct.High Energy Phys. 20, 303 (2009)]

- Comparison of theory and experiment: $\mathbf{3.6\sigma}$ deviation

$$\mathbf{\Delta a_\mu = a_\mu^{exp} - a_\mu^{SM} = 28.8(6.3)^{Exp}(4.9)^{SM} \times 10^{-10}}$$

- new physics?

Diagrams at $\mathcal{O}(\alpha)$

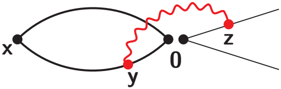


(+ quark-disconnected diagrams, where the photon couples to sea quarks)

- mass and wave function renormalisation of lepton
  $\rightarrow$ cancels in $\boldsymbol{\Gamma_0 - \Gamma_0^{\mathrm{pt}}}$
- hadronic and leptonic part can be factorised
  $\rightarrow$ "QED correction" to $\mathbf{f_{\pi/K}}$
  $\rightarrow$ obtained from similar correlation functions as for the meson masses
- hadronic and leptonic part cannot be factorised
  $\rightarrow$ cannot be written in terms of a decay constant

Diagrams with one photon vertex at quark and on vertex at lepton

- hadronic and leptonic part cannot be factorised
- need to calculate the complete diagram on the lattice



- correction to matrix element from this diagram can be calculated from the Euclidean correction function [N. Carrasco *et al.* Phys.Rev. **D91** (2015) no.7, 074506]

$$\mathbf{C}_{\alpha\beta}(\mathbf{x}_0) = -\sum_{\vec{\mathbf{x}}} \sum_{\mathbf{y},\mathbf{z}} \langle 0 | \mathbf{J}_{\mathbf{W}}^{\nu}(0)\, \mathbf{j}_{\mu}(\mathbf{y})\, \phi^{\dagger}(\mathbf{x}) | 0 \rangle \;\; \boldsymbol{\Delta}(\mathbf{y}-\mathbf{z})$$
$$\times \left(\gamma_{\nu}(1-\gamma_5)\, \mathbf{S}^{\ell}(0,\mathbf{z})\, \gamma_{\mu} \right)_{\alpha\beta} \mathbf{e}^{\mathbf{E}_{\ell}\mathbf{z}_0}\, \mathbf{e}^{-\mathbf{i}\vec{\mathbf{p}}_{\ell}\cdot\vec{\mathbf{z}}}$$

- $\mathbf{S}^{\ell}(0,\mathbf{z})$: Euclidean lepton propagator
- $\mathbf{j}_{\mu}(\mathbf{y}) = \mathbf{q_f}\bar{\mathbf{f}}\gamma_{\mu}\mathbf{f}$: electromagnetic current for quark with flavor $\mathbf{f}$
- $\mathbf{J}_{\mathbf{W}}^{\nu}(0)$: V-A current
- $\phi^{\dagger}(\mathbf{x})$ operator that creates a Kaon/Pion