Multi-hadron matrix elements from lattice QCD

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HIM

Helmholtz-Institut Mainz

Lattice QCD in a nutshell

In LQCD we evaluate the Feynman path-integral numerically

$$\textbf{observable} = \int \mathcal{D}\phi \; e^{iS} \left[\begin{array}{c} \textbf{quantum fields} \\ \textbf{of the observable} \end{array} \right]$$

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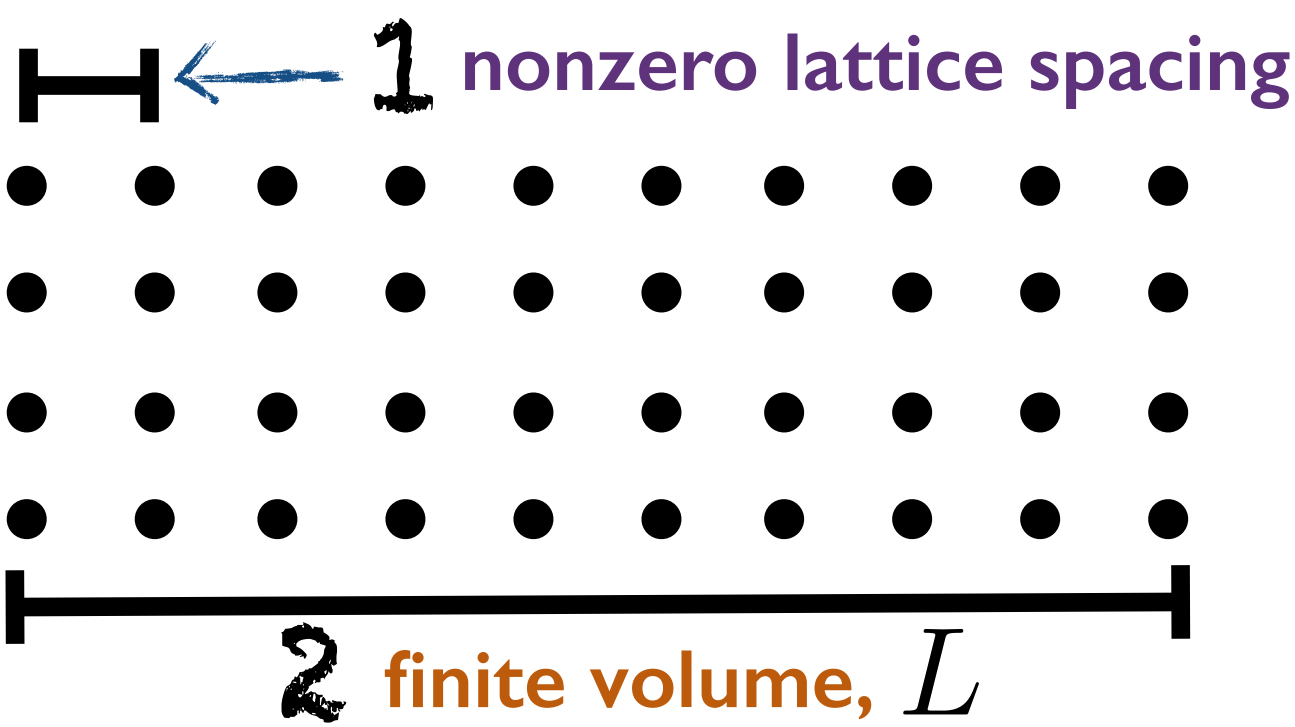

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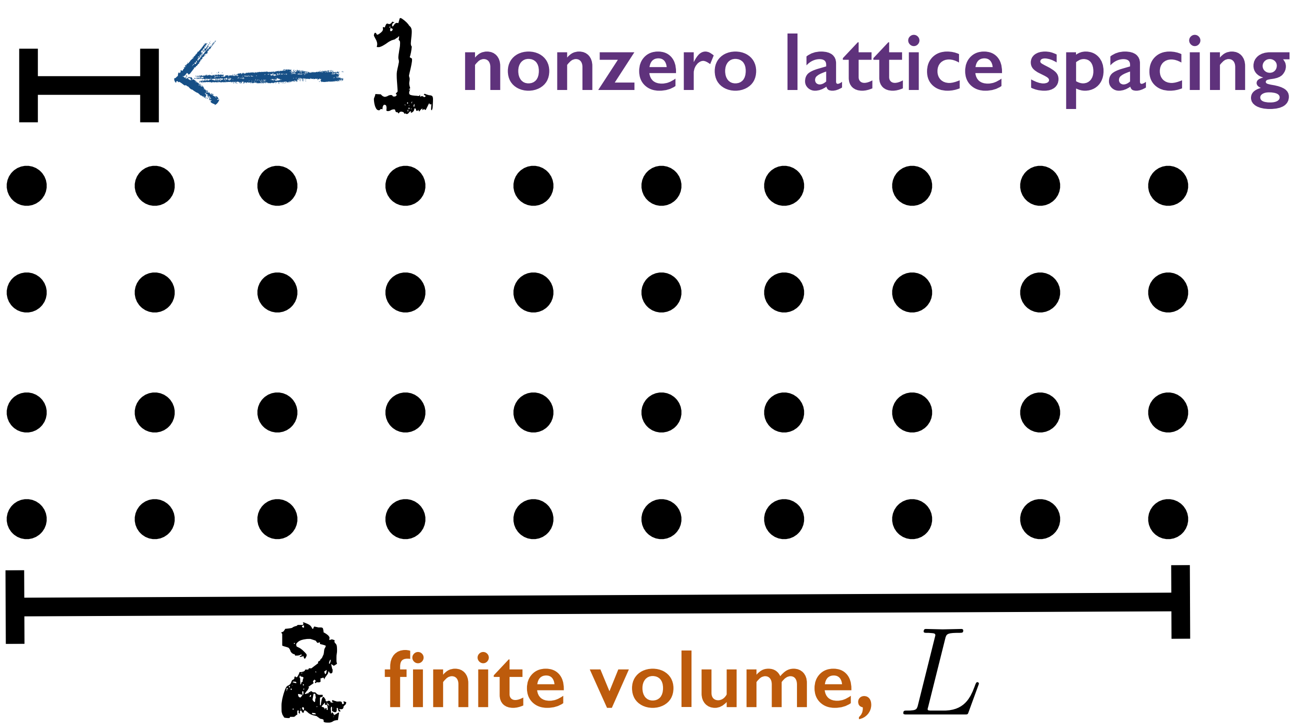

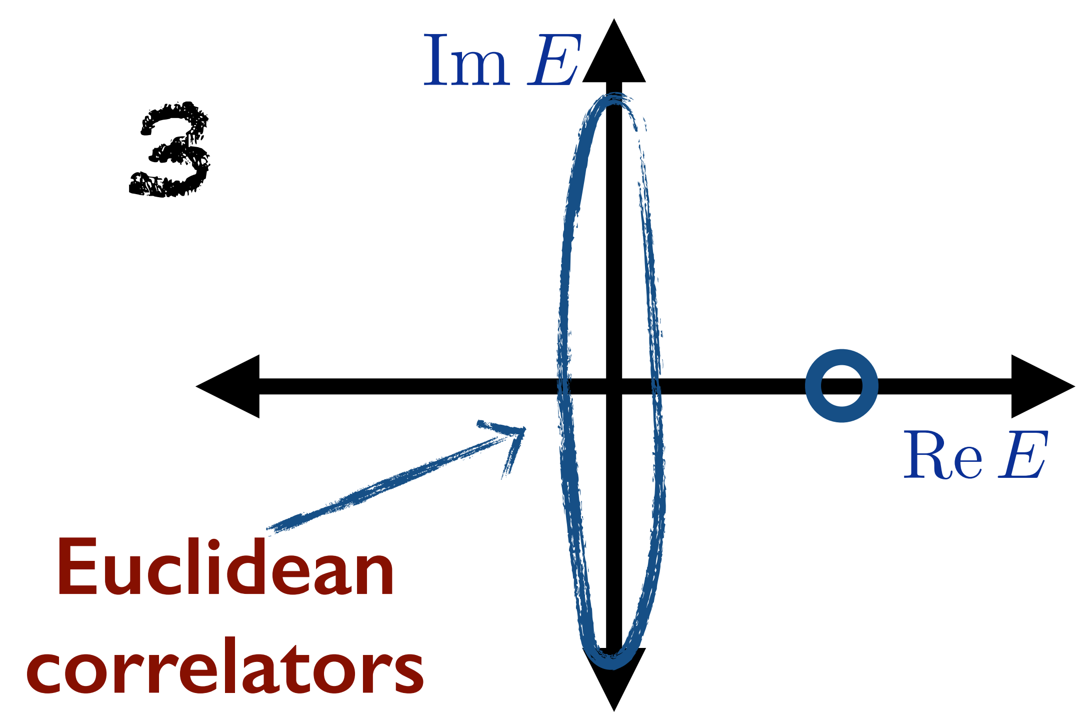

4 unphysical quark content $\quad M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

Calculations at the physical pion mass do now exist

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$$\begin{pmatrix}\textbf{observable?} \\ \text{discretized, finite volume,} \\ \text{Euclidean, heavy pions}\end{pmatrix} = \int \prod_i^N d\phi_i \; e^{-S\left[\begin{array}{c}\textbf{quantum fields} \\ \textbf{of the observable}\end{array}\right]}$$

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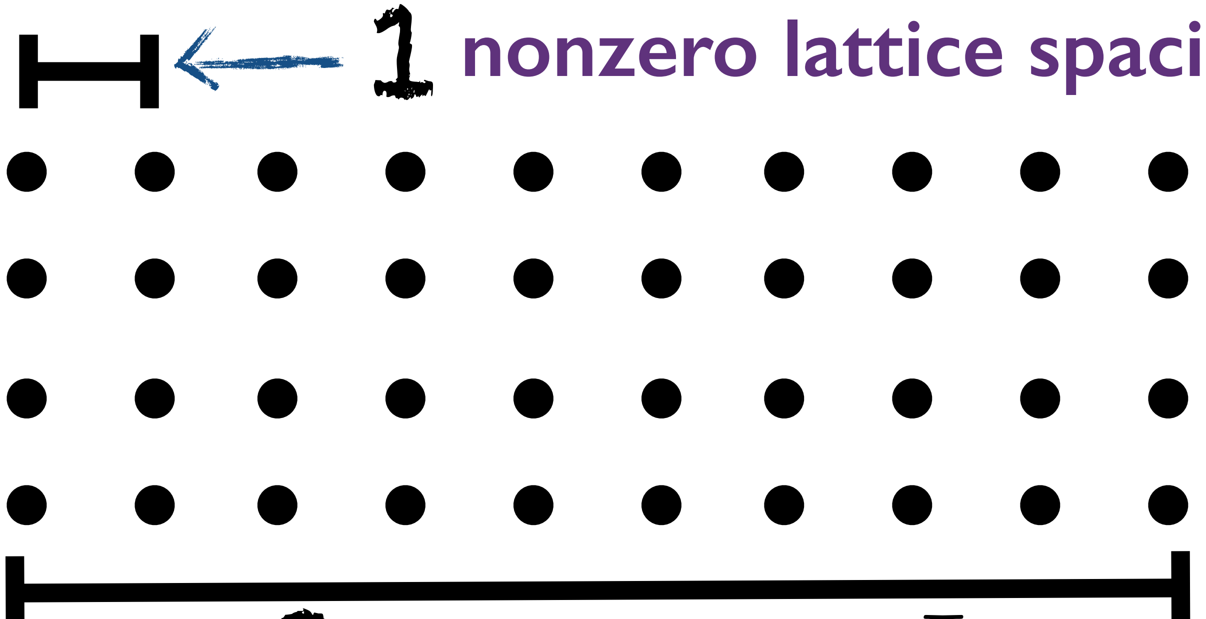
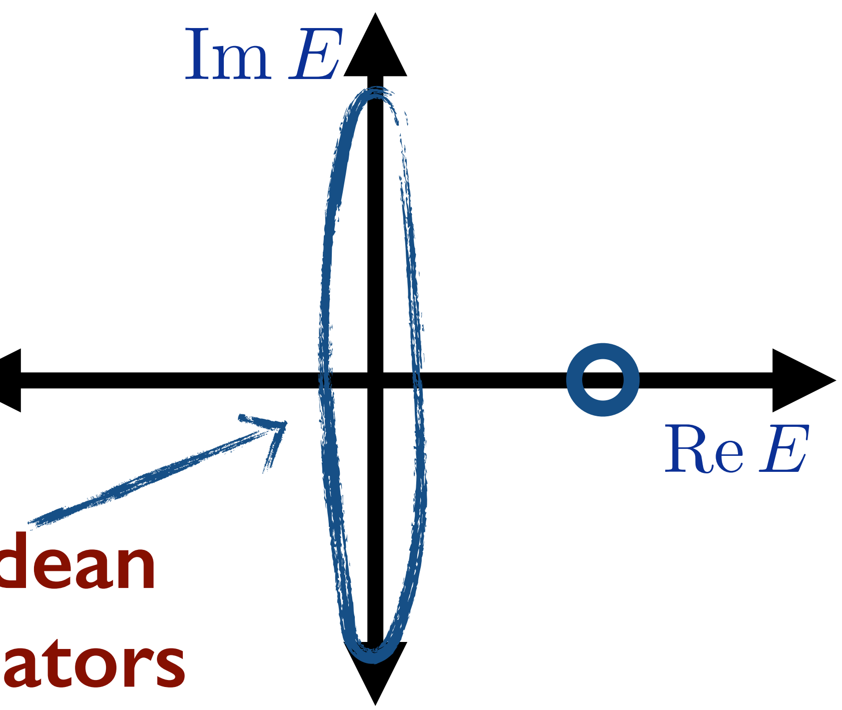
1 nonzero lattice spacing



2 finite volume, L

3



Euclidean correlators

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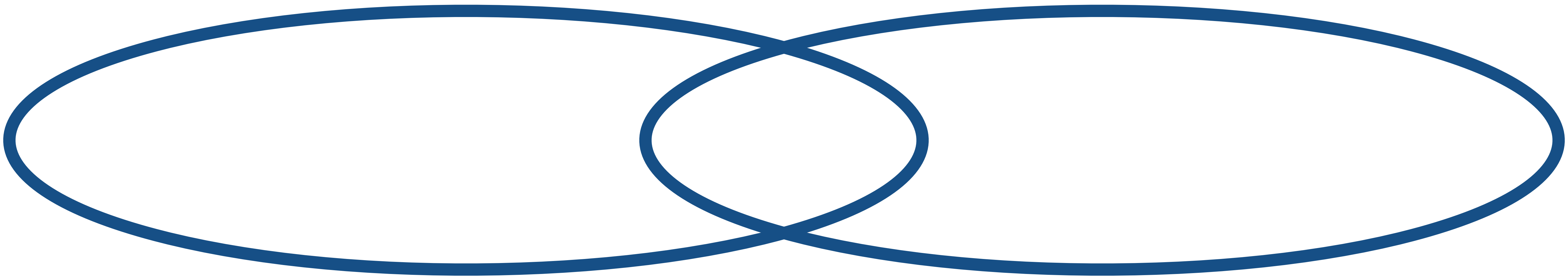
Extracting physical predictions…

Two basic approaches to handle these modifications

Perform multiple calculations and extrapolate

Use theoretical methods to understand the modification



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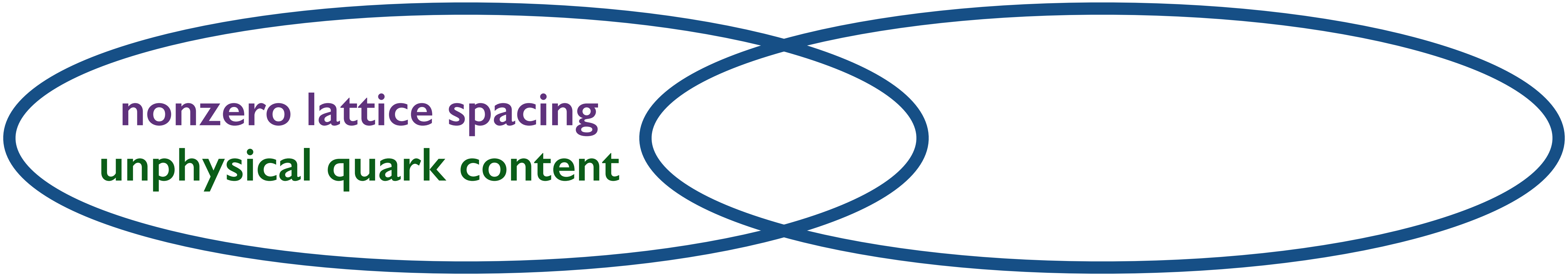

Modern calculations often have reliable chiral-continuum extrapolations (see e.g. FLAG)

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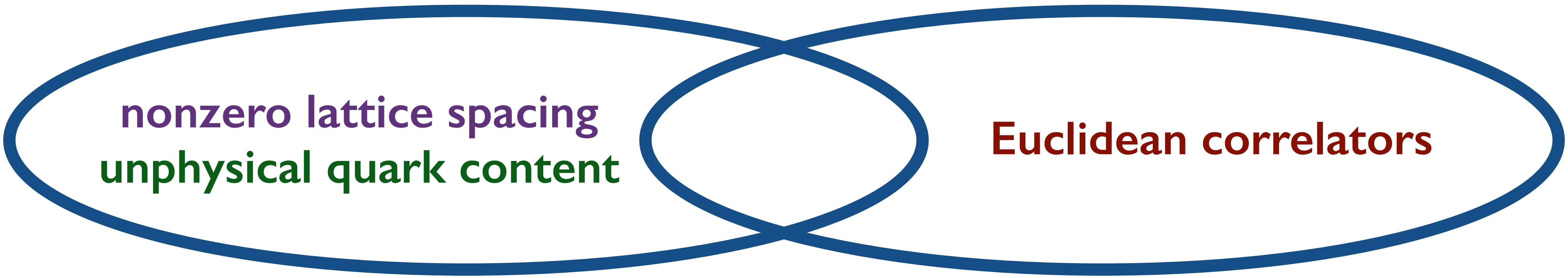

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For decay constants and form factors one should extrapolate to infinite-volume…

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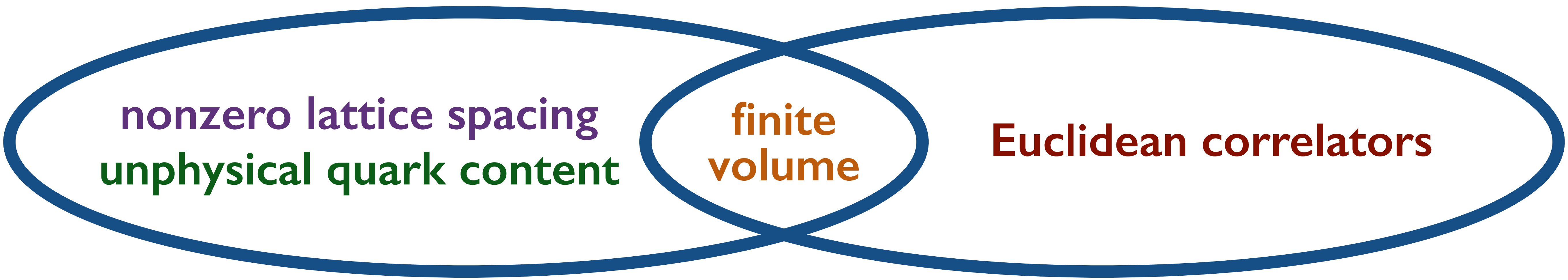

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To extract multi-hadron decay and scattering amplitudes we do not take the infinite-volume limit

$$N + \nu_\ell \longrightarrow \ell + X \qquad\qquad X \in \{N, N\pi, N\pi\pi, \cdots\}$$

This is the focus of this talk!

Multi-hadron processes from LQCD…

In a LQCD calculation it is possible to access

$$H_{\mathrm{QCD}}|n, \text{``}\pi\pi\text{''}, L\rangle = |n, \text{``}\pi\pi\text{''}, L\rangle \underline{E_n(L)}$$

$$\underline{\langle n, \text{``}N\pi\text{''}, L|\mathcal{J}_\mu|\text{``}N\text{''}, L\rangle}$$

finite-volume energies and matrix elements

(labels in quotes indicate quantum numbers)

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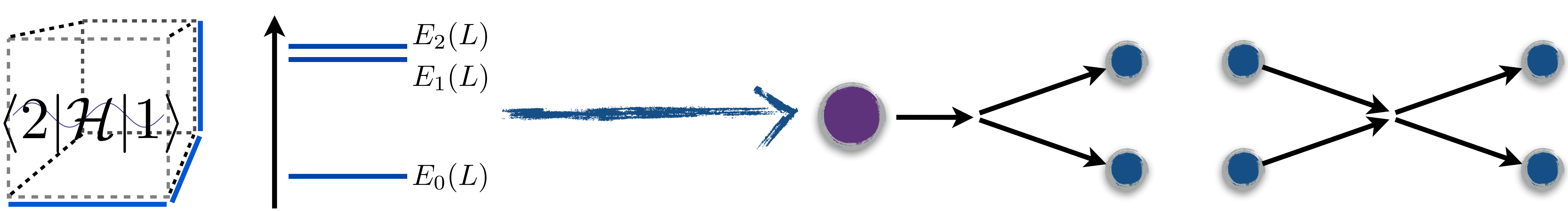
Lüscher (1991) + Lellouch and Lüscher (2001) derived relations between such finite-volume quantities and infinite-volume experimental observables



Neglect contributions scaling as $e^{-M_\pi L}$.

Status of multi-hadron matrix elements in LQCD…

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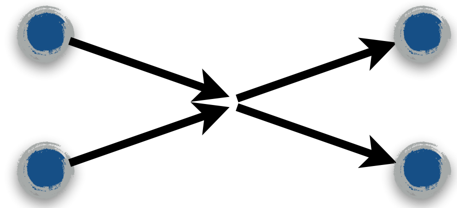
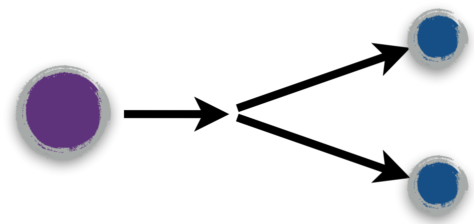
$\pi\pi \to \pi\pi, \ \sqrt{s} < 4M_\pi$ $(\mathbf{P} \neq 0$ in finite-volume frame)*		Lüscher (1986, 1991) Rummukainen and Gottlieb (1995)*
$K \to \pi\pi$ (relies on $M_K < 4M_\pi$) $(\mathbf{P} \neq 0$ in finite-volume frame)*		Lellouch and Lüscher (2001) Kim, Sachrajda and Sharpe (2005)*, Christ, Kim and Yamazaki (2005)*

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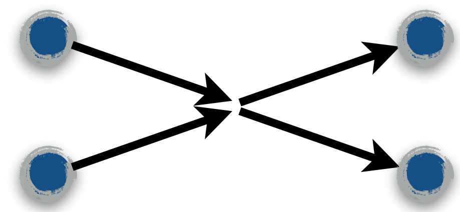
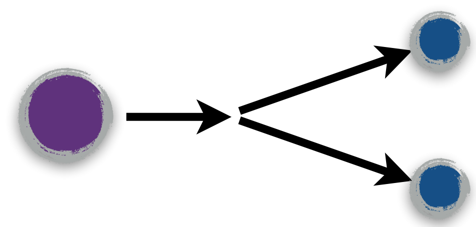
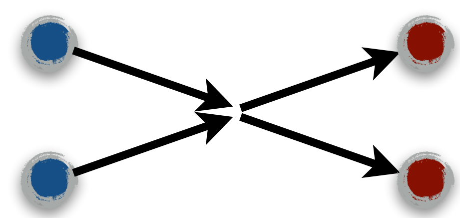
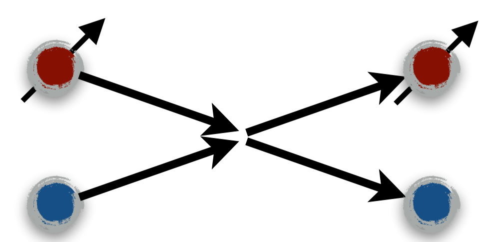
$\pi\pi \to \pi\pi, \ \sqrt{s} < 4M_\pi$ ($\mathbf{P} \neq 0$ in finite-volume frame)*		Lüscher (1986, 1991) Rummukainen and Gottlieb (1995)*
$K \to \pi\pi$ (relies on $M_K < 4M_\pi$) ($\mathbf{P} \neq 0$ in finite-volume frame)*		Lellouch and Lüscher (2001) Kim, Sachrajda and Sharpe (2005)*, Christ, Kim and Yamazaki (2005)*
$\pi\pi \to K\overline{K}, \ \sqrt{s} < 4M_\pi$ (not possible for physical masses)		Bernard et al. (2011), Fu (2012), Briceño and Davoudi (2012)
$NN \to NN, \ N\pi \to N\pi$ (energies below three-particle production)		Detmold and Savage (2004) Göckeler et al. (2012) Briceño (2014)

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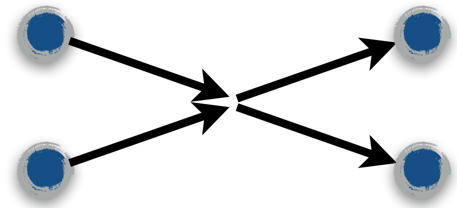
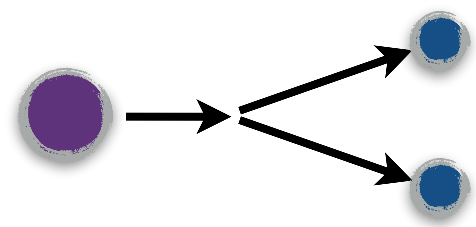
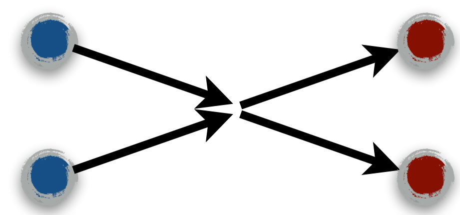
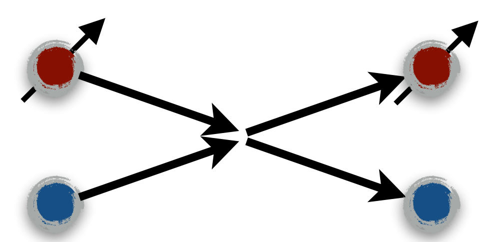
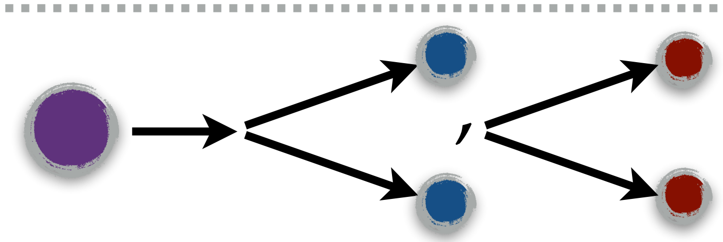
$\pi\pi \to \pi\pi,\ \sqrt{s} < 4M_\pi$ $(\mathbf{P} \neq 0$ in finite-volume frame)*		Lüscher (1986, 1991) Rummukainen and Gottlieb (1995)*
$K \to \pi\pi$ (relies on $M_K < 4M_\pi$) $(\mathbf{P} \neq 0$ in finite-volume frame)*		Lellouch and Lüscher (2001) Kim, Sachrajda and Sharpe (2005)*, Christ, Kim and Yamazaki (2005)*
$\pi\pi \to K\overline{K},\ \sqrt{s} < 4M_\pi$ (not possible for physical masses)		Bernard et al. (2011), Fu (2012), Briceño and Davoudi (2012)
$NN \to NN,\ N\pi \to N\pi$ (energies below three-particle production)		Detmold and Savage (2004) Göckeler et al. (2012) Briceño (2014)
$D \to \pi\pi,\ K\overline{K}$ (ignores four-particle states)		MTH and Sharpe (2012)

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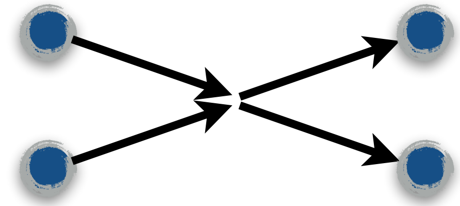
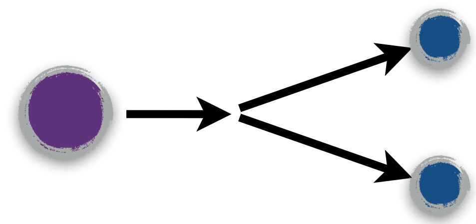
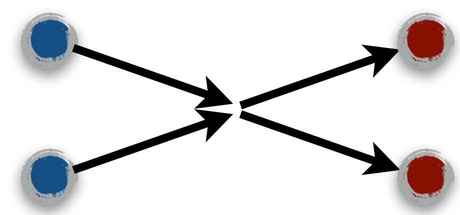
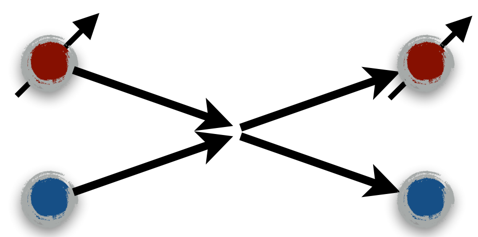
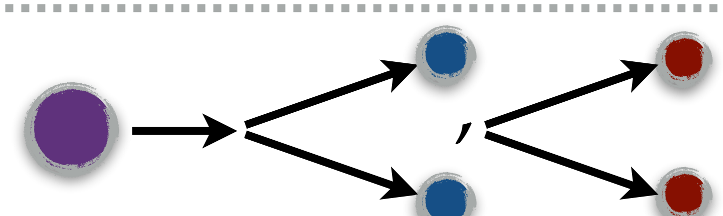
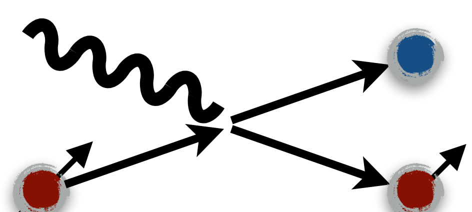
$\pi\pi \to \pi\pi,\ \sqrt{s} < 4M_\pi$ $(\mathbf{P} \neq 0$ in finite-volume frame)*		Lüscher (1986, 1991) Rummukainen and Gottlieb (1995)*
$K \to \pi\pi$ (relies on $M_K < 4M_\pi$) $(\mathbf{P} \neq 0$ in finite-volume frame)*		Lellouch and Lüscher (2001) Kim, Sachrajda and Sharpe (2005)*, Christ, Kim and Yamazaki (2005)*
$\pi\pi \to K\overline{K},\ \sqrt{s} < 4M_\pi$ (not possible for physical masses)		Bernard et al. (2011), Fu (2012), Briceño and Davoudi (2012)
$NN \to NN,\ N\pi \to N\pi$ (energies below three-particle production)		Detmold and Savage (2004) Göckeler et al. (2012) Briceño (2014)
$D \to \pi\pi,\ K\overline{K}$ (ignores four-particle states)		MTH and Sharpe (2012)
$\gamma^* \to \pi\pi,\ \pi\gamma^* \to \pi\pi,$ $N\nu \to N\pi\ell$ $B \to K^*(\to K\pi)\ell\ell$ (energies below three-particle production)		Meyer (2011), Bernard et al. (2012), A. Agadjanov et al. (2014), Briceño, MTH and Walker-Loud (2014) Briceño and MTH (2015)

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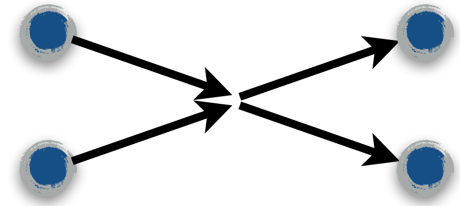
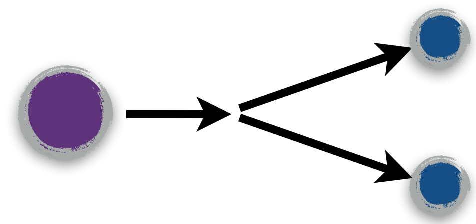
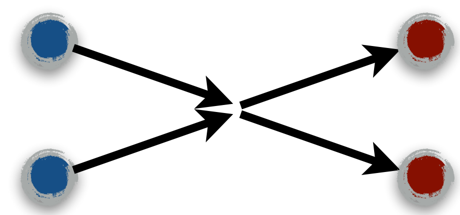
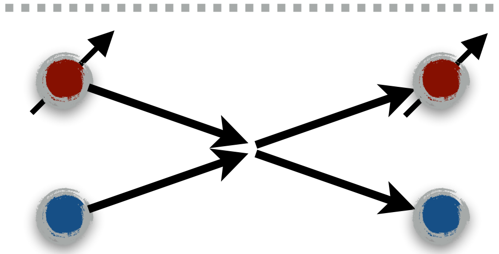
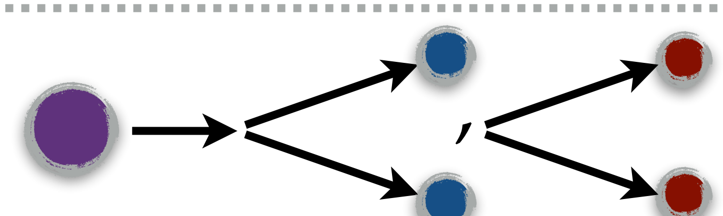
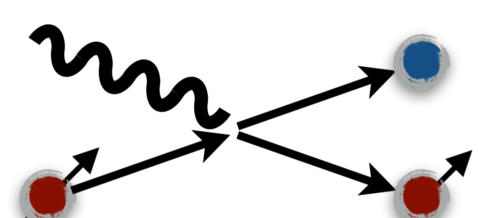
elastic scattering of identical scalars		Lüscher (1986, 1991) Rummukainen and Gottlieb (1995)*
decay into identical scalars (no other open decay channels)		Lellouch and Lüscher (2001) Kim, Sachrajda and Sharpe (2005)*, Christ, Kim and Yamazaki (2005)*
non-identical scalars, multiple coupled channels*		Bernard et al. (2011), Fu (2012), Briceño and Davoudi (2012)
scattering of particles with intrinsic spin*		Detmold and Savage (2004) Göckeler et al. (2012) Briceño (2014)
decay into multiple, coupled two-particle channels*		MTH and Sharpe (2012)
particle production mediated by a generic local current*		Meyer (2011), Bernard et al. (2012), A. Agadjanov et al. (2014), Briceño, MTH and Walker-Loud (2014) Briceño and MTH (2015)

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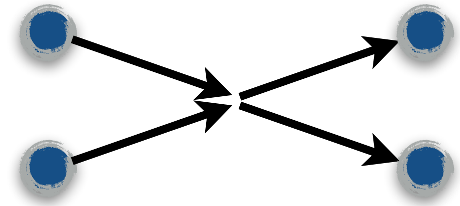
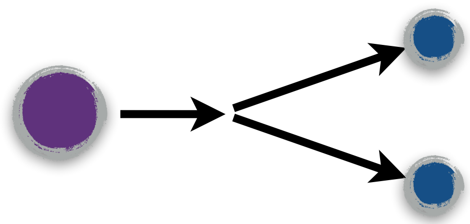
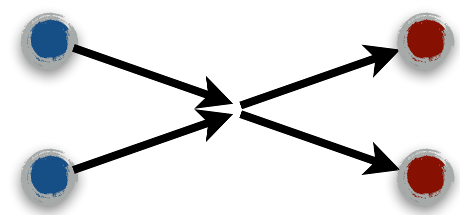
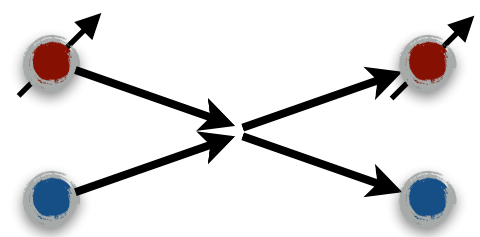
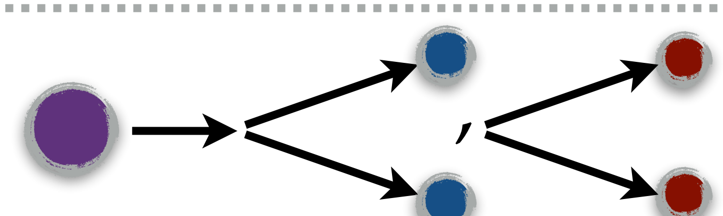
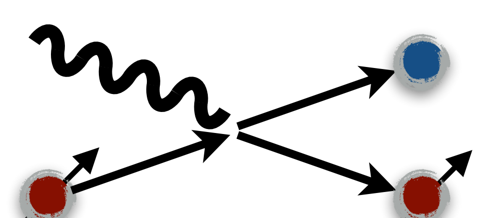
elastic scattering of identical scalars		Lüscher (1986, 1991) Rummukainen and Gottlieb (1995)*
decay into identical scalars (no other open decay channels)		Lellouch and Lüscher (2001) Kim, Sachrajda and Sharpe (2005)*, Christ, Kim and Yamazaki (2005)*
non-identical scalars, multiple coupled channels*		Bernard et al. (2011), Fu (2012), Briceño and Davoudi (2012)
scattering of particles with intrinsic spin*		Detmold and Savage (2004) Göckeler et al. (2012) Briceño (2014)
decay into multiple, coupled two-particle channels*		MTH and Sharpe (2012)
particle production mediated by a generic local current*		Meyer (2011), Bernard et al. (2012), A. Agadjanov et al. (2014), Briceño, MTH and Walker-Loud (2014) Briceño and MTH (2015)

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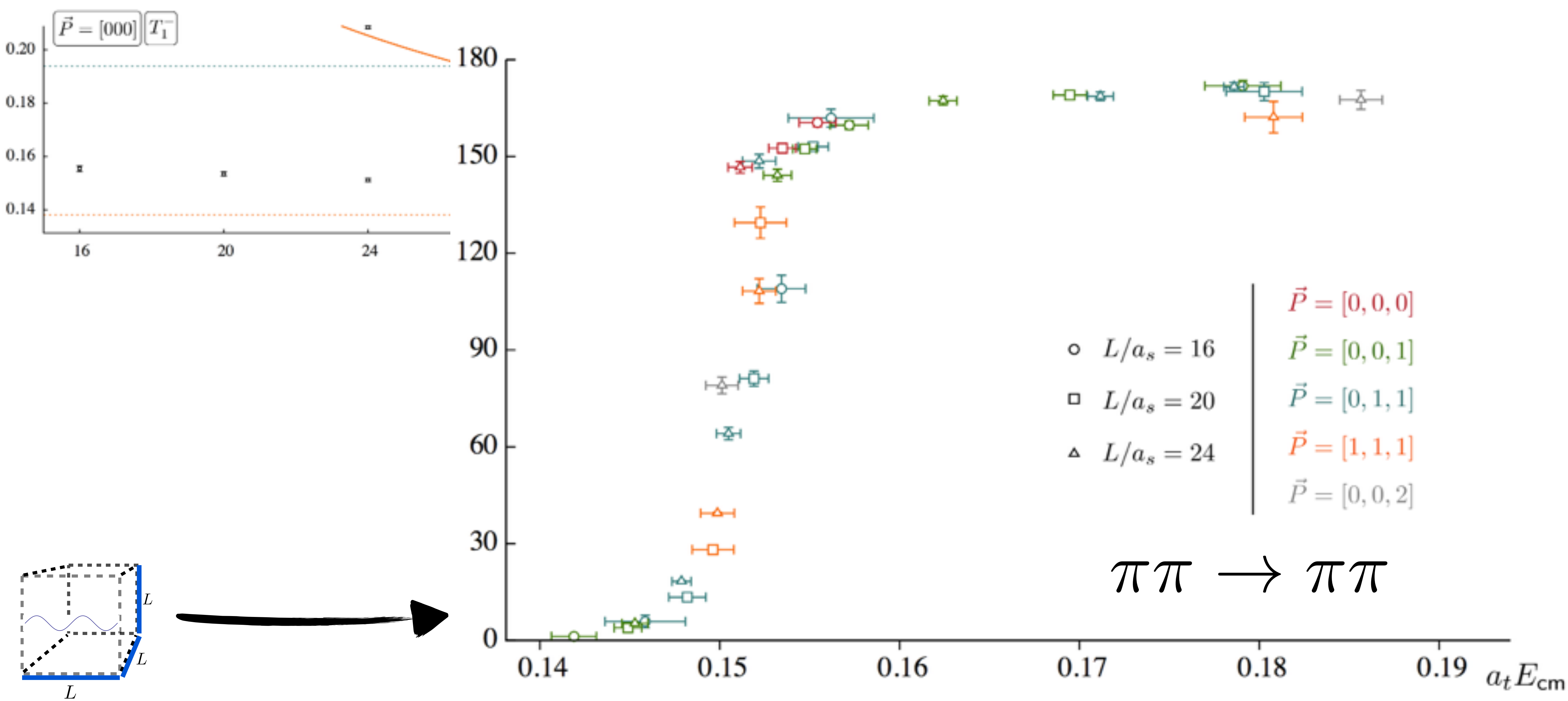
Example: pion scattering in the ρ channel…

$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$

scattering phase **known geometric function**

Lüscher (1986, 1991), Rummukainen and Gottlieb (1995)



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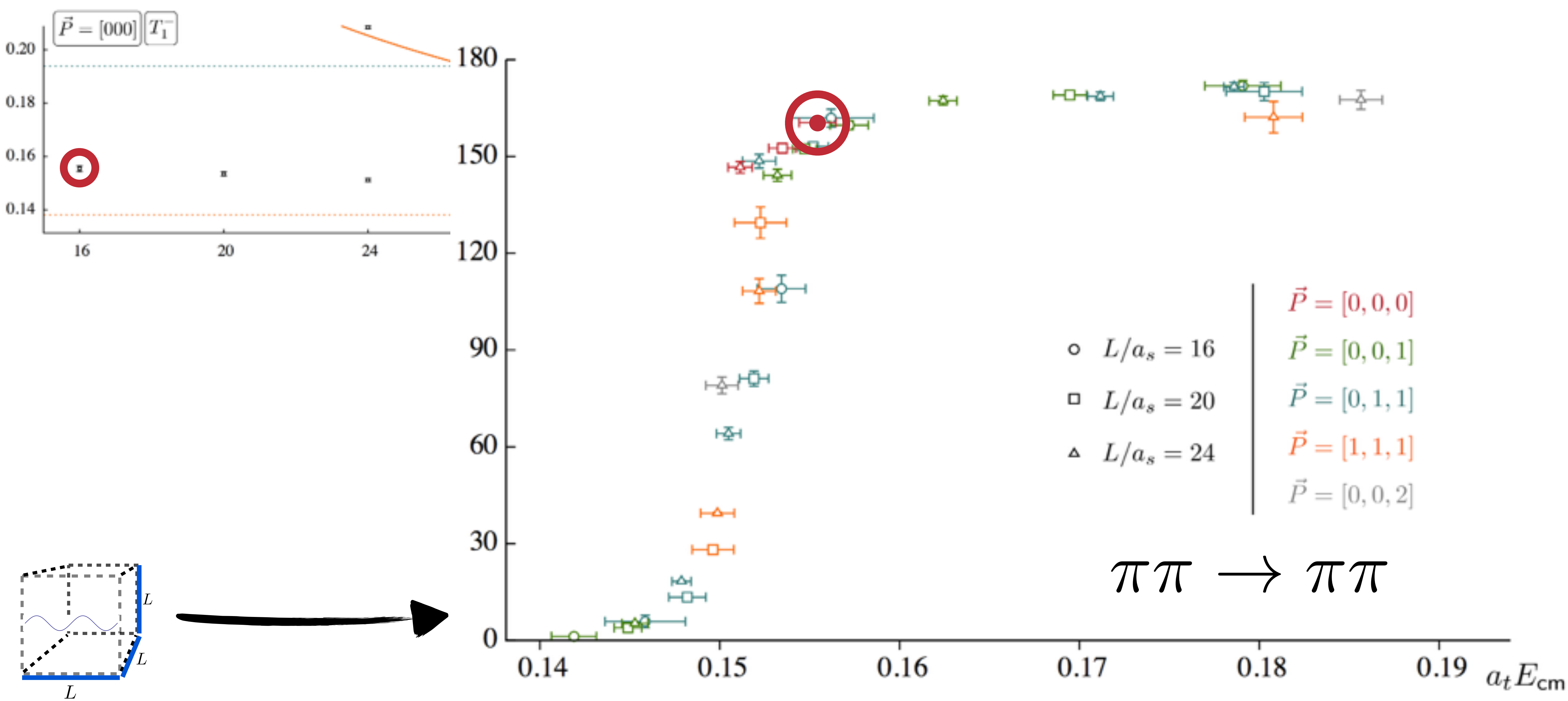

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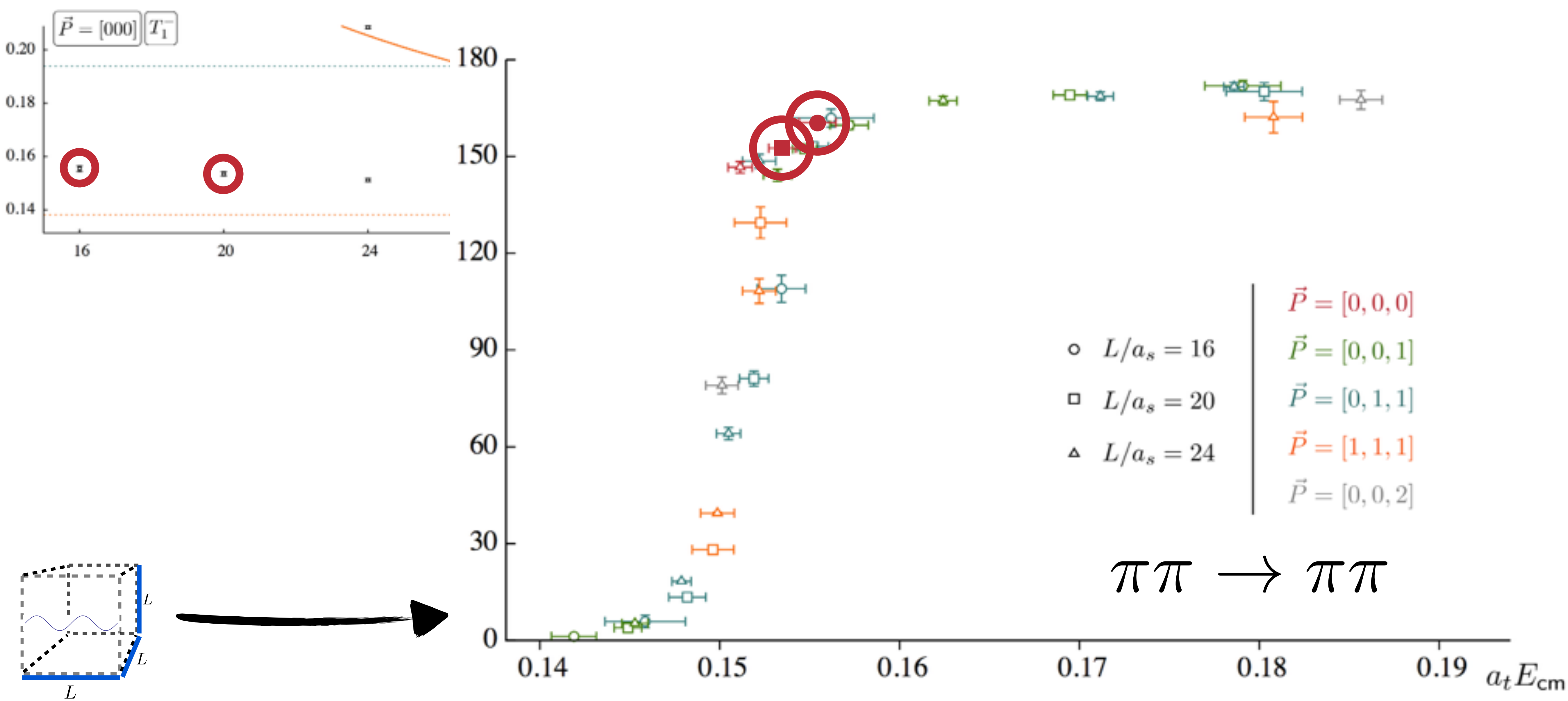

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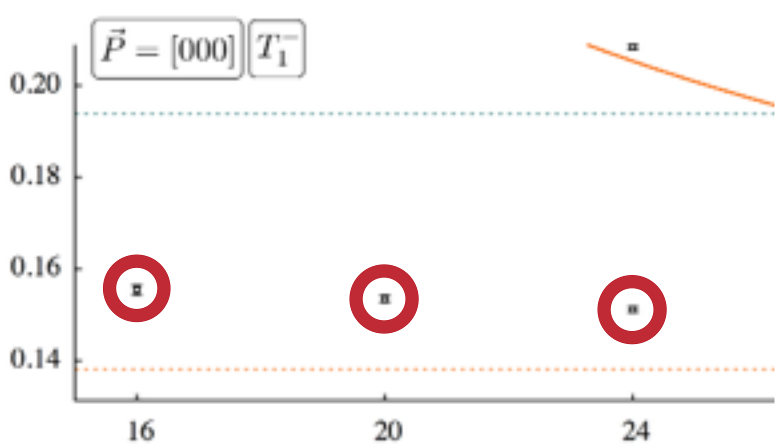

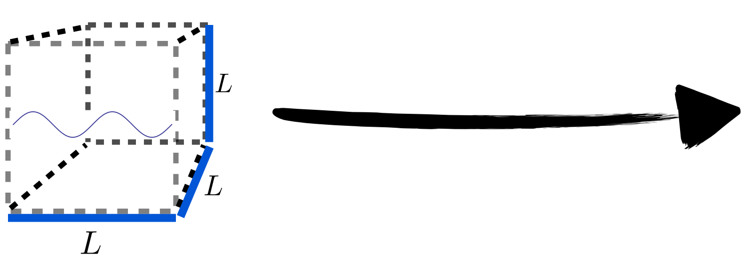

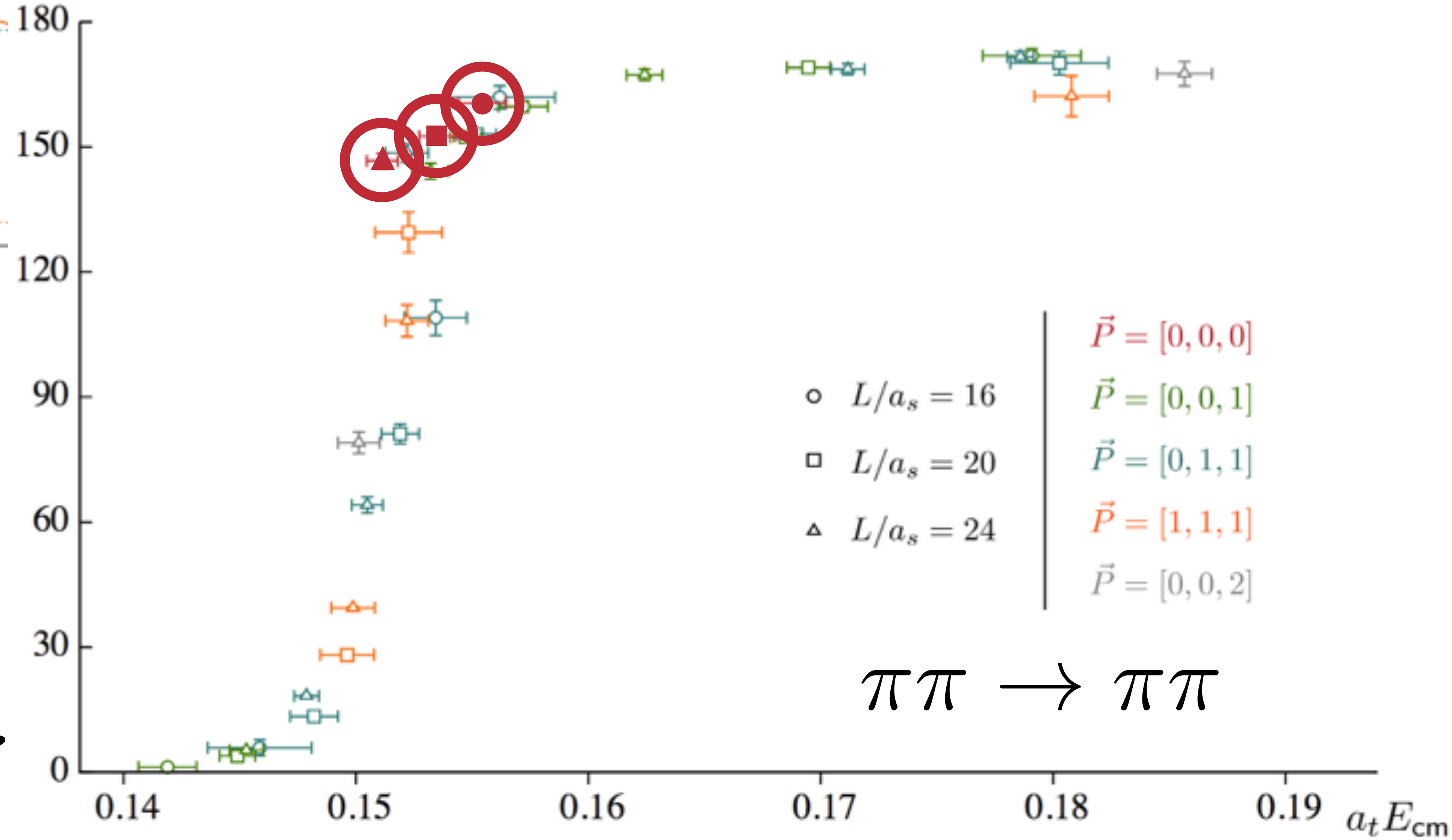

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$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$

scattering phase **known geometric function**

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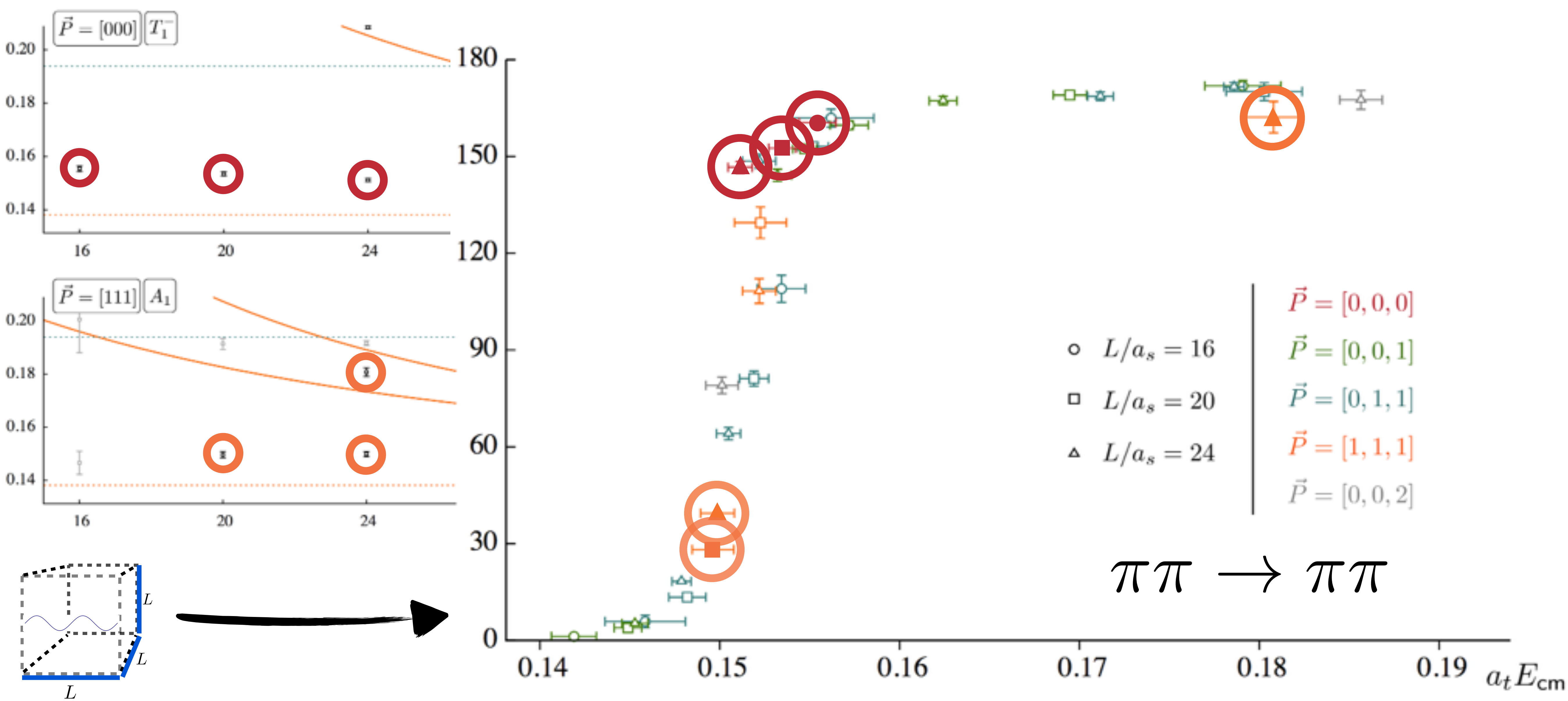

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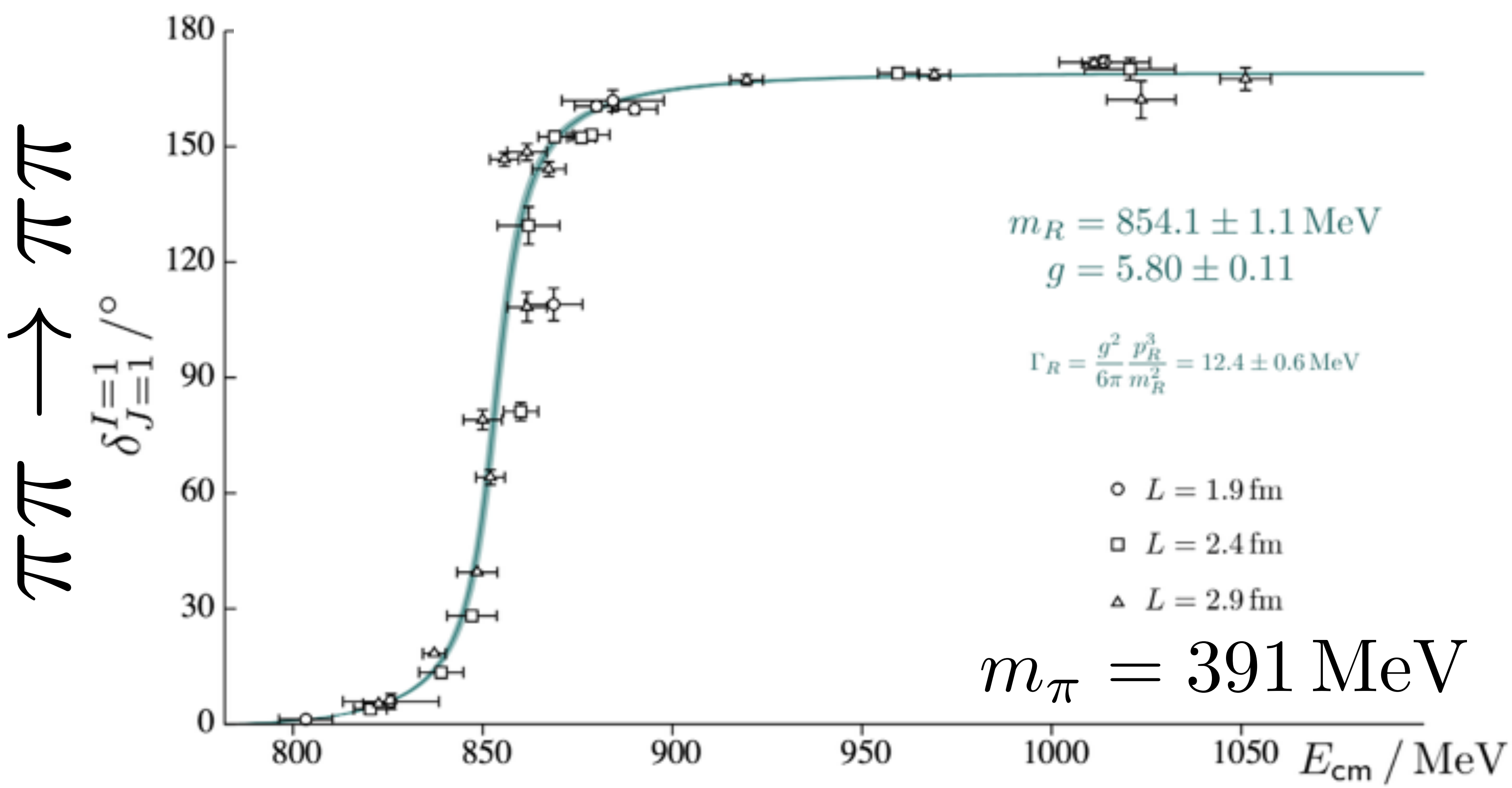

Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

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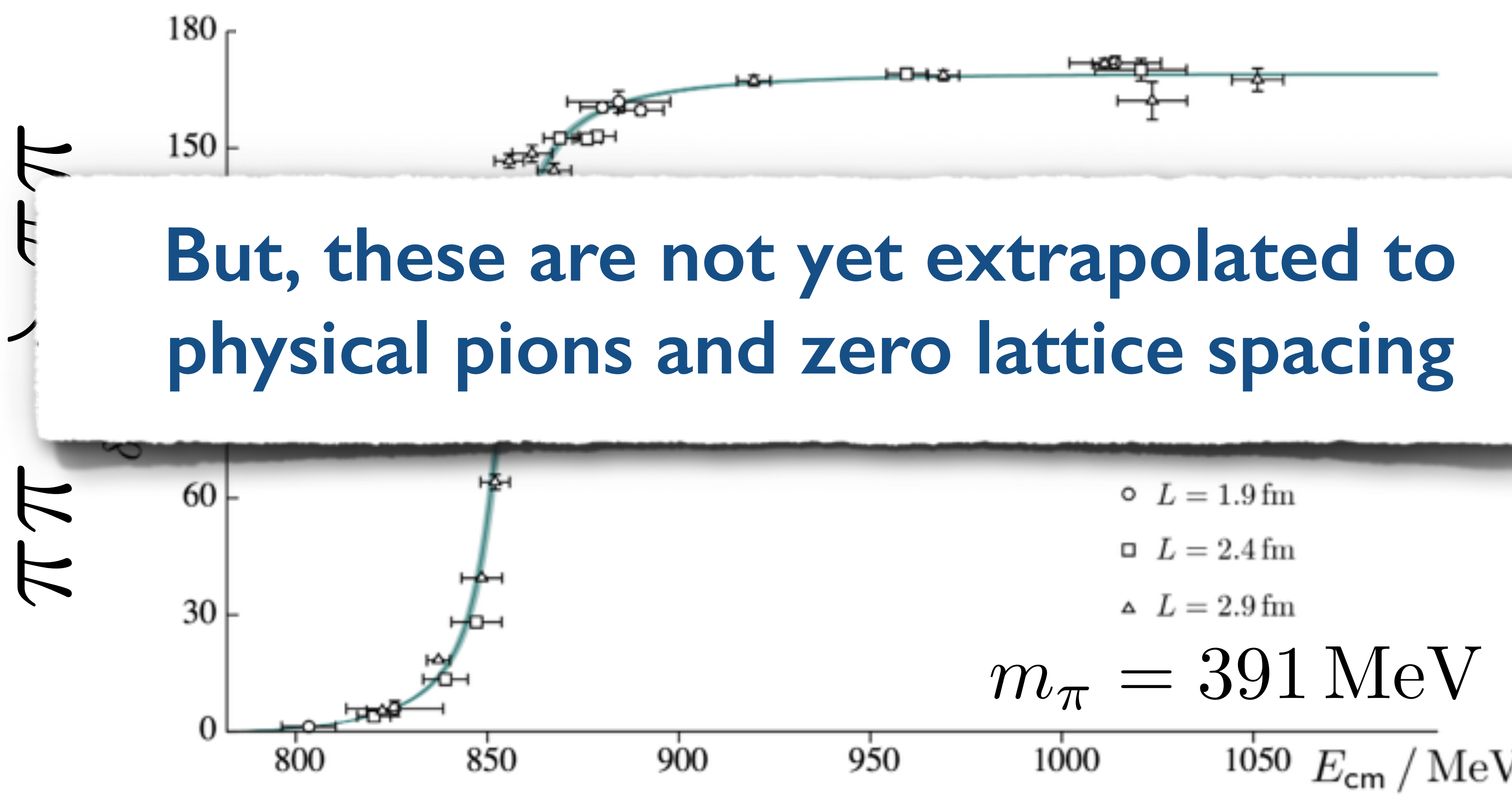
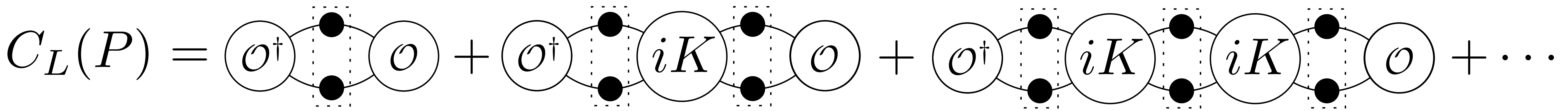


But, these are not yet extrapolated to physical pions and zero lattice spacing

Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505

Derivation in a nut shell

1 Define finite-volume correlator and relate to skeleton expansion



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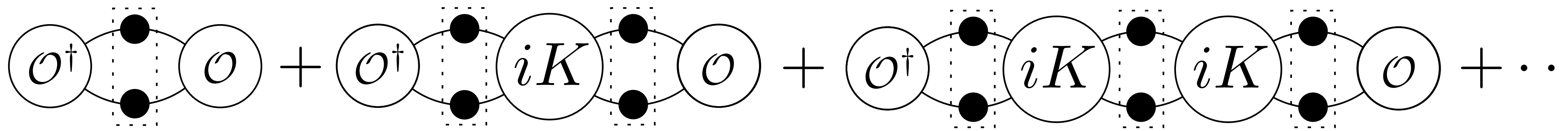
$C_L(P) =$ 

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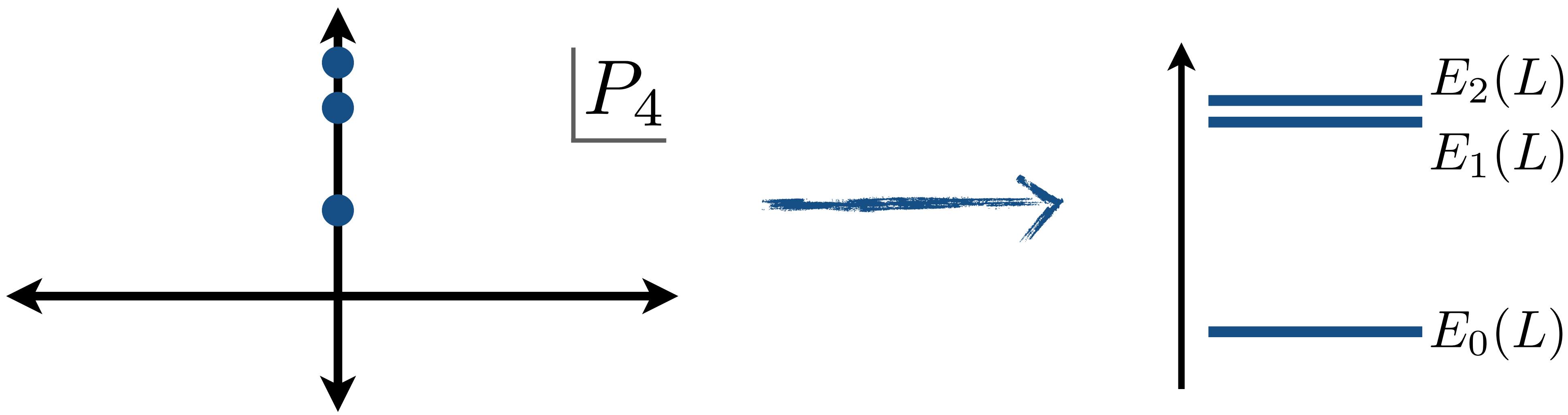

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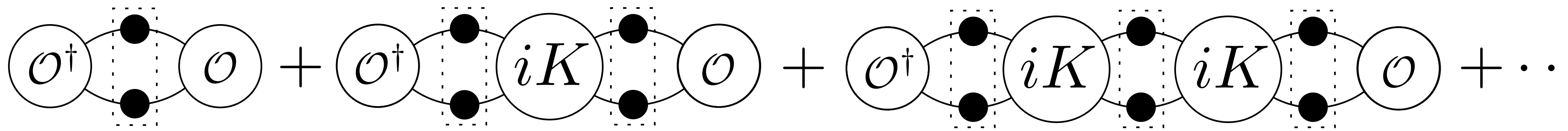
$C_L(P) =$ 

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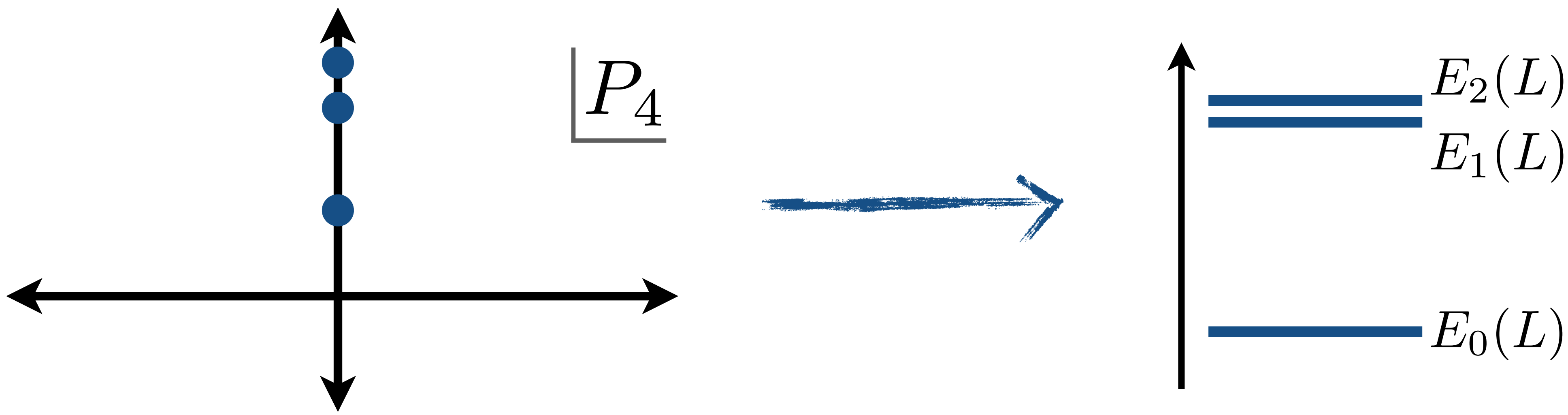

3 **Break diagrams into finite- and infinite-volume parts**

Derivation in a nut shell

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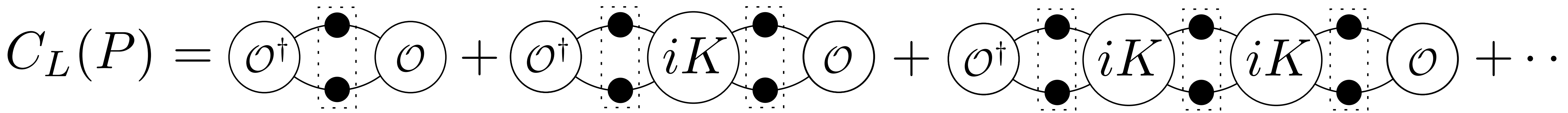

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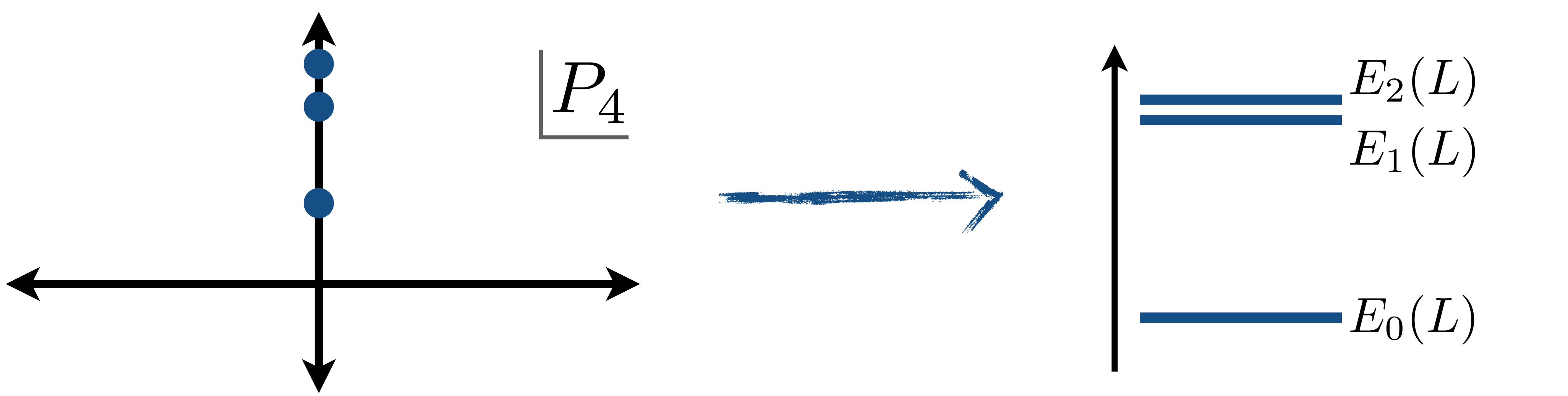


3 Break diagrams into finite- and infinite-volume parts

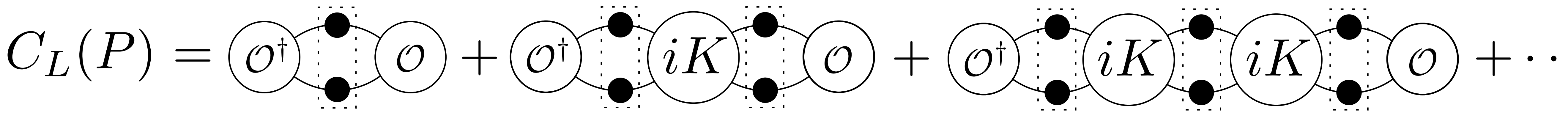
4 Sum resulting series and identify poles in C_L to reach

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2\to 2} F} A$$

Matrix of known geometric functions

Derivation in a nut shell

1 **Define finite-volume correlator and relate to skeleton expansion**



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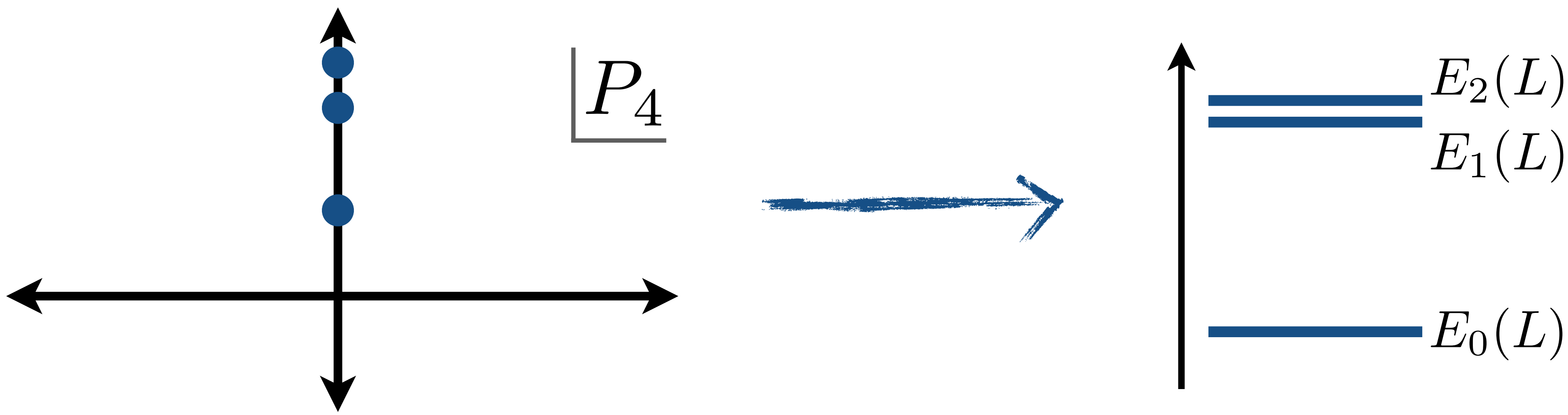

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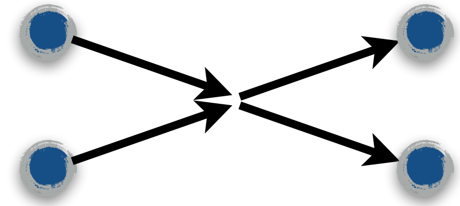
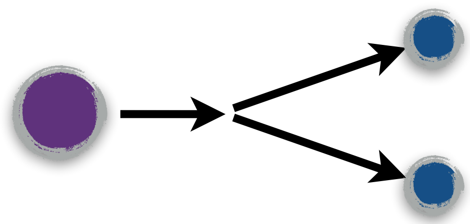
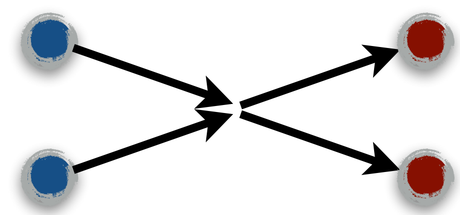
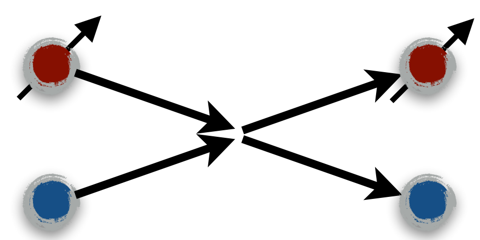
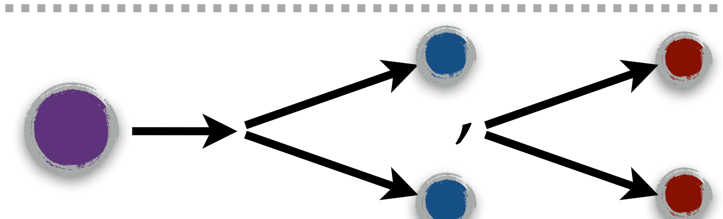
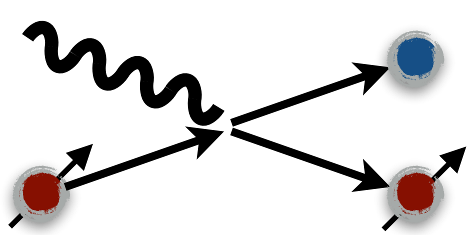
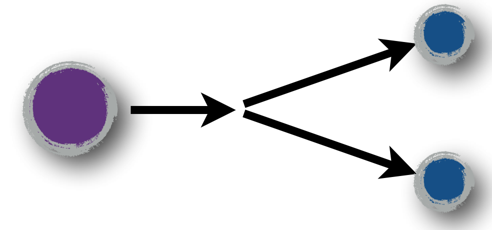
Determinant over angular momenta

Status of multi-hadron matrix elements in LQCD…

Physical system	Method to get it from LQCD	
elastic scattering of identical scalars		Lüscher (1986, 1991) Rummukainen and Gottlieb (1995)*
decay into identical scalars (no other open decay channels)		Lellouch and Lüscher (2001) Kim, Sachrajda and Sharpe (2005)*, Christ, Kim and Yamazaki (2005)*
non-identical scalars, multiple coupled channels*		Bernard et al. (2011), Fu (2012), Briceño and Davoudi (2012)
scattering of particles with intrinsic spin*		Detmold and Savage (2004) Göckeler et al. (2012) Briceño (2014)
decay into multiple, coupled two-particle channels*		MTH and Sharpe (2012)
particle production mediated by a generic local current*		Meyer (2011), Bernard et al. (2012), A. Agadjanov et al. (2014), Briceño, MTH and Walker-Loud (2014) Briceño and MTH (2015)

*(assumes no three or four-particle channels open)

$K \to \pi\pi$ from LQCD



$$|\langle \pi\pi, L|\widetilde{\mathcal{H}}_W|K, L\rangle|^2 = \mathcal{B}[\delta_{\pi\pi}]\ |\langle \pi\pi, \mathrm{out}|\mathcal{H}_W|K\rangle|^2$$

$$\mathcal{B}[\delta_{\pi\pi}] = \frac{p}{32\pi M_K^2}\left[\frac{\partial}{\partial E}\left(\phi + \delta_{\pi\pi}\right)\right]^{-1}_{E=M_K}$$

Lellouch and Lüscher (2001)

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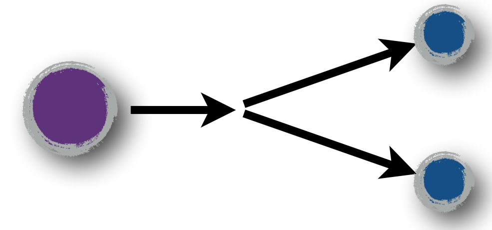

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To convert finite-volume LQCD matrix elements to physically observable decay amplitudes one uses the Lellouch-Lüscher conversion factor $\mathcal{B}[\delta_{\pi\pi}]$.

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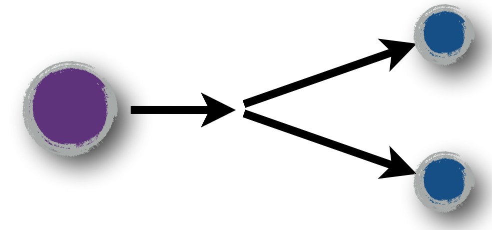

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(1). Determine finite-volume energies

(2). Use these to determine the (derivative of the) scattering phase

(3). Calculate the finite-volume matrix element

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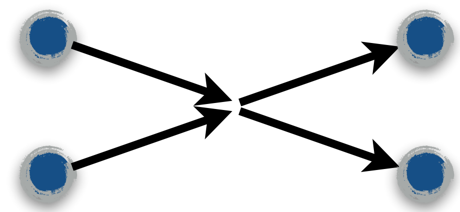
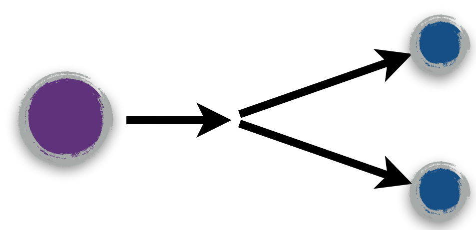
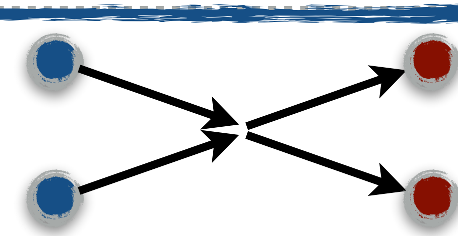
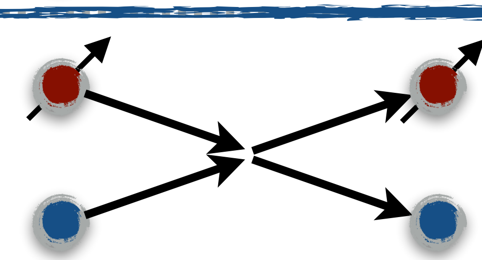
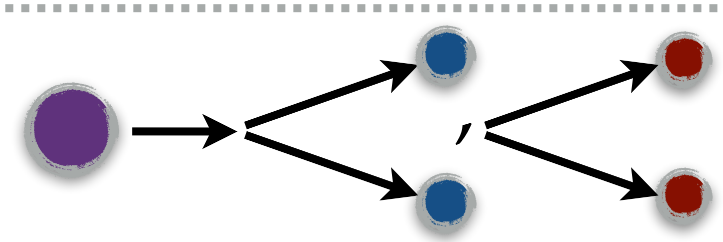
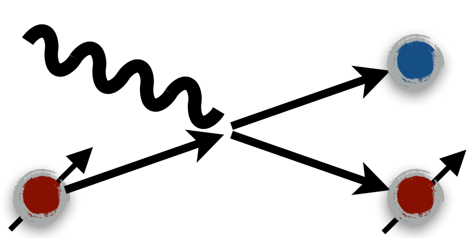
Status of multi-hadron matrix elements in LQCD…

Physical system	Method to get it from LQCD	
elastic scattering of identical scalars		Lüscher (1986, 1991) Rummukainen and Gottlieb (1995)*
decay into identical scalars (no other open decay channels)		Lellouch and Lüscher (2001) Kim, Sachrajda and Sharpe (2005)*, Christ, Kim and Yamazaki (2005)*
non-identical scalars, multiple coupled channels*		Bernard et al. (2011), Fu (2012), Briceño and Davoudi (2012)
scattering of particles with intrinsic spin*		Detmold and Savage (2004) Göckeler et al. (2012) Briceño (2014)
decay into multiple, coupled two-particle channels*		MTH and Sharpe (2012)
particle production mediated by a generic local current*		Meyer (2011), Bernard et al. (2012), A. Agadjanov et al. (2014), Briceño, MTH and Walker-Loud (2014) Briceño and MTH (2015)

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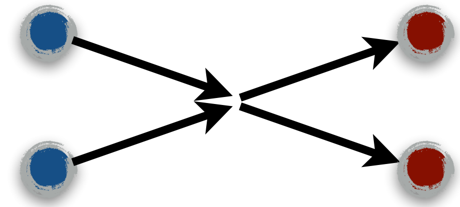
Multiple two-particle channels



Must now include a channel index

MTH and Sharpe/Briceño and Davoudi

$$\det\left[\begin{pmatrix}\mathcal{M}_{a\to a} & \mathcal{M}_{a\to b}\\ \mathcal{M}_{b\to a} & \mathcal{M}_{b\to b}\end{pmatrix}^{-1}+\begin{pmatrix}F_a & 0\\ 0 & F_b\end{pmatrix}\right]=0$$

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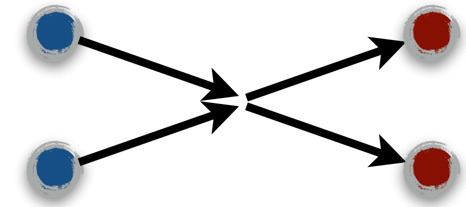

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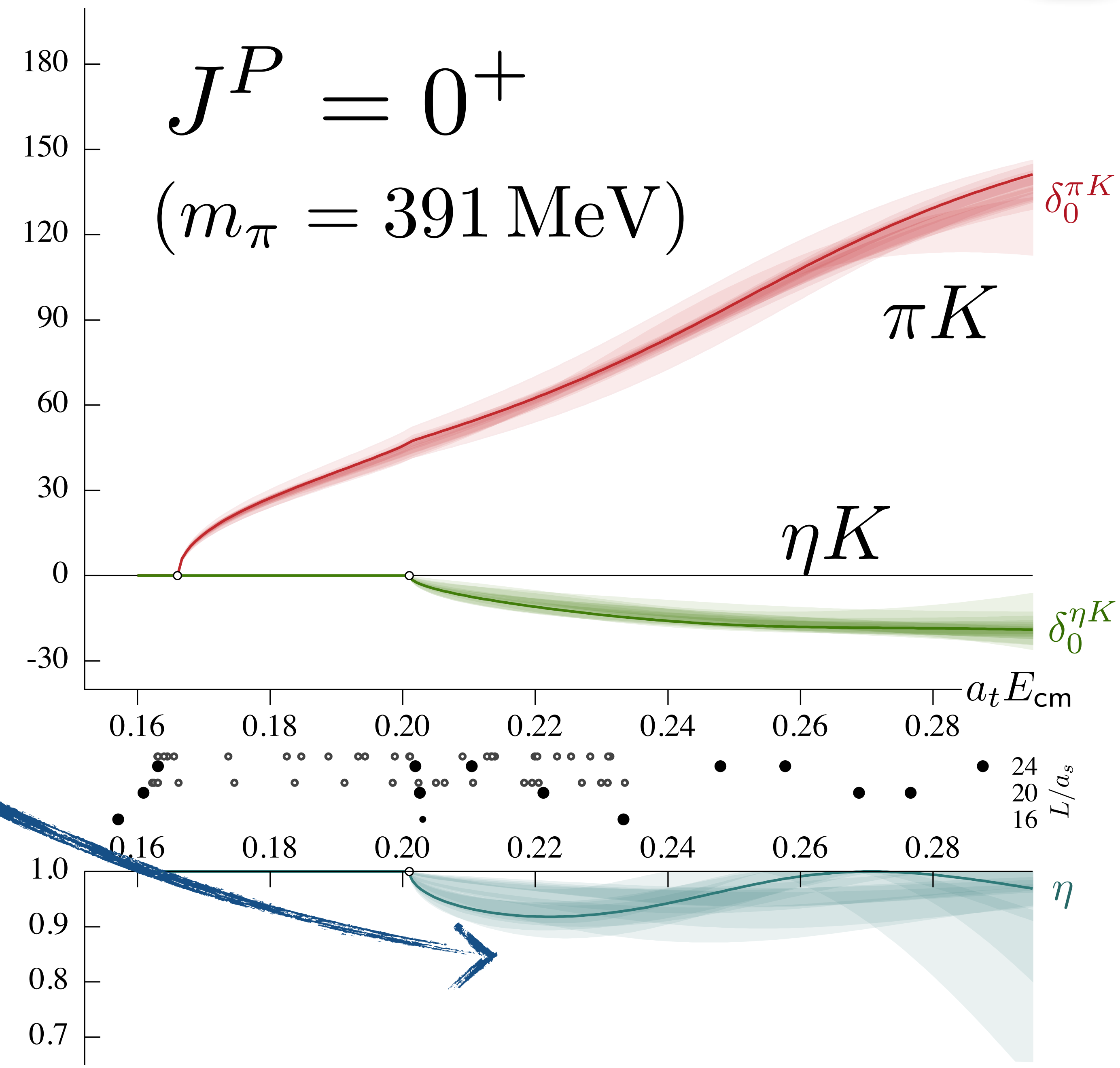
MTH and Sharpe/Briceño and Davoudi

First used in HadSpec study of

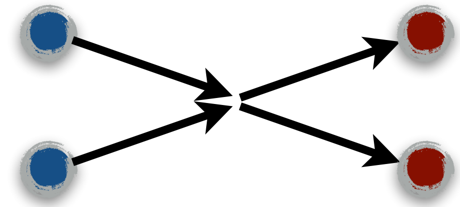
$$\pi K,\ \eta K$$

$$\mathcal{M}(\pi K \to \eta K)\sim\sqrt{1-\eta^2}$$



Wilson, Dudek, Edwards, Thomas, *Phys. Rev.* D 91, 054008 (2015) arXiv: 1411.2004

Multiple two-particle channels



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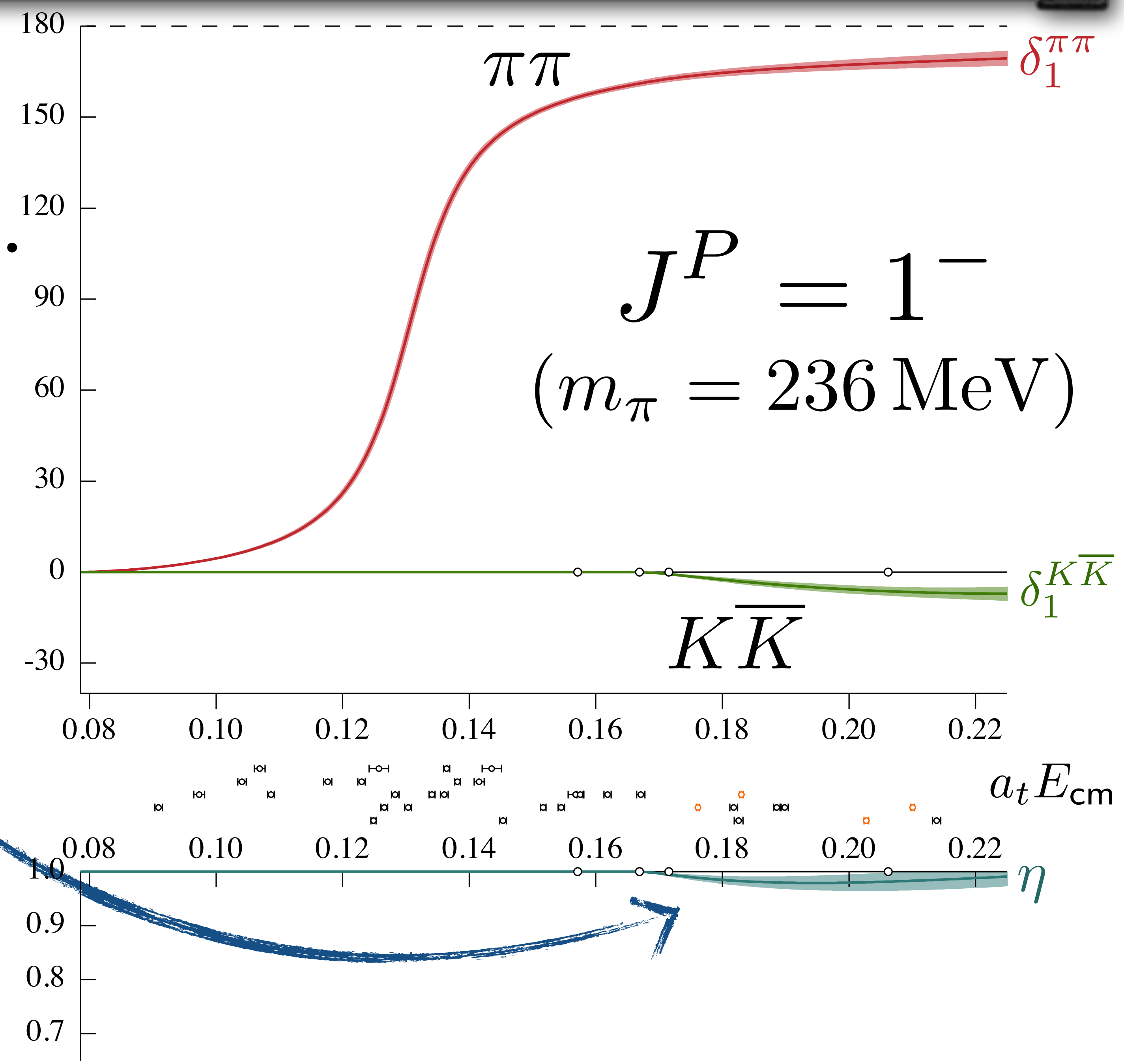
MTH and Sharpe/Briceño and Davoudi

Since then in other channels, e.g...

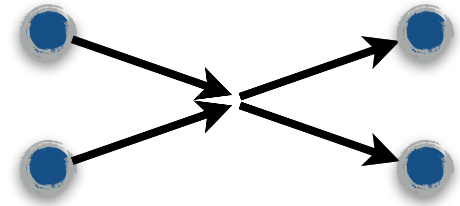
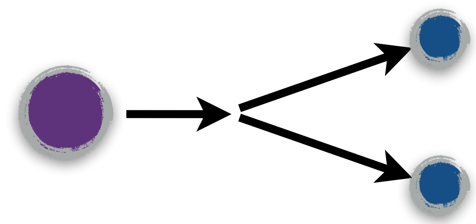
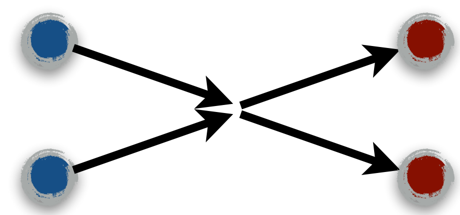
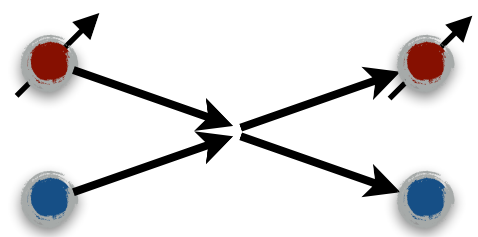
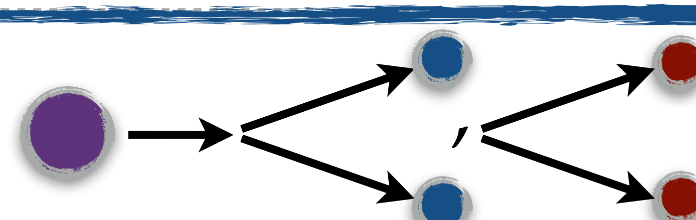
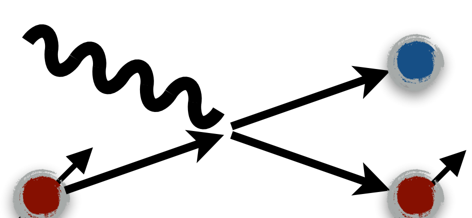
$$\pi\pi,\ \ K\overline{K}$$

$$\mathcal{M}(\pi\pi \to K\overline{K}) \sim \sqrt{1-\eta^2}$$



Wilson, Briceño, Dudek, Edwards, Thomas,
Phys. Rev. D 92, 094502 (2015)
arXiv:1507:02599

Status of multi-hadron matrix elements in LQCD…

Physical system		Method to get it from LQCD
elastic scattering of identical scalars		Lüscher (1986, 1991) Rummukainen and Gottlieb (1995)*
decay into identical scalars (no other open decay channels)		Lellouch and Lüscher (2001) Kim, Sachrajda and Sharpe (2005)*, Christ, Kim and Yamazaki (2005)*
non-identical scalars, multiple coupled channels*		Bernard et al. (2011), Fu (2012), Briceño and Davoudi (2012)
scattering of particles with intrinsic spin*		Detmold and Savage (2004) Göckeler et al. (2012) Briceño (2014)
decay into multiple, coupled two-particle channels*		MTH and Sharpe (2012)
particle production mediated by a generic local current*		Meyer (2011), Bernard et al. (2012), A. Agadjanov et al. (2014), Briceño, MTH and Walker-Loud (2014) Briceño and MTH (2015)

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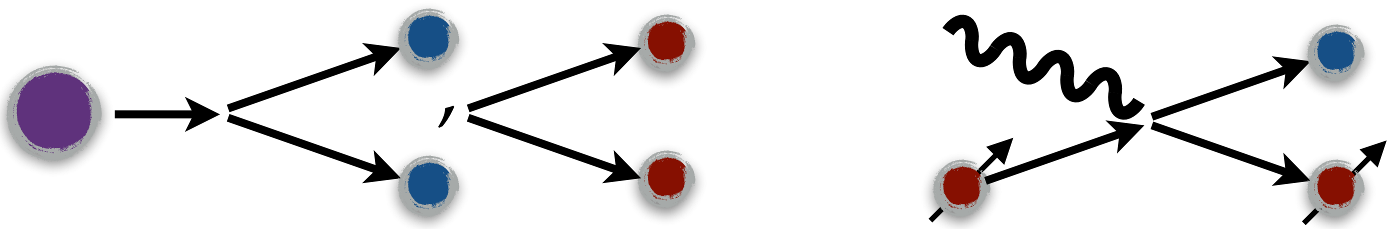
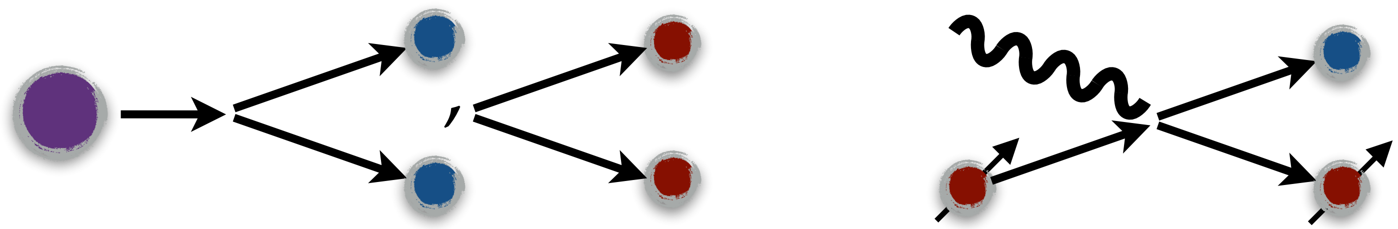
General two-hadron matrix elements



Formalism is now available for all one-to-two matrix elements of local currents

$$\langle n, L|\mathcal{J}_\mu|N, L\rangle = \left|\mathcal{C}_{N\pi}(L)\langle N\pi, \mathrm{out}|\mathcal{J}_\mu|N\rangle + \mathcal{C}_{N\eta}(L)\langle N\eta, \mathrm{out}|\mathcal{J}_\mu|N\rangle + \cdots\right|$$

General two-hadron matrix elements



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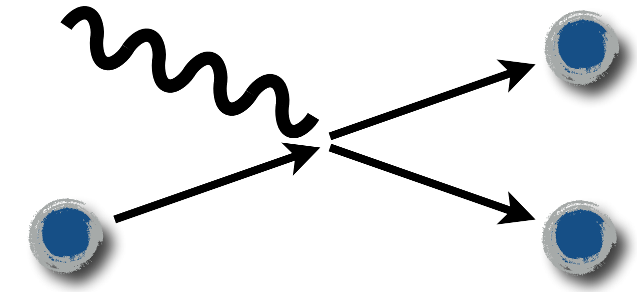
(1). **Determine finite-volume energies**

(2). **Use these to determine the (derivatives of) all scattering parameters in the coupled-channel sector**

(3). **Calculate multiple finite-volume matrix elements**

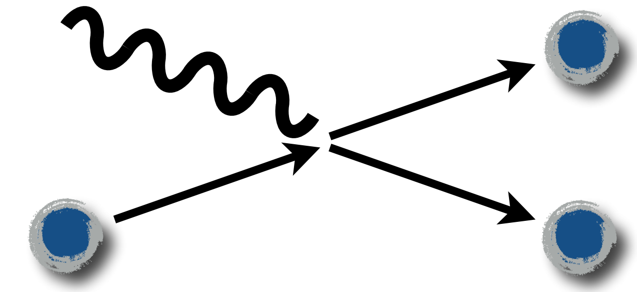
(4). **Deduce multiple, linearly independent relations between finite- and infinite-volume matrix elements**

(5). **Solve for the infinite-volume transition amplitudes**

Derivation in a nut shell $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

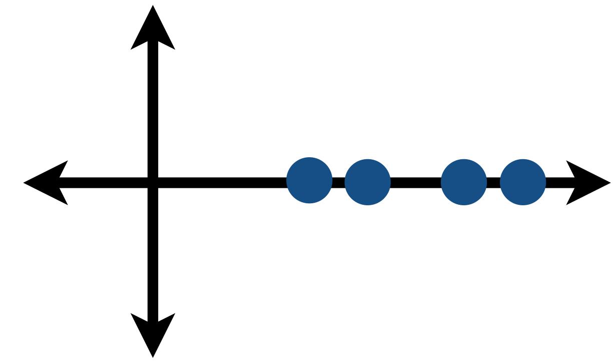
How can we get this from finite-volume observables?

Derivation in a nut shell $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$



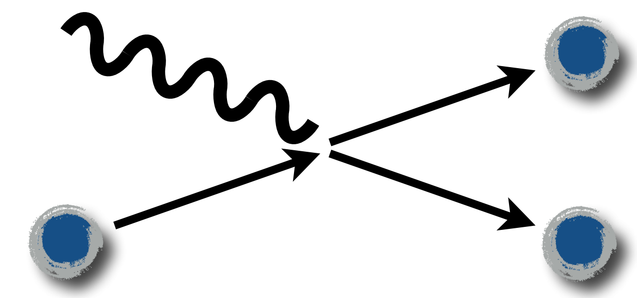
How can we get this from finite-volume observables?

Why did we expect $C_L(P)$ ***to have poles?***



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

Derivation in a nut shell $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$



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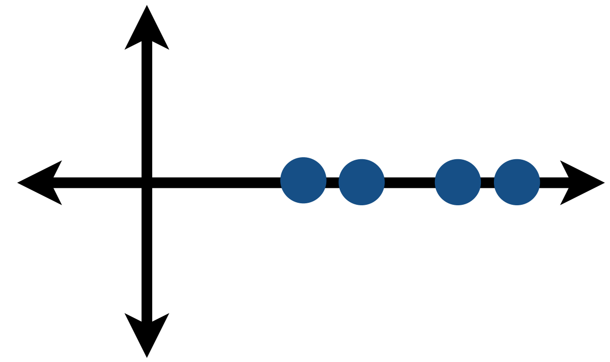

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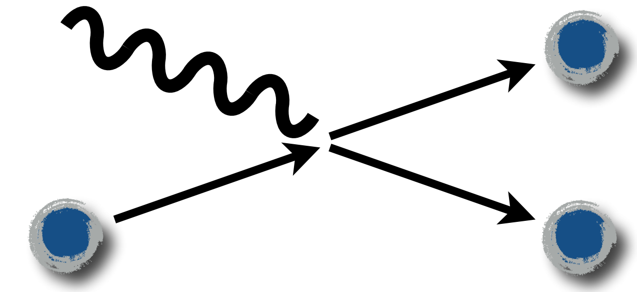
Insert a complete set finite-volume of states

Derivation in a nut shell $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$



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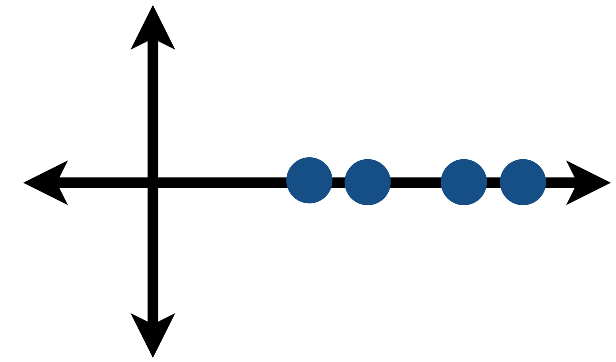
Why did we expect $C_L(P)$ *to have poles?*



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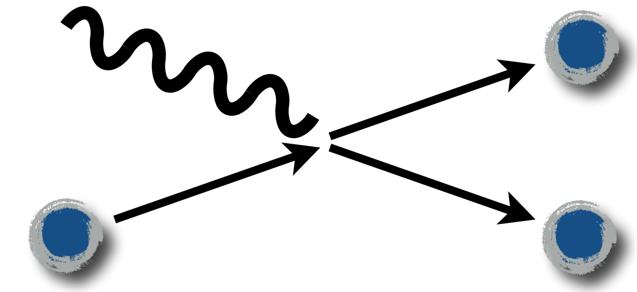
Insert a complete set finite-volume of states

$$C_L(P) \xrightarrow[E \to E_n]{} -\frac{L^3 \langle 0|\mathcal{O}(0)|n, \vec{P}, L\rangle\langle n, \vec{P}, L|\mathcal{O}^\dagger(0)|0\rangle}{E - E_n}$$

Derivation in a nut shell $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

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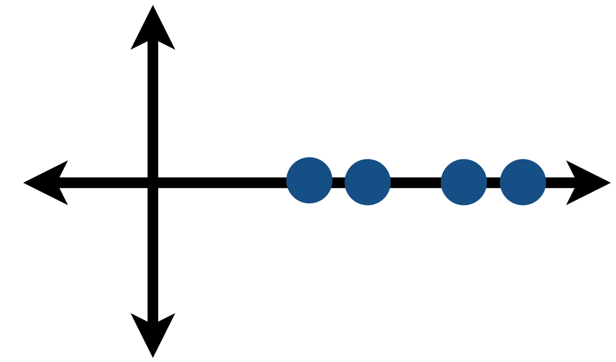

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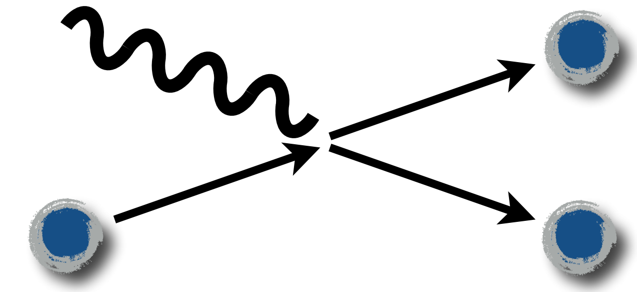
Derivation in a nut shell $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

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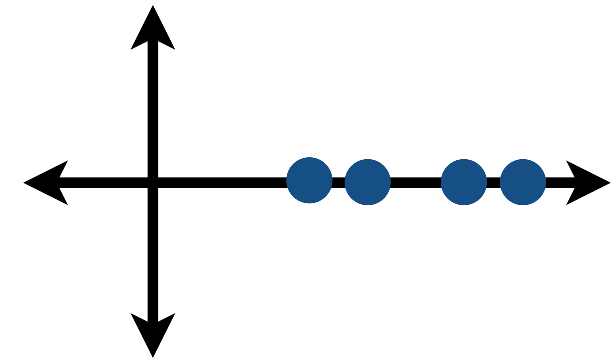

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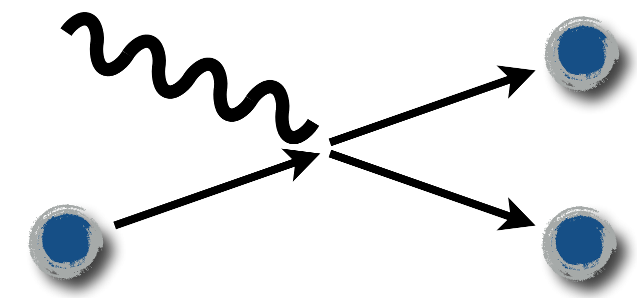
Derivation in a nut shell $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

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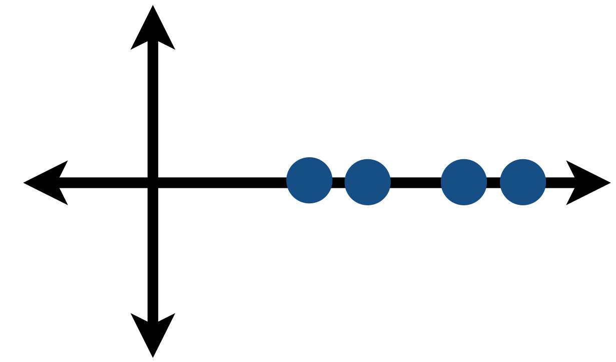

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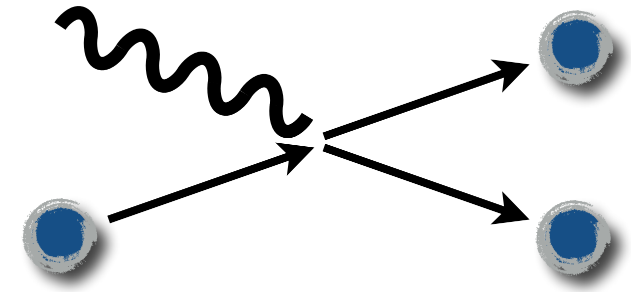
Derivation in a nut shell $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

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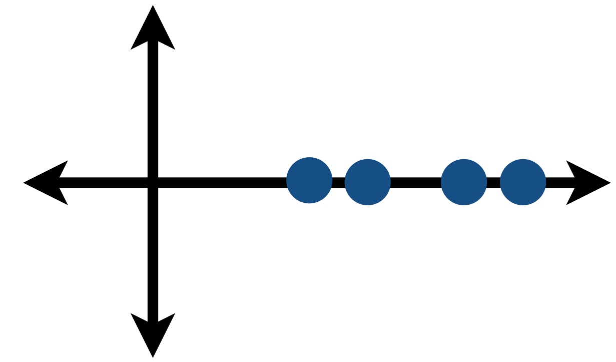

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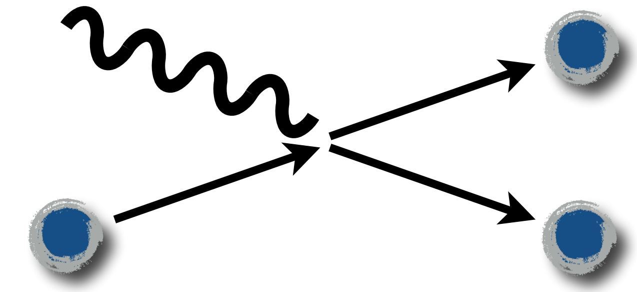
Derivation in a nut shell $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$ 

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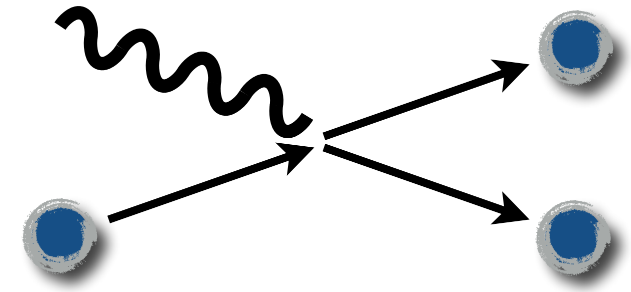
Derivation in a nut shell $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

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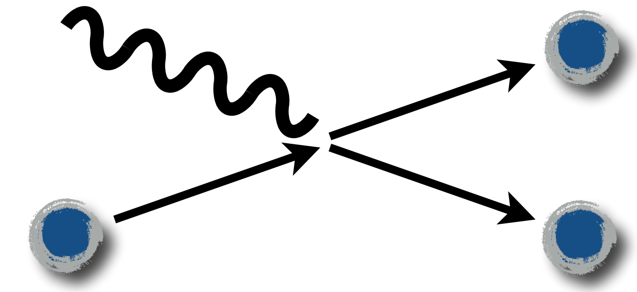
This can then be re-expressed as...

get this from the lattice **experimental observable**

$$\langle n, L|\mathcal{J}_\mu|\pi, L\rangle = \left| \mathcal{C}_{\pi\pi}(L) \langle \pi\pi, J = 1, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle + \cdots \right|$$

R. A. Briceño, MTH, A. Walker-Loud, 2015

Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$ 

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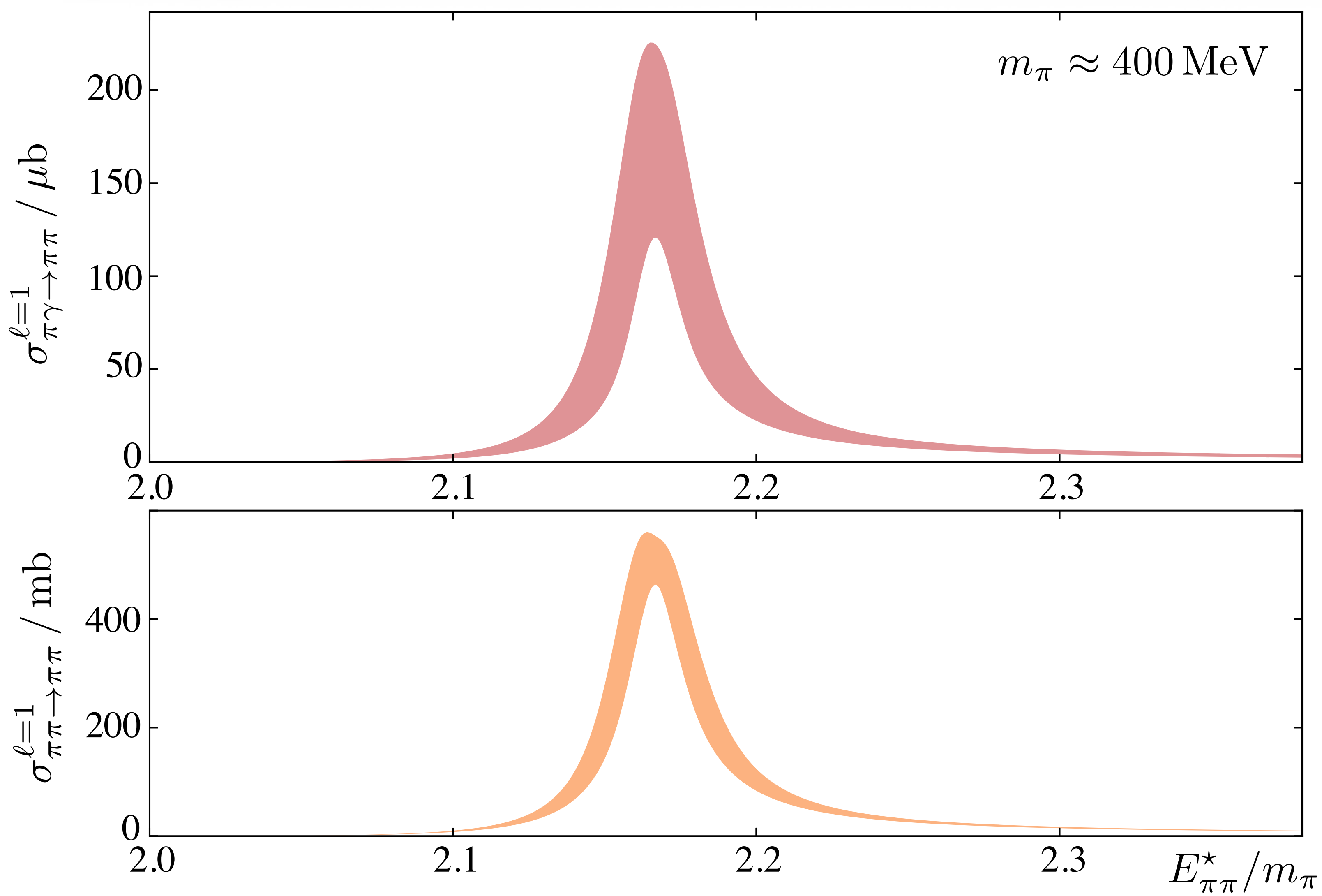

Photoproduction in the rho channel

Briceño, Dudek, Edwards, Schultz, Thomas, Wilson, Phys. Rev. D 93, 114508 (2016)

Most important limitation:

Formalism is only available for final state energies below three-particle production threshold

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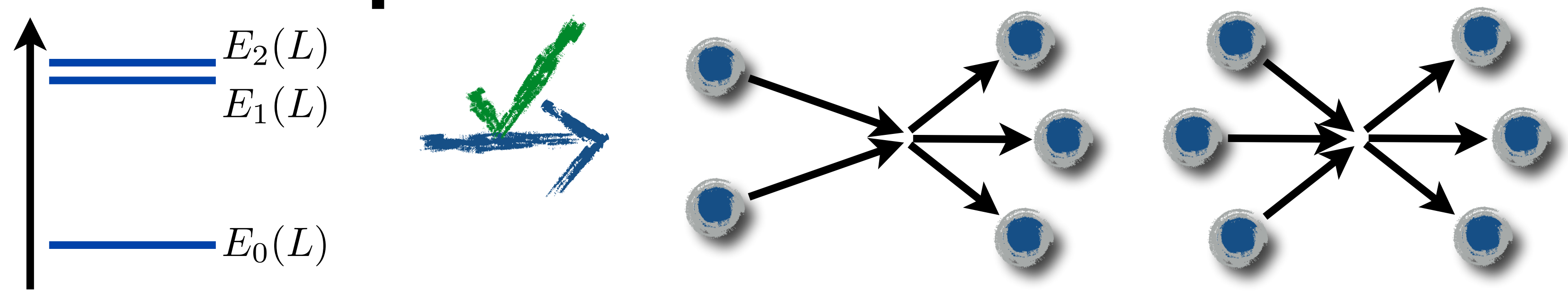
Major focus over the last few years

Three particles: Current status



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Model-independent relation between

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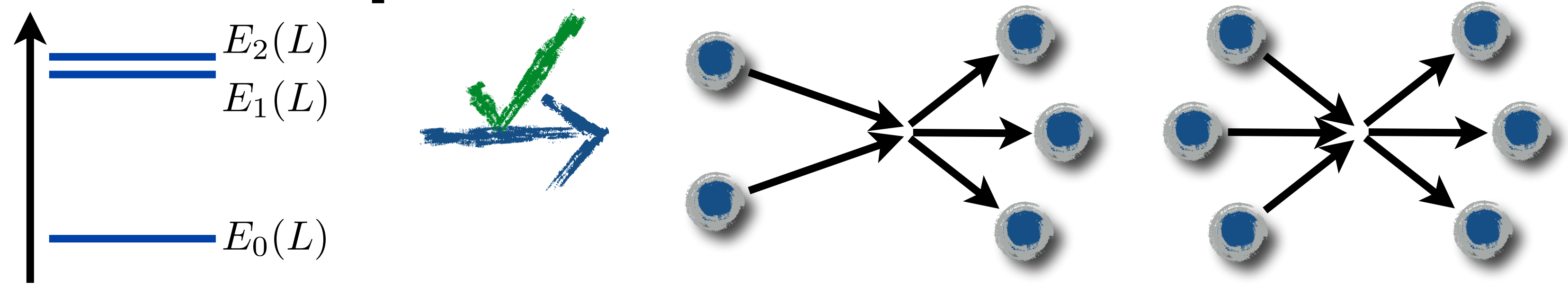

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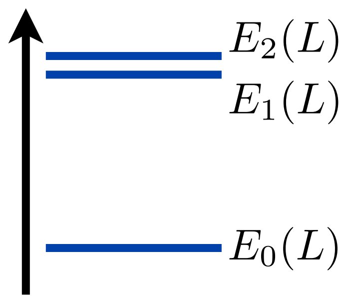

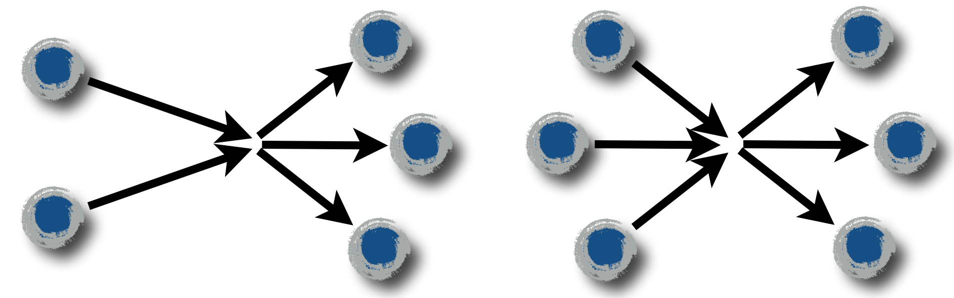

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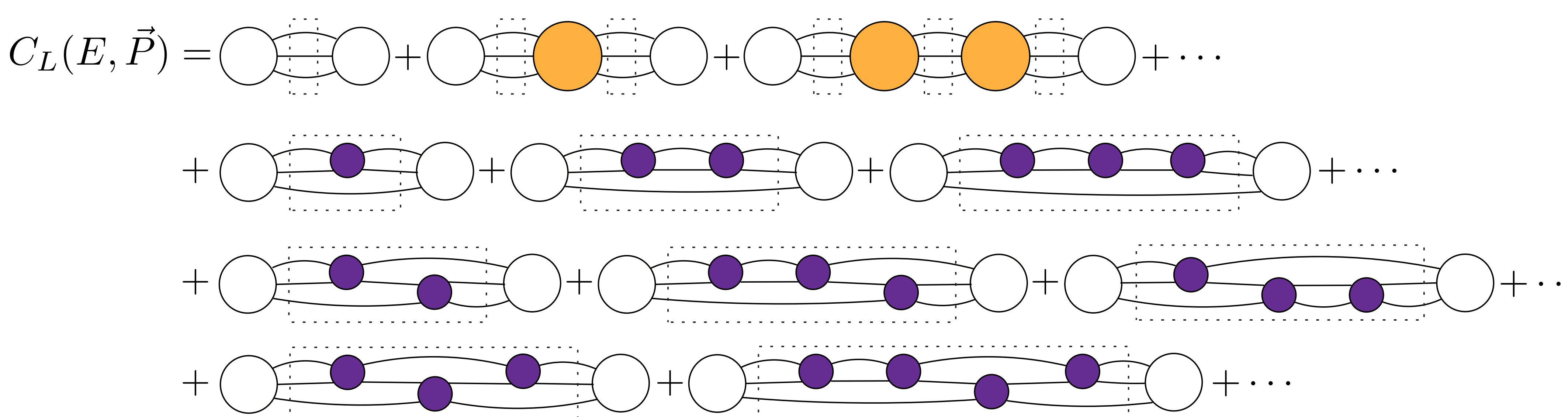

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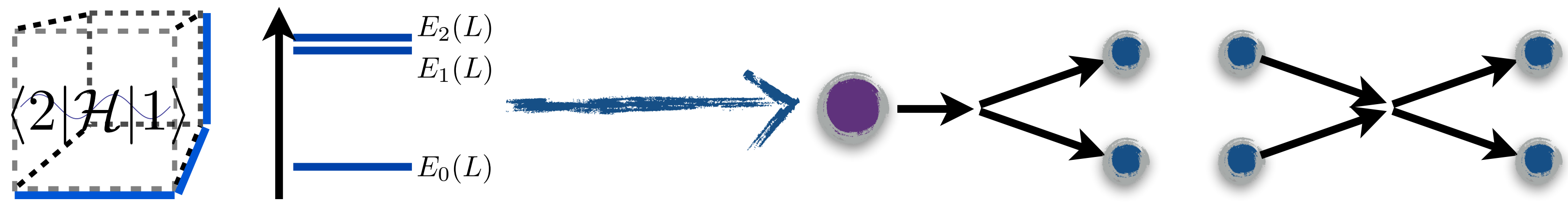

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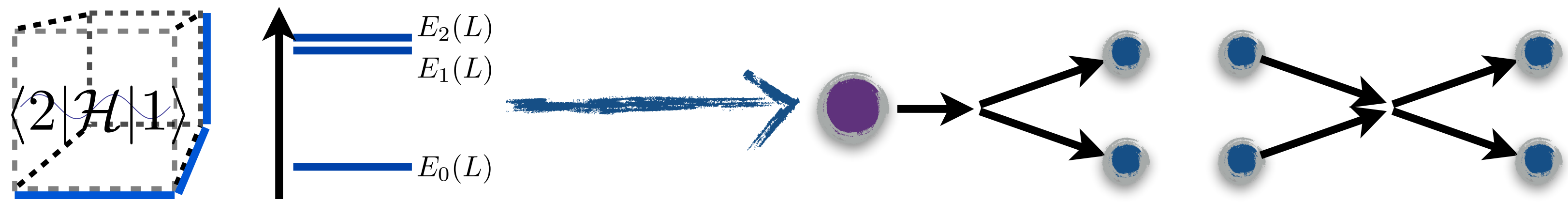

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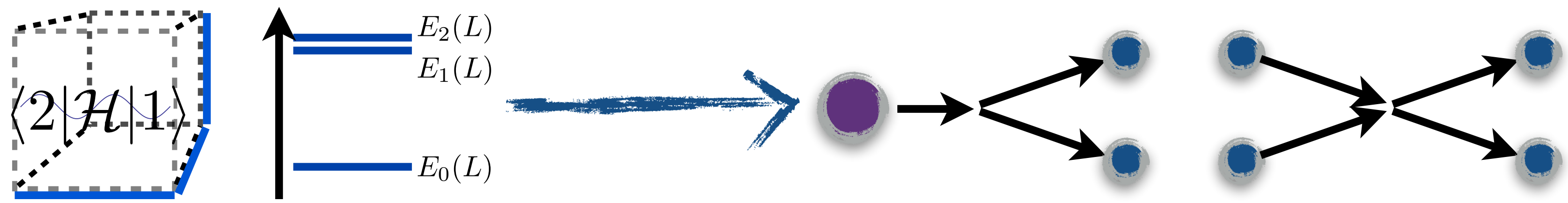

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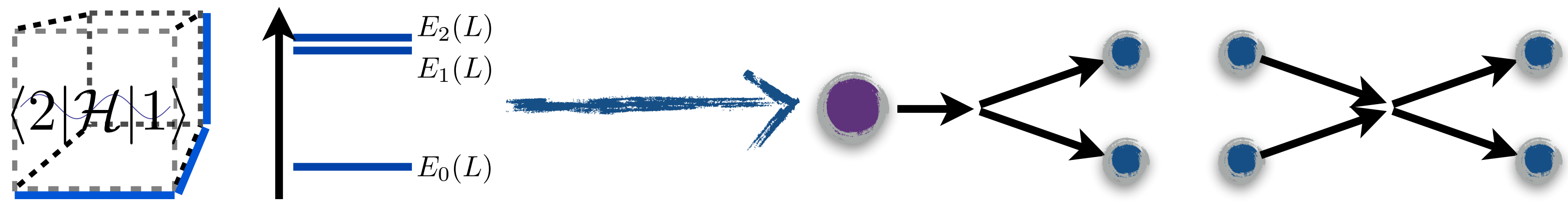

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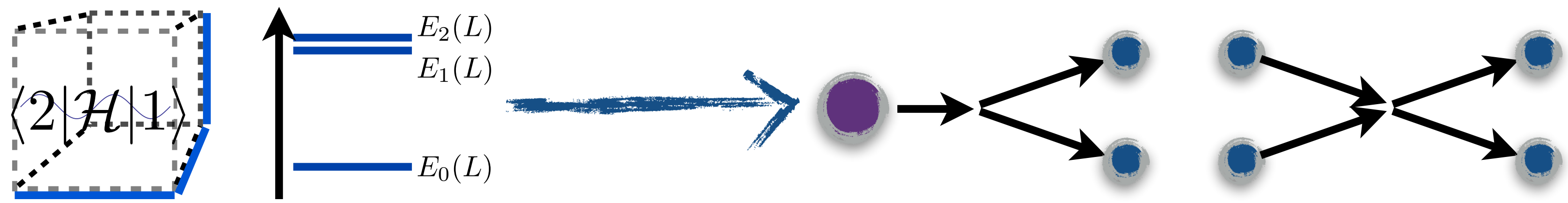

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One then aims to extract

$$\rho(E) = \lim_{\Delta\to 0}\, \lim_{L\to\infty}\, \widehat{\rho}\,(E, L, \Delta)$$

Order of limits is important

Backus-Gilbert for total rates $C(\tau, L) \to \widehat{\rho}(E, L, \Delta) \to \rho(E)$

One can construct $C(\tau, L)$ such that

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Here I do not explain the algorithm but only summarize key points:

(1). **Developed by geophysicists Backus and Gilbert to study seismic activity**

(2). **Technique to solve the inverse problem:** $G(\tau, L) = \int_0^\infty \frac{d\omega}{2\pi} e^{-\omega\tau}\rho(\omega, L)$

(3). **Gives a smoothened version of $\rho(\omega, L)$ with characteristic width Δ**

(4). **Preliminary evidence shows reasonable values of Δ and L could give a good estimate of the infinite-volume, zero-width limit**

Summary and Conclusions

Relation between finite-volume matrix elements and transition amplitudes is well understood for two-particle states (and three particle states are on the way)

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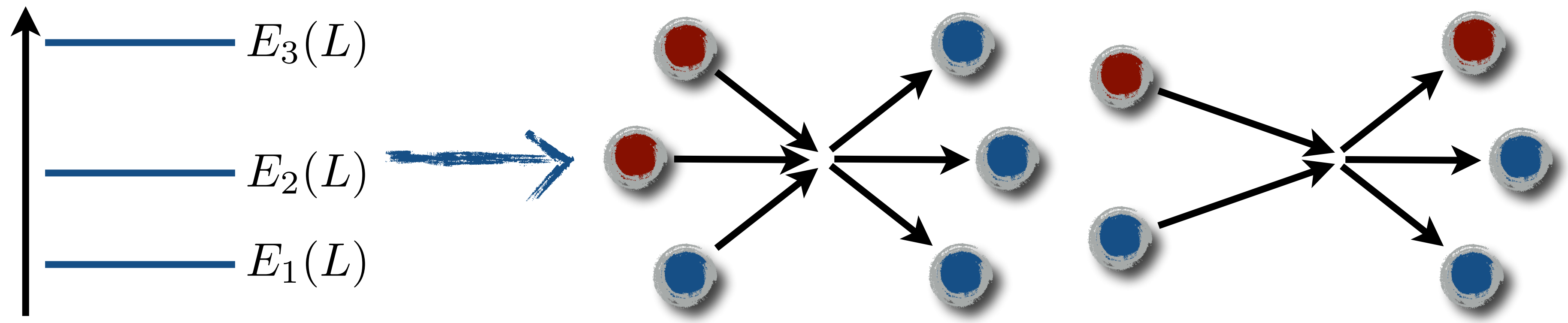
Stay tuned for a new approach that directly extracts inclusive transition rates

Our aim is to derive the generalization for arbitrary two- and three-particle systems



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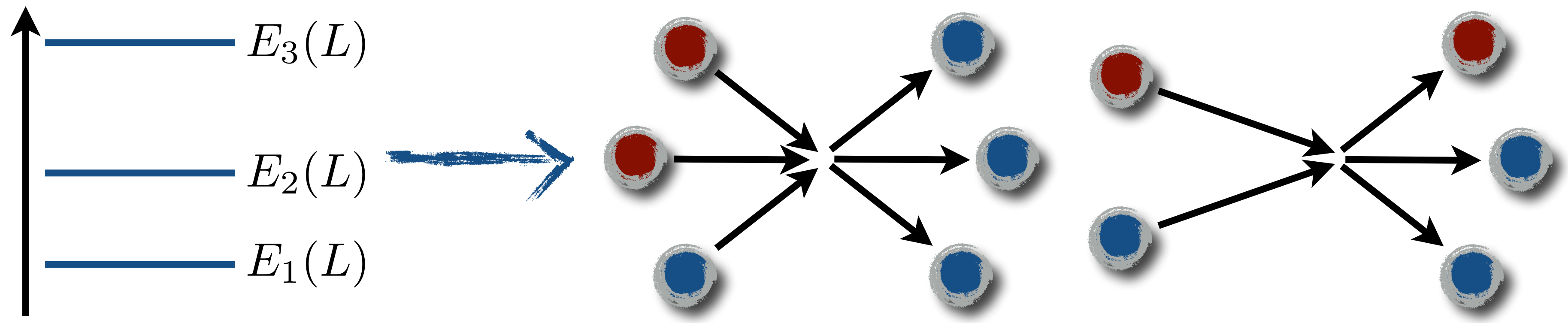
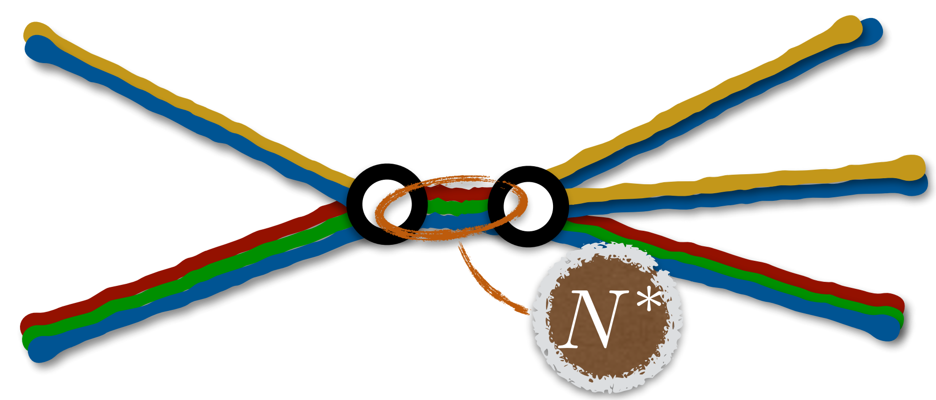


Potential applications…

Studying three-particle resonances

$$\omega(782) \to \pi\pi\pi$$

$$N(1440) \to N\pi, N\pi\pi$$

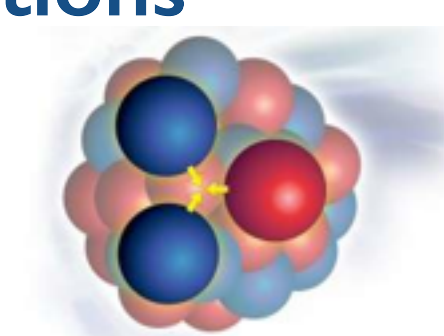


Calculating weak decay amplitudes and form factors

$$K \to \pi\pi\pi$$

Determining three-body interactions

NNN three-body forces needed as EFT input for studying larger nuclei and nuclear matter



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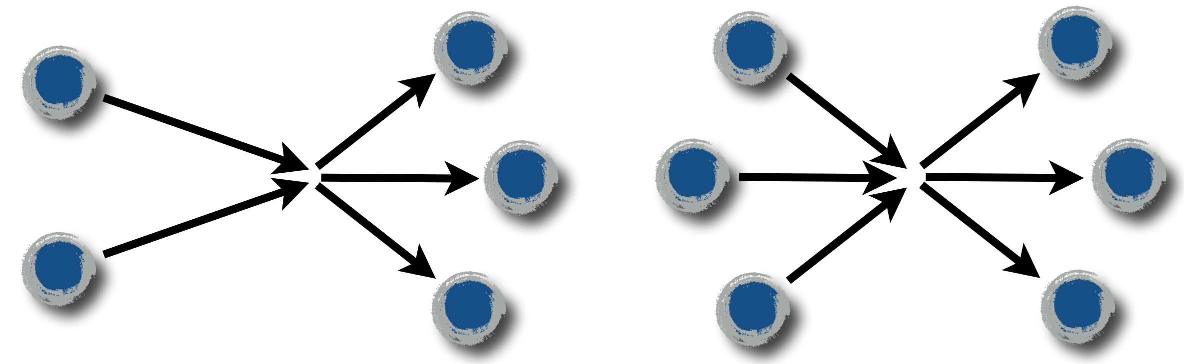

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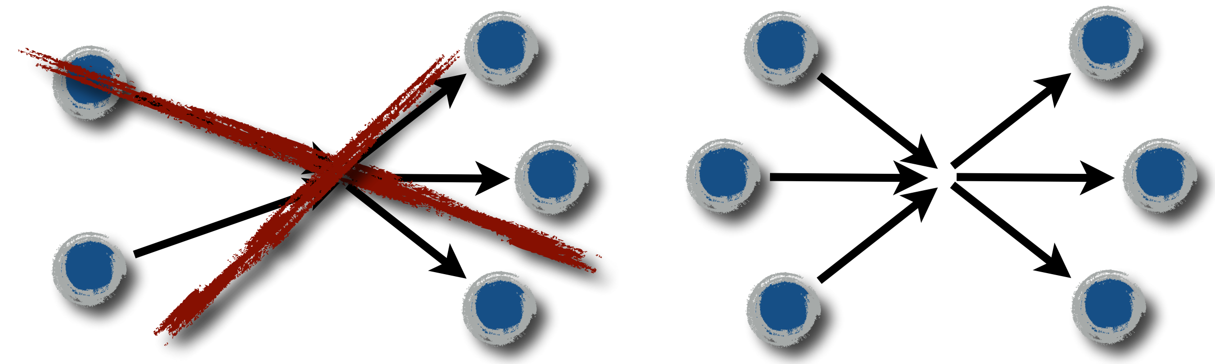

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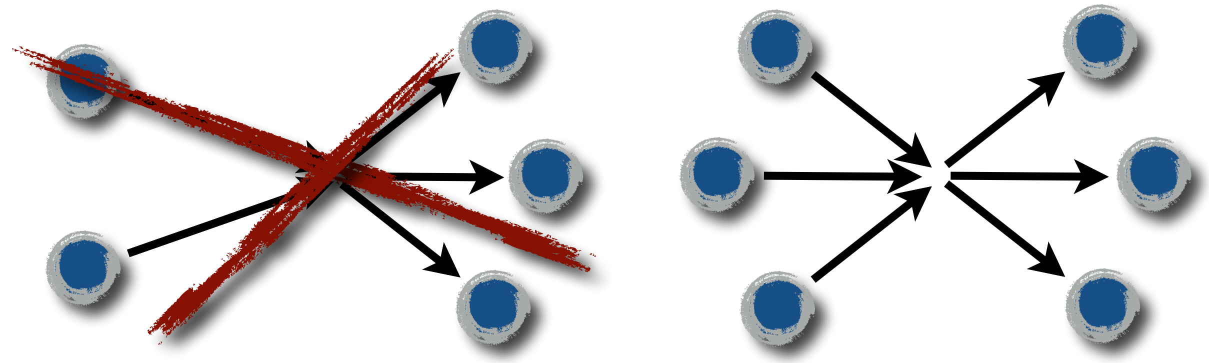

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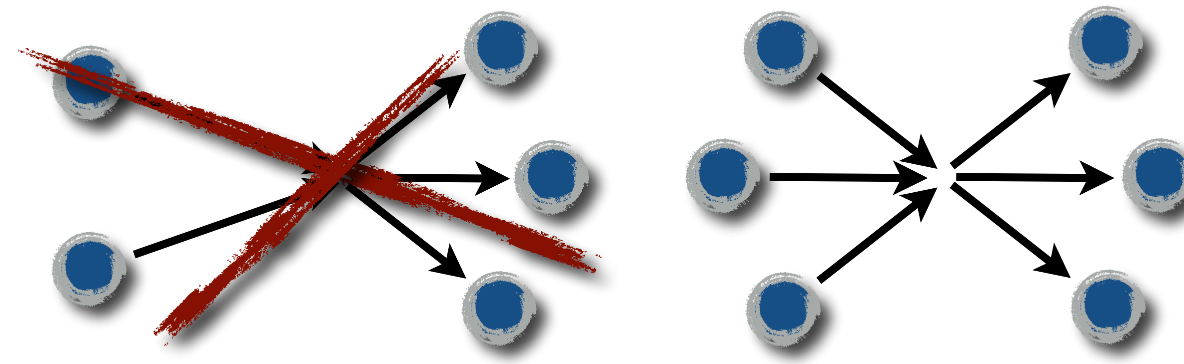

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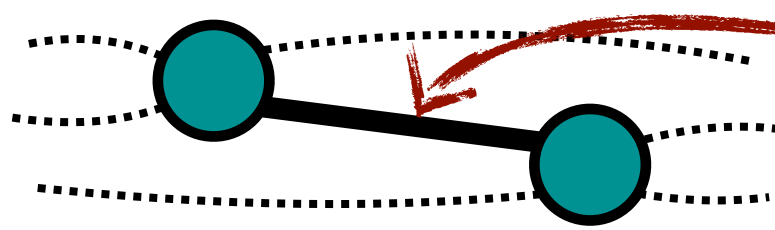

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Certain external momenta put this on-shell!

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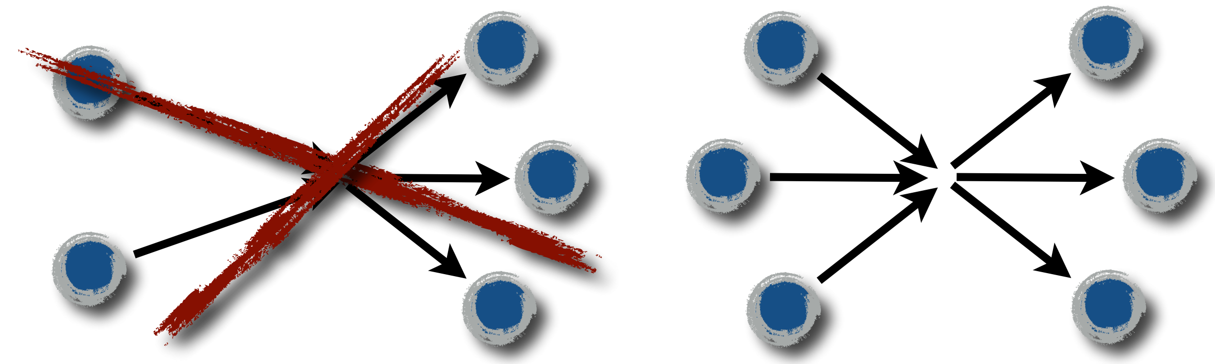

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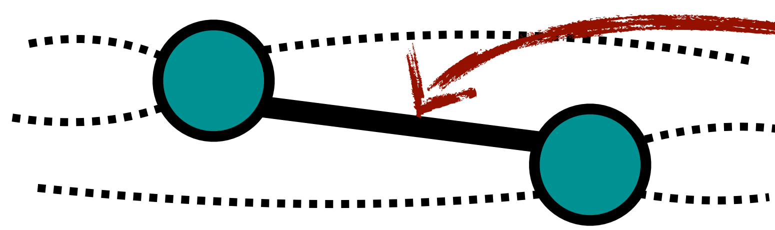
$$=$$  $+\cdots$

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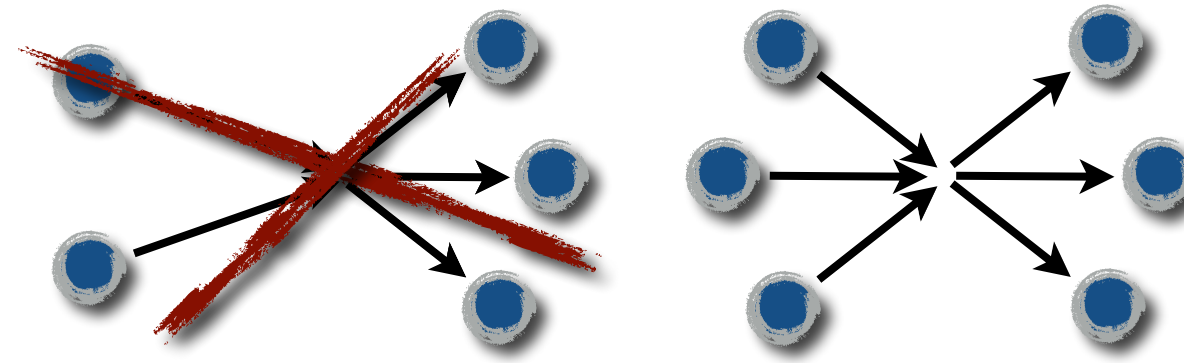

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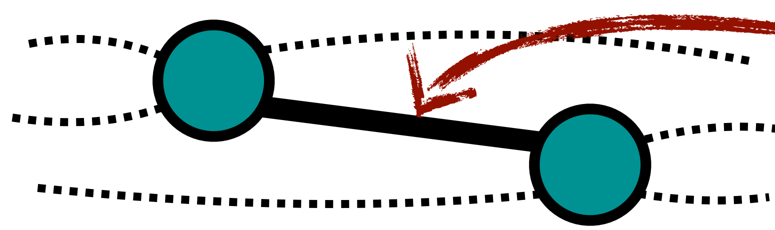
$$=$$  $$+\cdots$$

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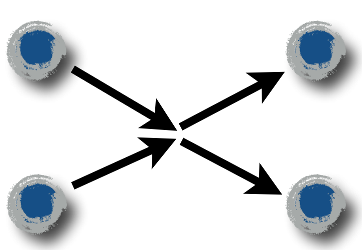

12 momentum components

-10 Poincaré generators

2 degrees of freedom

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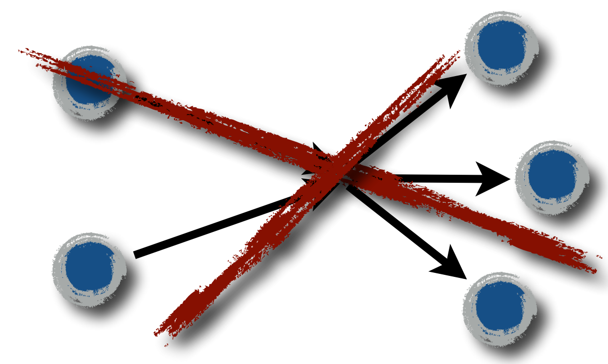

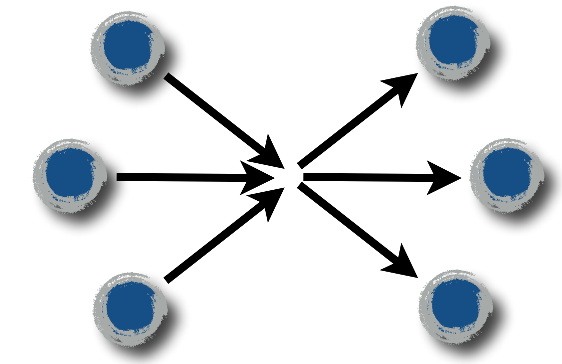

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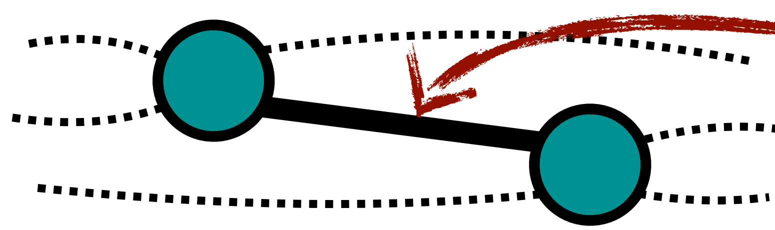
$=$  $+\cdots$

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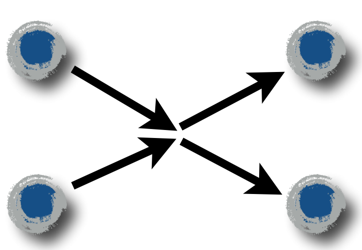

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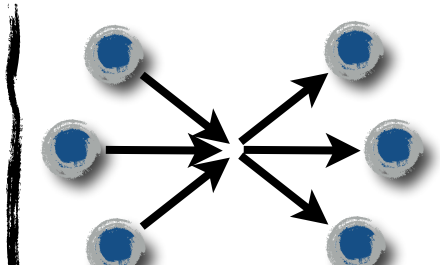

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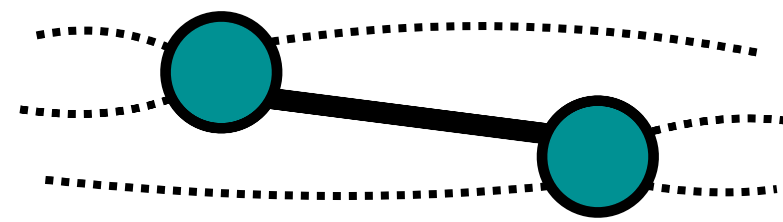

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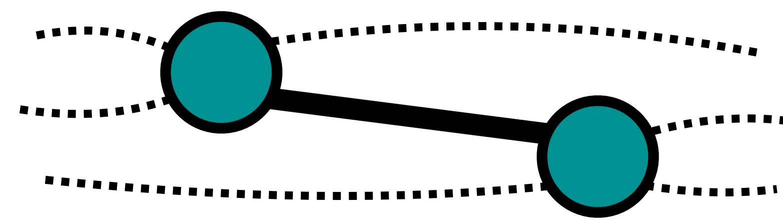

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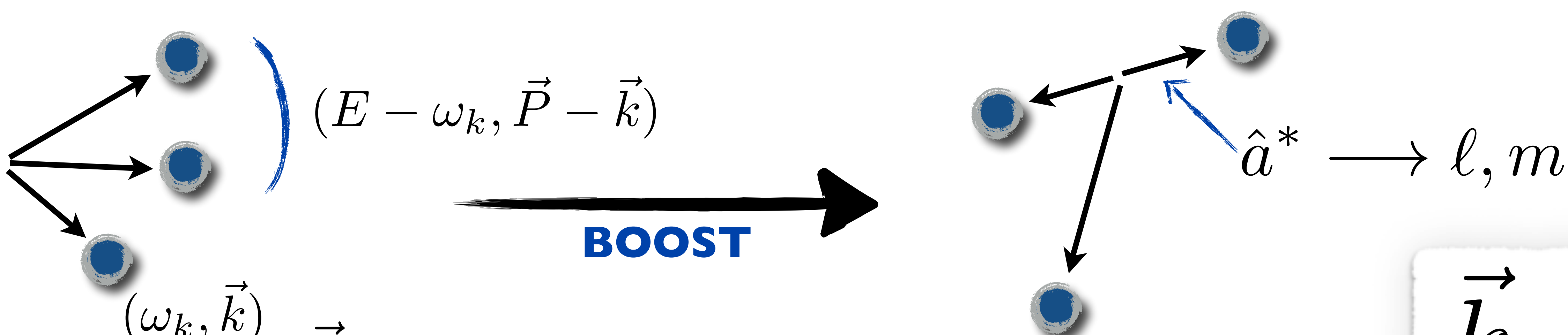

($\vec{k}$ is restricted to finite-volume momenta)

$$\boxed{\vec{k}, \ell, m}$$

Three-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to $\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$

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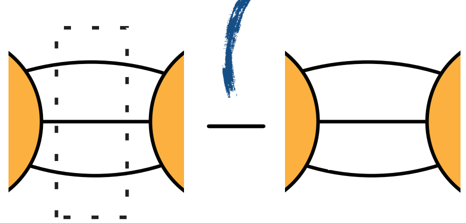

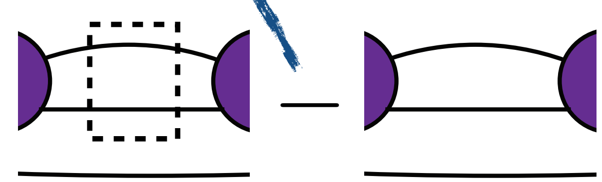

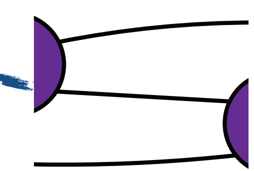

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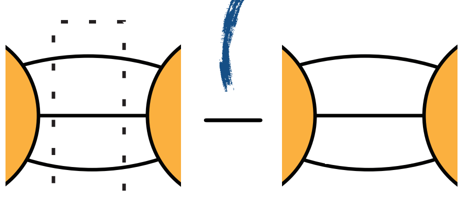

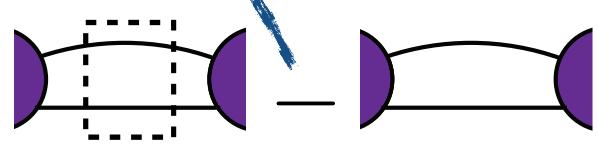

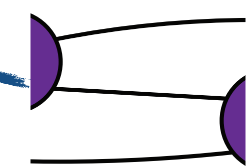

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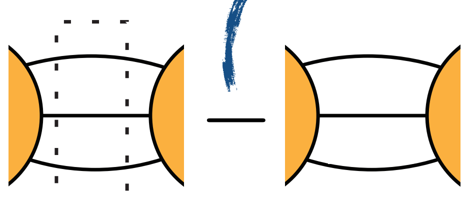

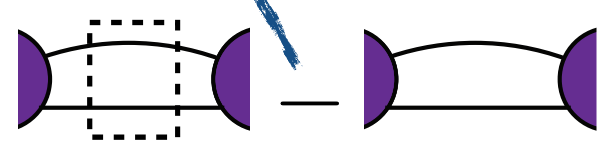

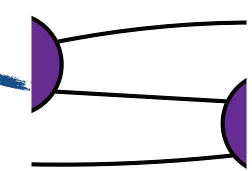

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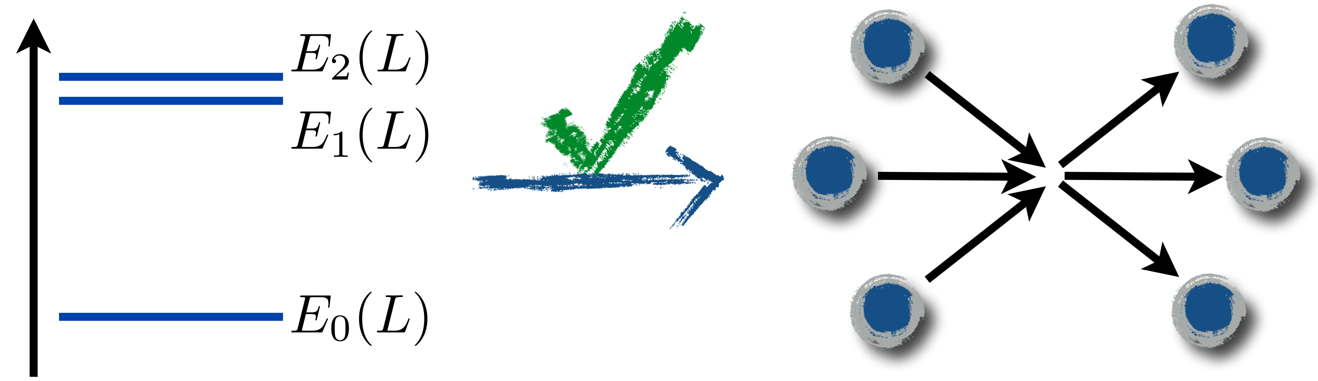
This completes the formal story and confirms that the **three-particle spectrum** is determined by **physical scattering amplitudes**

Three-to-three scattering



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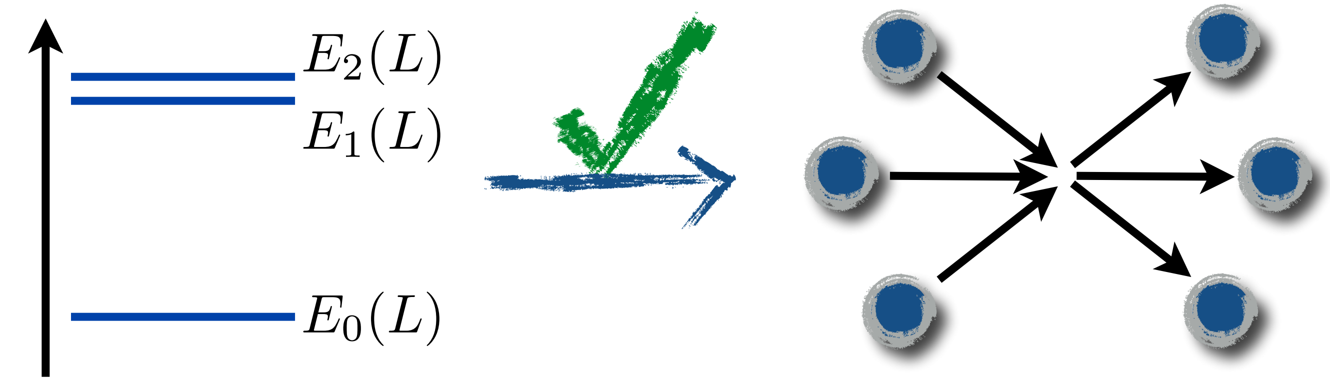

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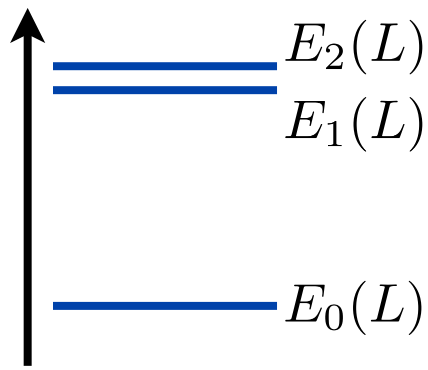

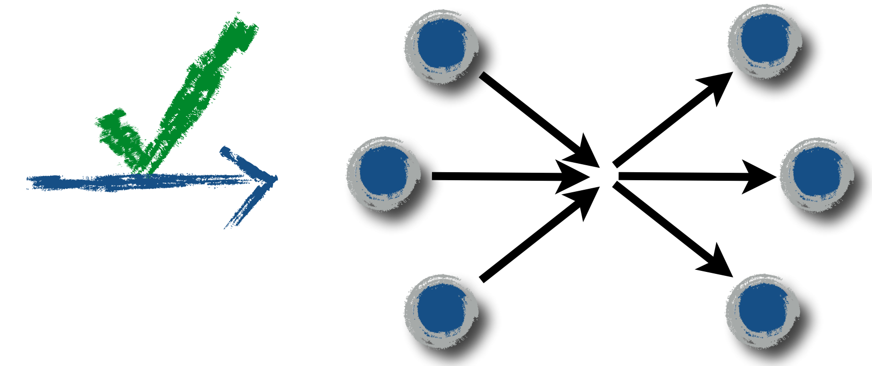

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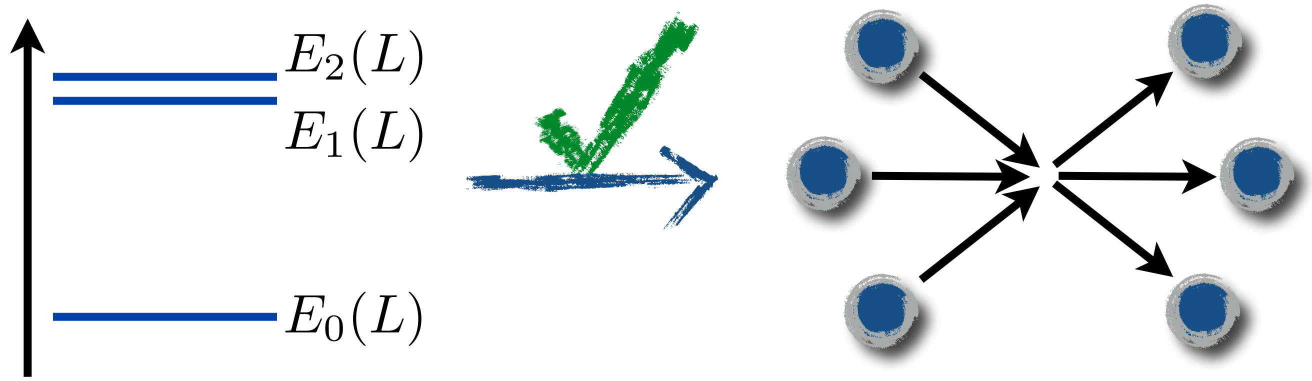

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Reproduced and generalized earlier work based in non-relativistic quantum mechanics

K. Huang and C. Yang, *Phys. Rev.* 105 (1957) 767-775
Beane, Detmold, Savage, *Phys. Rev.* D76 (2007) 074507

Meißner, Rìos and Rusetsky,
Phys. Rev. Lett. 114, 091602 (2015)

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$$c = -96.351\cdots$$

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We reproduce the exponent, leading power and overall constant using our relativistic formalism

Reproducing the result…

1. **Show that the relativistic quantization predicts (at leading order in 1/L)**

$$\Delta E(L) = -\frac{1}{2E_B}\left[\frac{1}{L^3}\sum_{\vec{k}} - \int_{\vec{k}}\right]\frac{\overline{\Gamma}^{(u)}(k)\Gamma^{(u)}(k)}{2\omega_k \mathcal{M}_2(k)}$$

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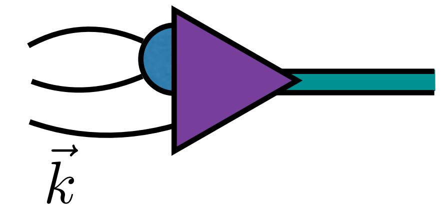

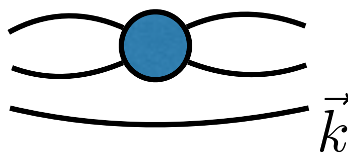

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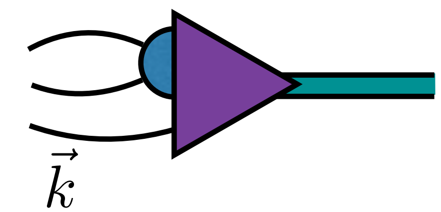
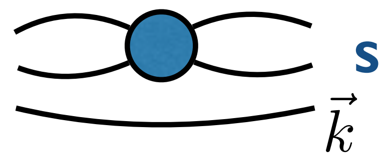




s-wave scattering amplitude

2. Derive the functional forms of the infinite-volume quantities

$$\Gamma^{(u)}(k) = \frac{3^{3/8}\pi^{1/4}}{4} A\sqrt{-c}\,\mathcal{M}_2(k) \qquad \mathcal{M}_2(k) = \frac{32\pi m}{\kappa}\left[1 + \frac{3k^2}{4\kappa^2}\right]^{-1/2}$$

follows from matching to Effimov wavefunction

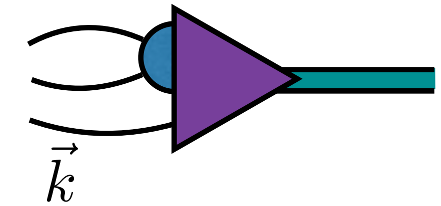
unitary amplitude with spectator "stealing" some momentum

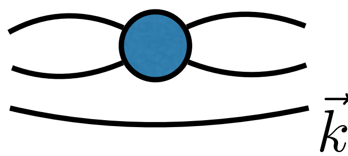
Reproducing the result…

1. **Show that the relativistic quantization predicts (at leading order in 1/L)**

$$\Delta E(L) = -\frac{1}{2E_B}\left[\frac{1}{L^3}\sum_{\vec{k}} - \int_{\vec{k}}\right]\frac{\overline{\Gamma}^{(u)}(k)\Gamma^{(u)}(k)}{2\omega_k \mathcal{M}_2(k)}$$

usymmetrized residue factor





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unitary amplitude with spectator "stealing" some momentum

3. **Evaluate the sum-integral difference with Poisson summation**

$$\Delta E(L) = c|A|^2\frac{3^{3/4}\pi^{3/2}}{3\kappa}6\int_{\vec{k}} e^{iL\hat{x}\cdot\vec{k}}\frac{1}{2\omega_k}\left[1+\frac{3k^2}{4\kappa^2}\right]^{-1/2}$$

$$= c|A|^2\frac{\kappa^2}{m}\frac{1}{(\kappa L)^{3/2}}e^{-2\kappa L/\sqrt{3}} + \cdots$$