The Surprising Simplicity of Scattering Amplitudes

Jacob Bourjaily
University of Copenhagen

Annual UK Theory Christmas Meeting
IPPP, Durham University

The Niels Bohr International Academy

VILLUM FONDEN

erc

Organization and Outline

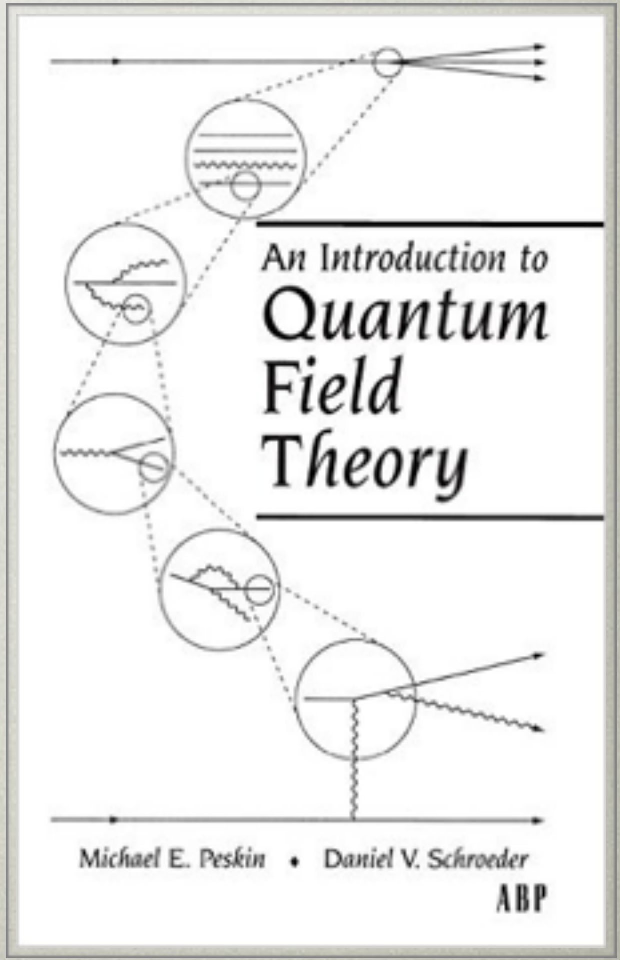
- ***Spiritus Movens*****:** the *surprising* simplicity of QFT
- **Basic Building Blocks: *on-shell functions***
  - tree amplitudes (and graphs of trees)
  - the Grassmannian duality (massless, 4d)
- **Constructing Loop Amplitude *Integrands***
  - on-shell, all-loop recursion relations
  - generalized and prescriptive unitarity
  - amplitude/correlator bootstrap
- **Loop Integration & Future Directions**

Surprising Simplicities of Quantum Field Theory

Traditional Description of QFT

- **Quantum Field Theory**: the marriage of (special) *relativity* with *quantum mechanics*
- Theories (can be) specified by Lagrangians—or equivalently, by Feynman rules for virtual particles

$$\mathcal{L} \equiv -\frac{1}{4}\sum_i (F^a_{i\mu\nu})^2 + \sum_J \overline{\psi}_J (i\not{D})\psi_J$$



- Predicted probability (*amplitudes*) from path integrals (over virtual 'histories'):

$$\int \mathcal{D}A\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\;e^{i\int d^4x\mathcal{L}}$$

Perturbation Theory and Loops

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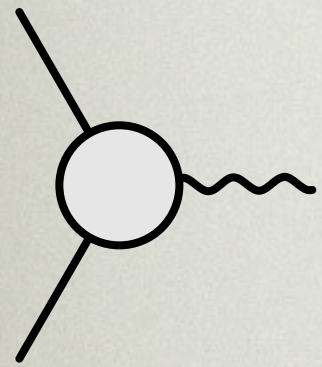

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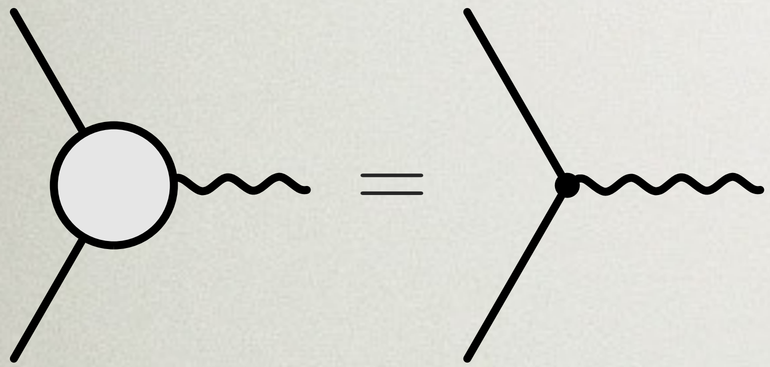

[Dirac (1933)]

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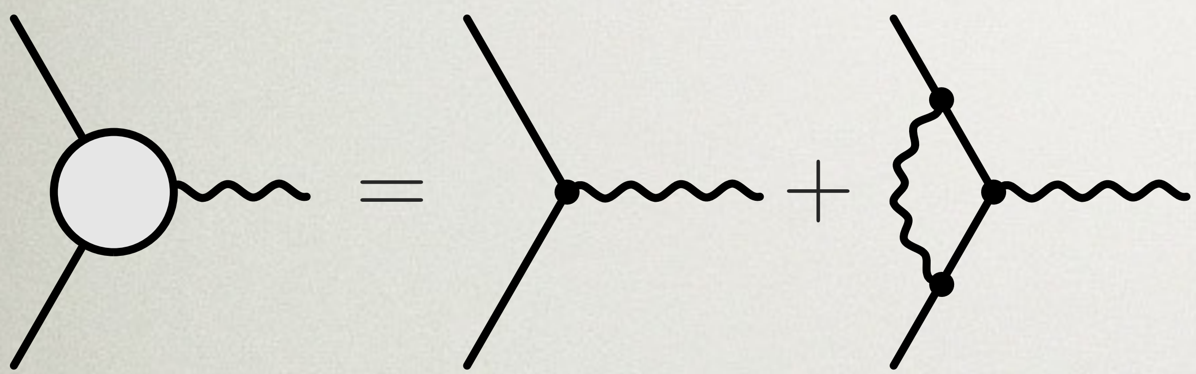

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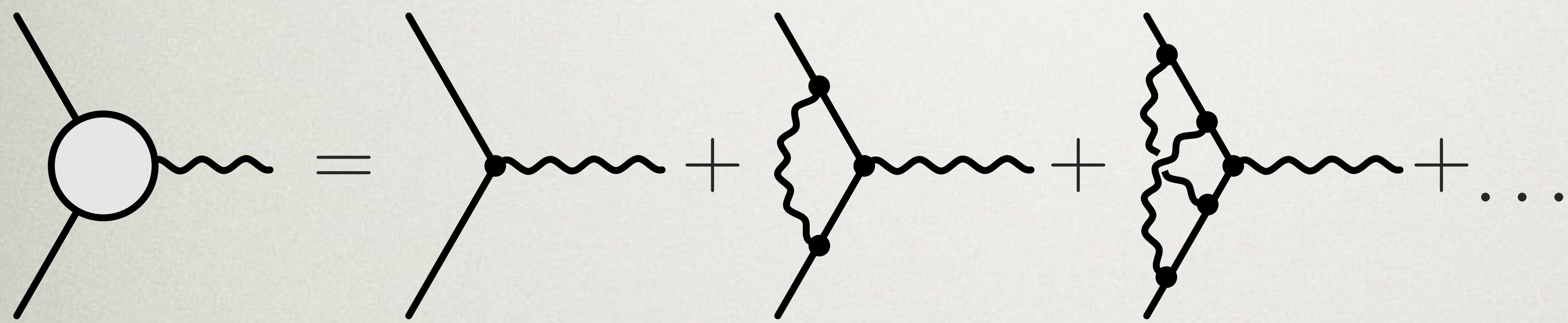

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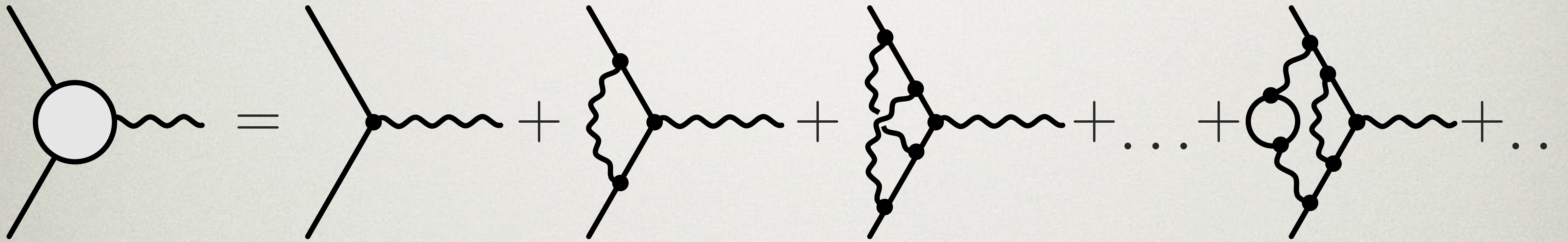

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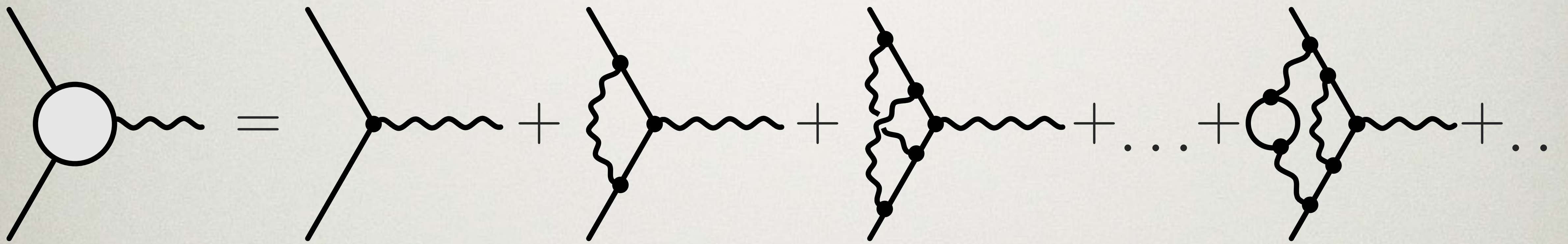
$\alpha \approx 1/137.036$



$$g_e^{\text{thy}} = 2 + \frac{\alpha}{\pi}(1) + \frac{\alpha^2}{\pi^2}\left(\frac{3}{2}\zeta_3 - \pi^2\log(2) + \zeta_2 + \frac{197}{72}\right) + \ldots$$

$$= 2.00231930435801\ldots$$

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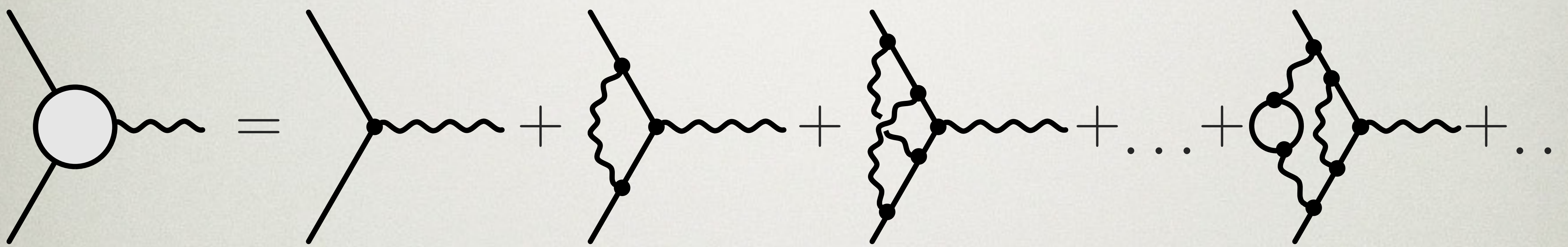

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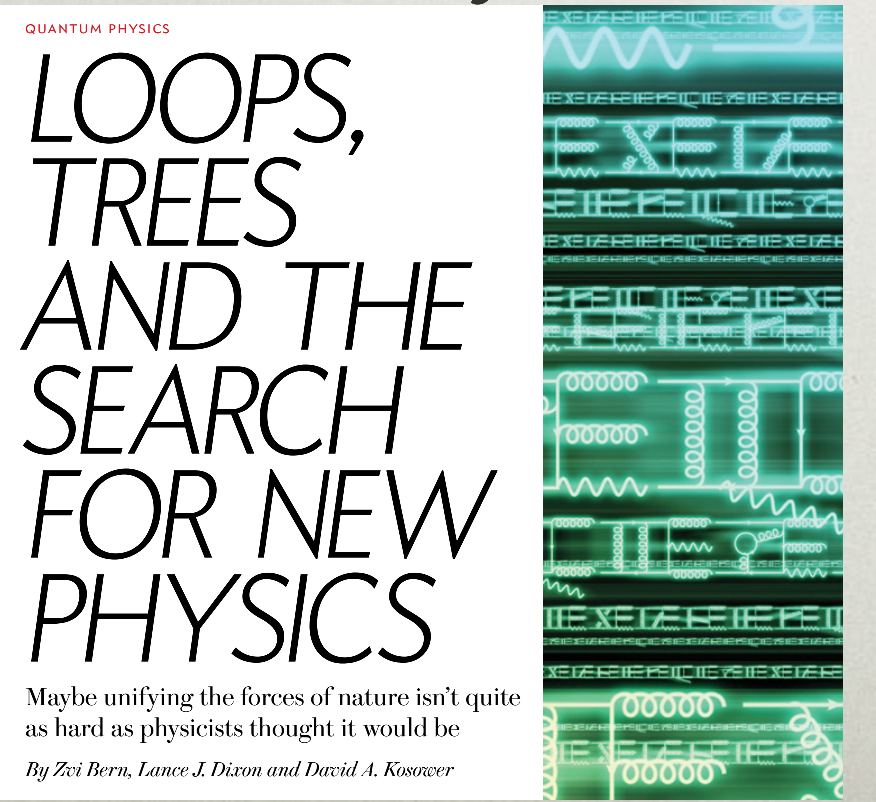
- *the most precisely tested idea in all of science!*

Explosions of Complexity

- While ultimately correct, the Feynman expansion renders *all but the **simplest*** predictions—

  those involving the **fewest particles**, at the **lowest orders of perturbation**—

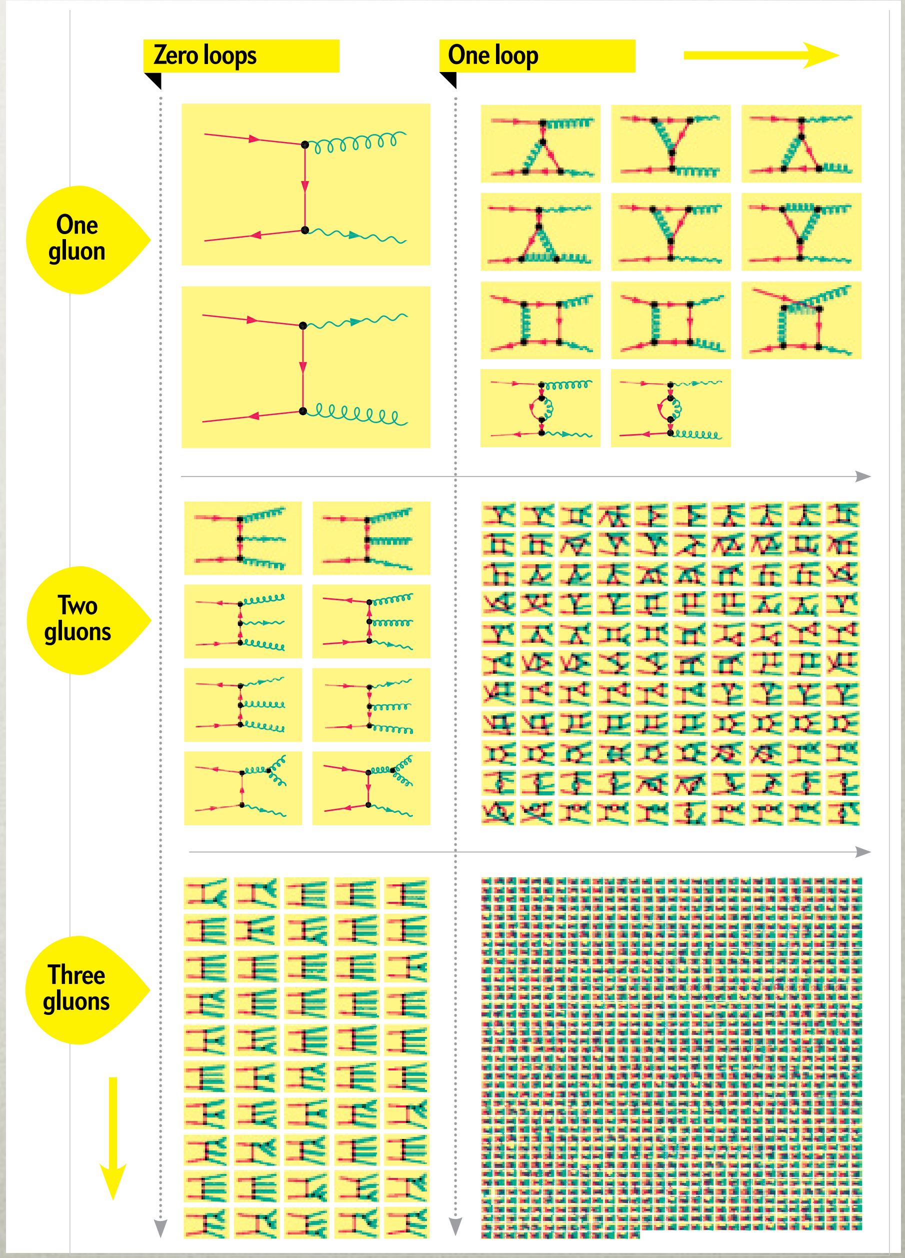
  computationally ***intractable*** or theoretically ***inscrutable***


QUANTUM PHYSICS

LOOPS, TREES AND THE SEARCH FOR NEW PHYSICS

Maybe unifying the forces of nature isn't quite as hard as physicists thought it would be

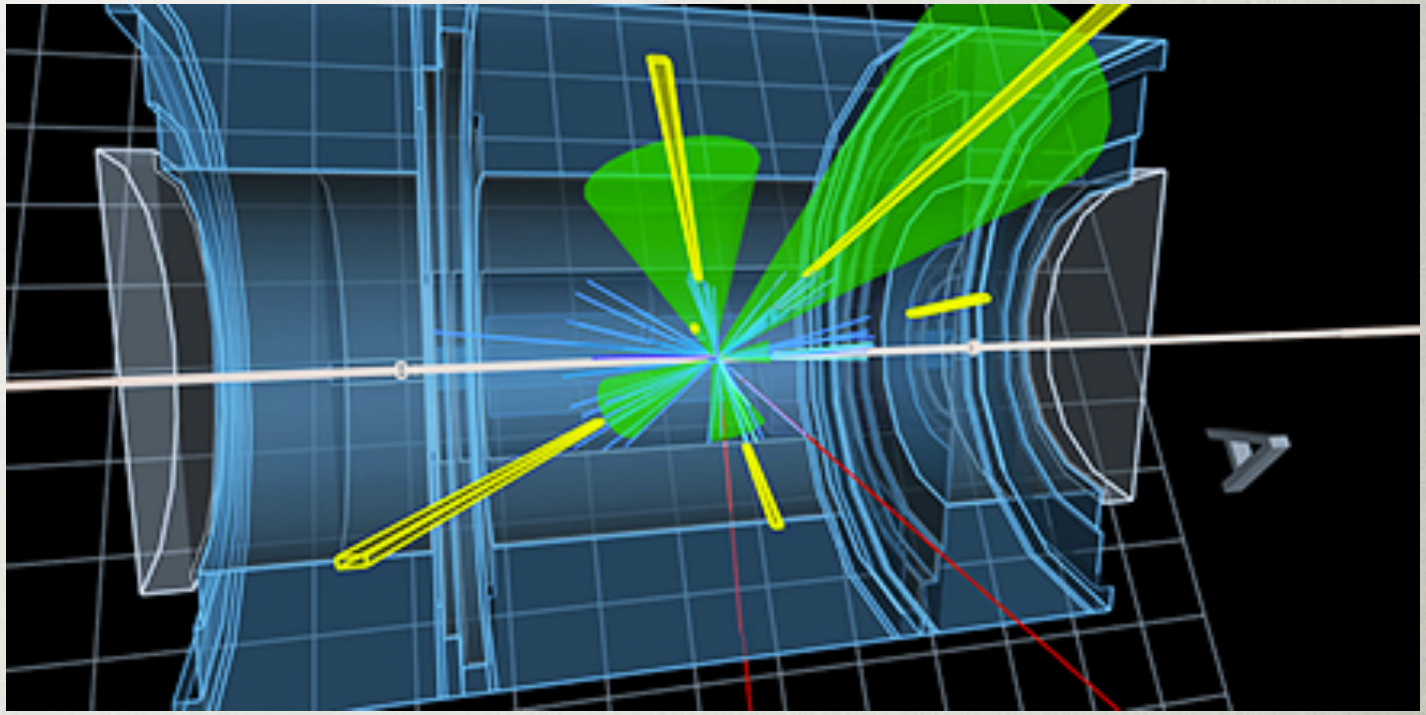
By Zvi Bern, Lance J. Dixon and David A. Kosower



[Bern, Dixon, Kosower, *Scientific American* (2012)]

Needs (Once) Beyond Our Reach

- Background amplitudes **crucial** for *e.g.* colliders



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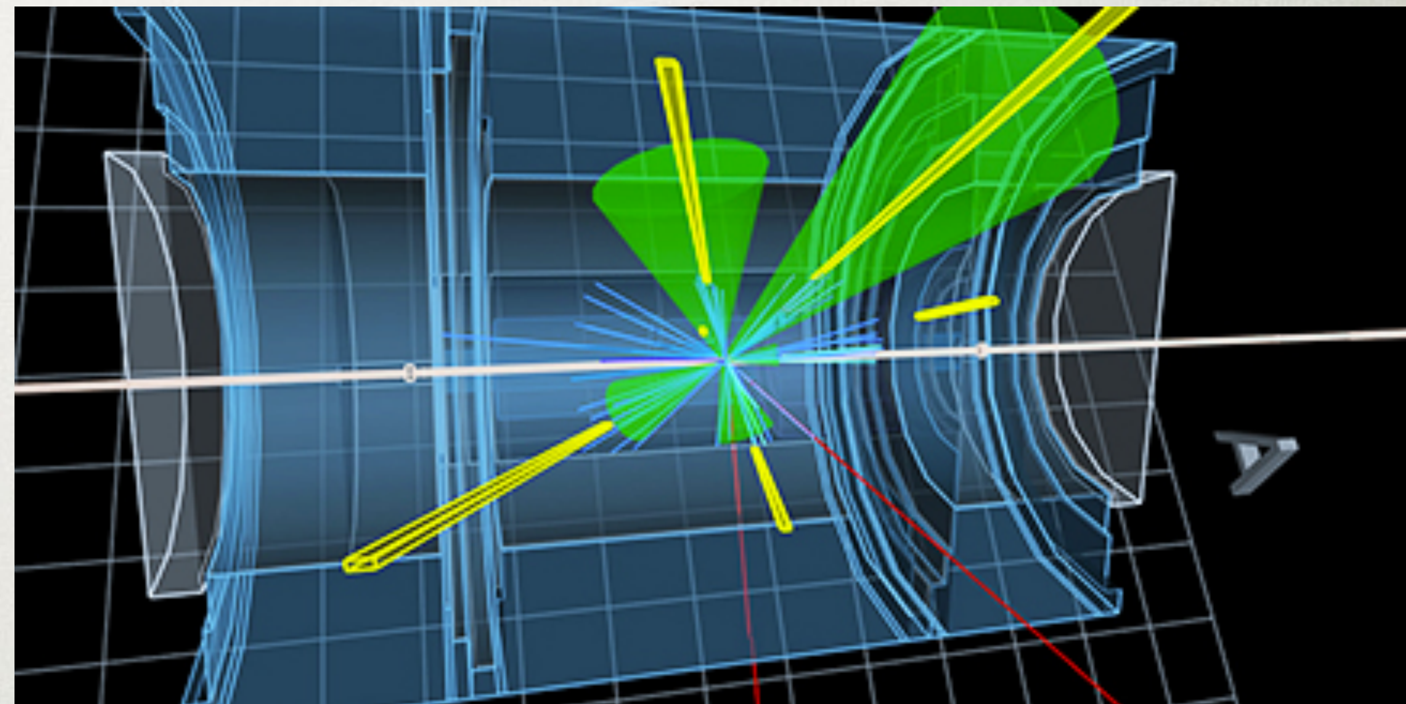

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Supercollider physics [*Rev.Mod.Phys*. **56** (1984)]

E. Eichten

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

I. Hinchliffe

Lawrence Berkeley Laboratory, Berkeley, California 94720

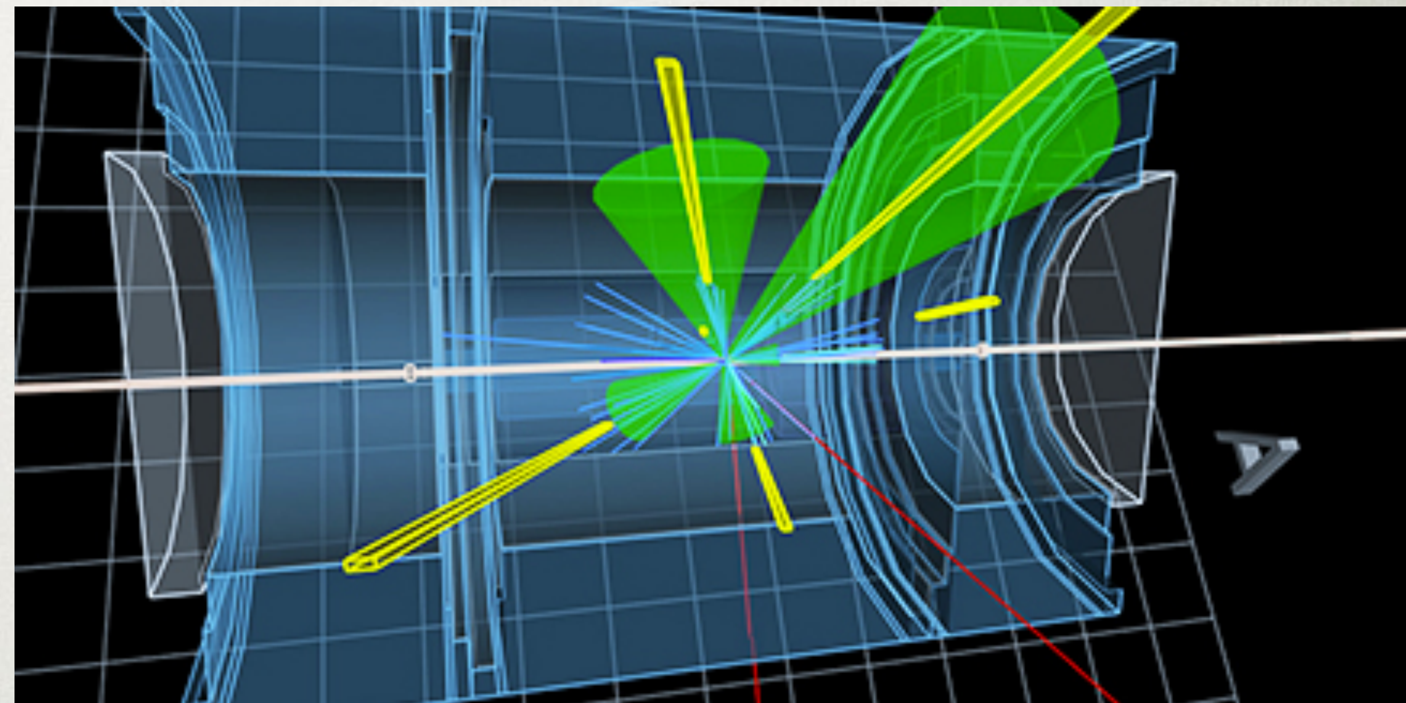
K. Lane

The Ohio State University, Columbus, Ohio 43210

C. Quigg

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

Eichten *et al.* summarize the motivation for exploring the 1-TeV ($=10^{12}$ eV) energy scale in elementary particle interactions and explore the capabilities of proton-(anti)proton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.



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For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two$\rightarrow$four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

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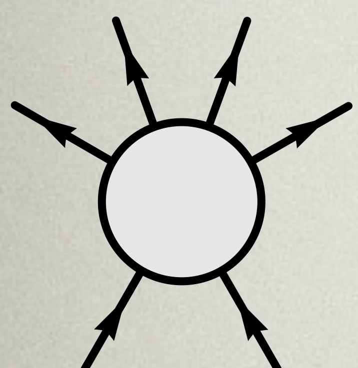

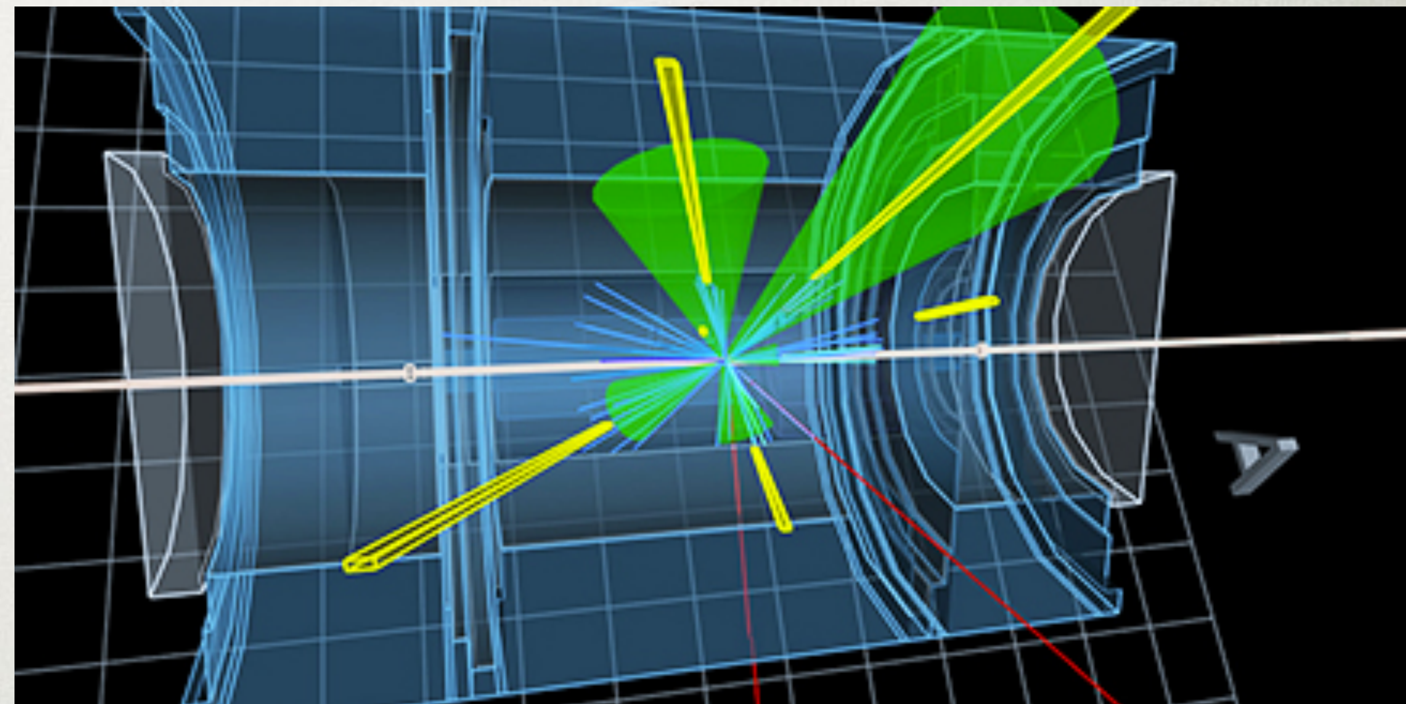

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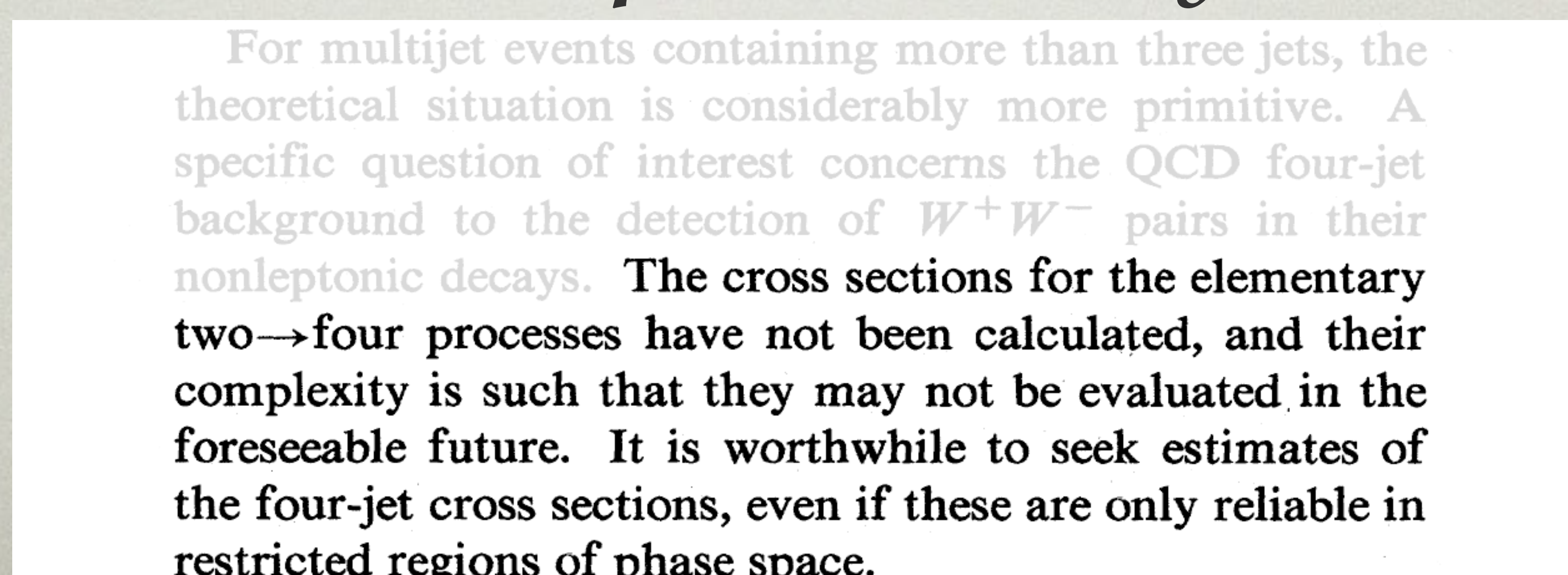

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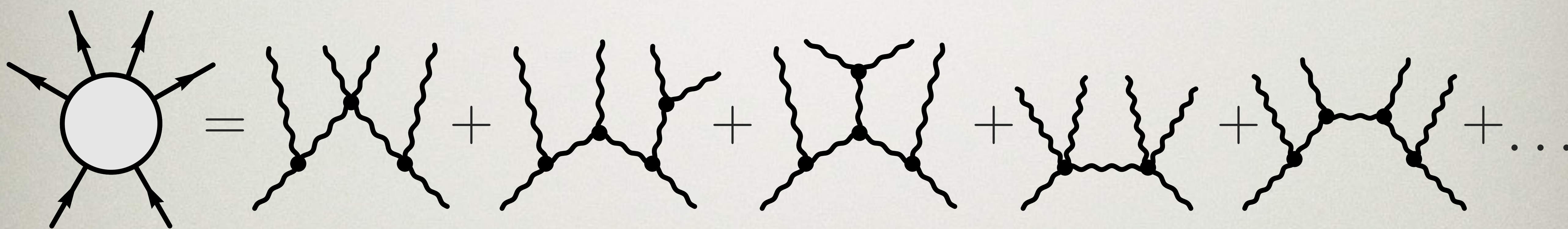

220 diagrams—thousands of terms

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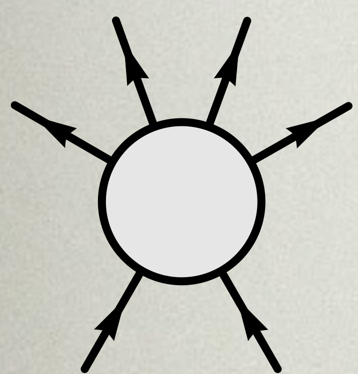

THE CROSS SECTION FOR FOUR-GLUON PRODUCTION BY GLUON–GLUON FUSION

Stephen J. PARKE and T.R. TAYLOR

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510 USA

Received 13 September 1985

The cross section for two-gluon to four-gluon scattering is given in a form suitable for fast numerical calculations.

[*Nucl.Phys.* **B269** (1985)]

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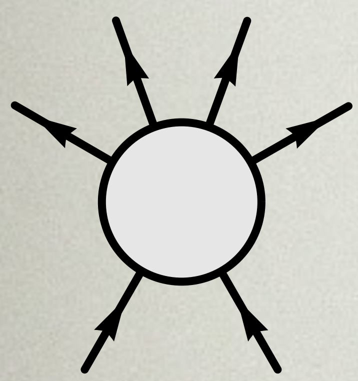

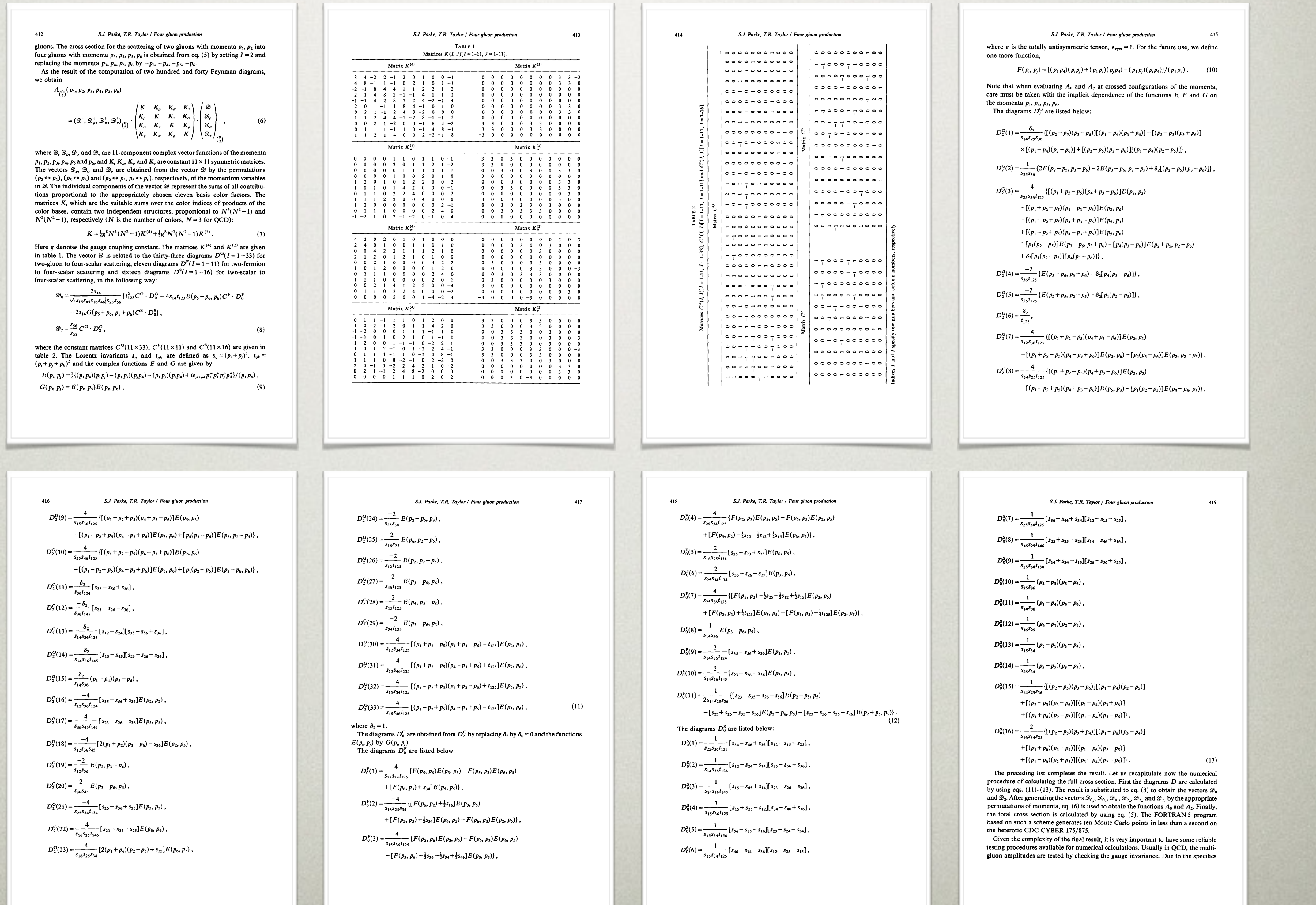

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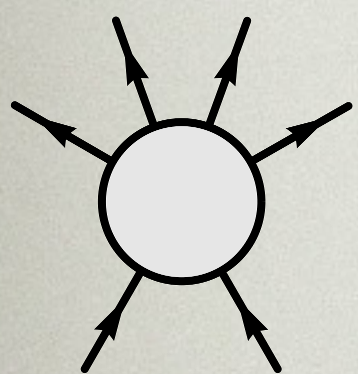

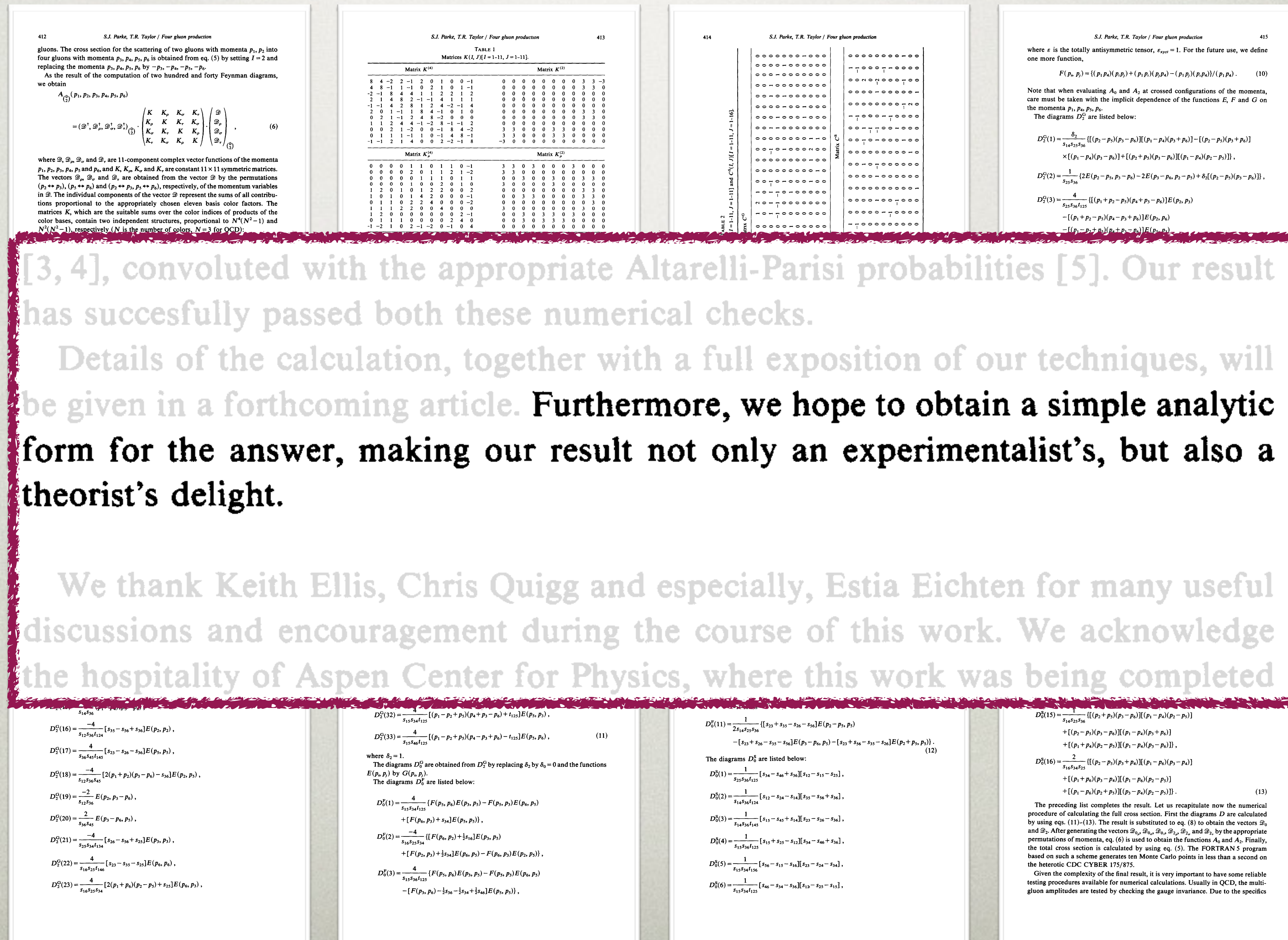
[3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. **Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.**

We thank Keith Ellis, Chris Quigg and especially, Estia Eichten for many useful discussions and encouragement during the course of this work. We acknowledge the hospitality of Aspen Center for Physics, where this work was being completed

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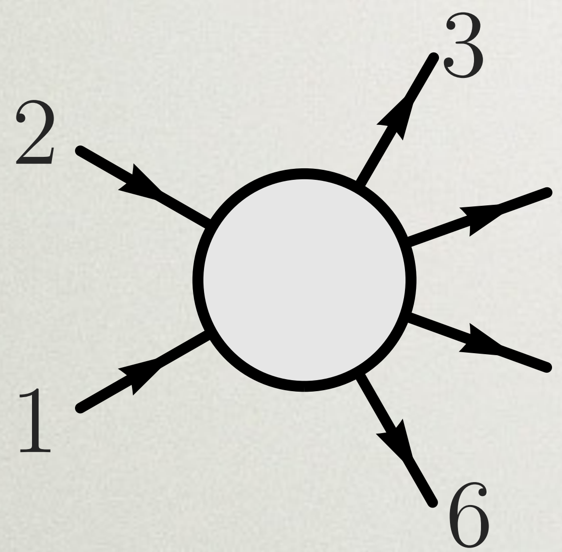

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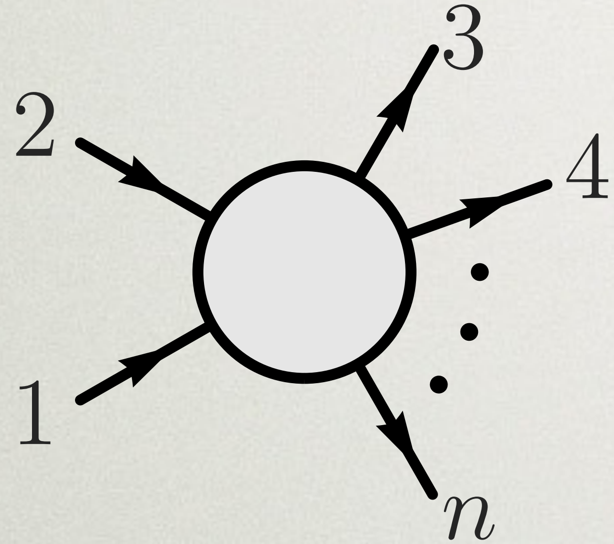

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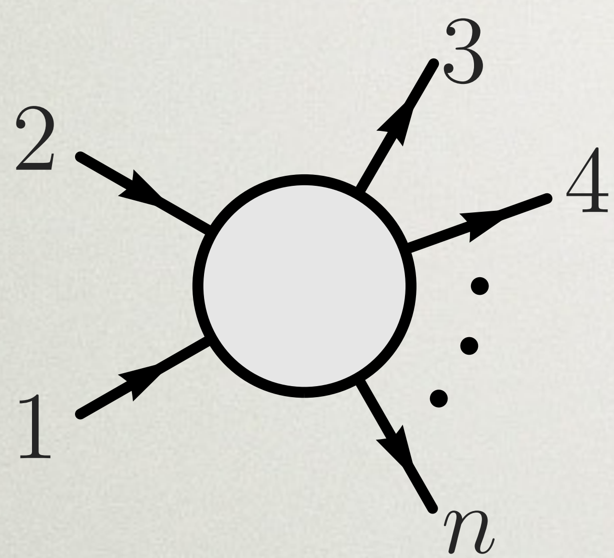

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Goal: make the simplicity of amplitudes **manifest** in the way we compute them, *dramatically* extending the reach of the predictions we can make for experiment

Simplex Sigillum Veri

To what extent can QFT be reformulated without reference to (simplicity-spoiling) redundancies?

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- **Loop Integration**: symbology, motives, bootstraps, symmetry-preserving regularization/evaluation, …

Basic Building Blocks: On-Shell Functions

Η απλότητα του σχεδίου κάνει εύκολη την κατασκευή

(the simplicity of the design makes it easy to build)

The Vernacular of the S-Matrix

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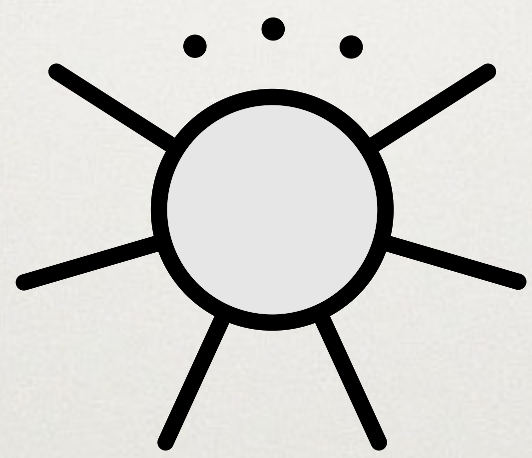

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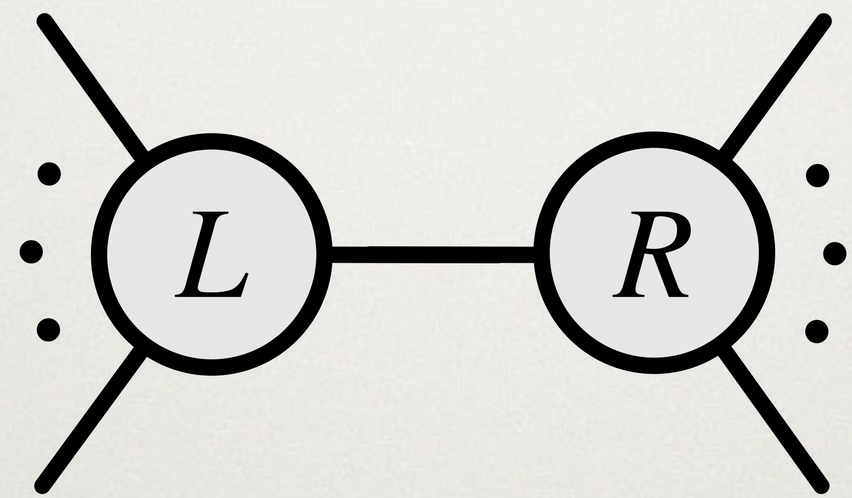

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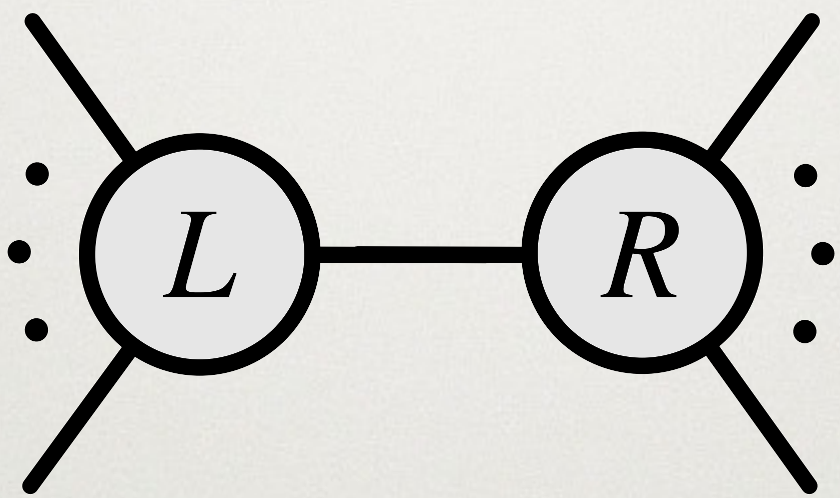

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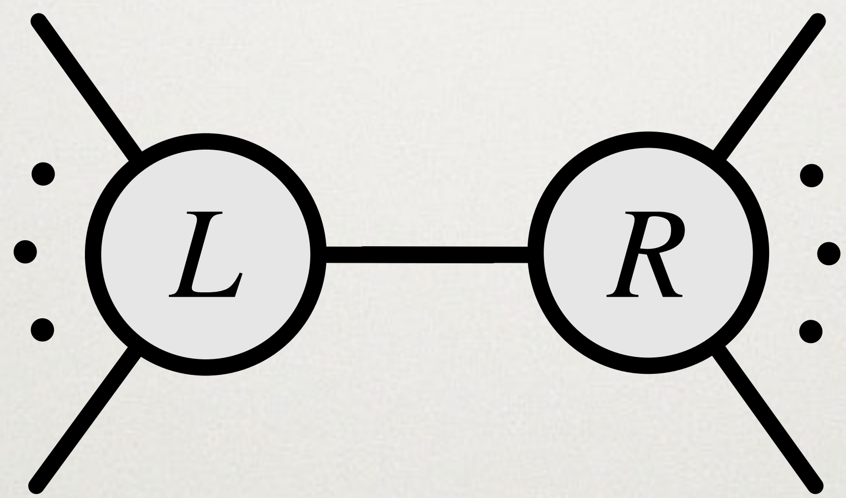

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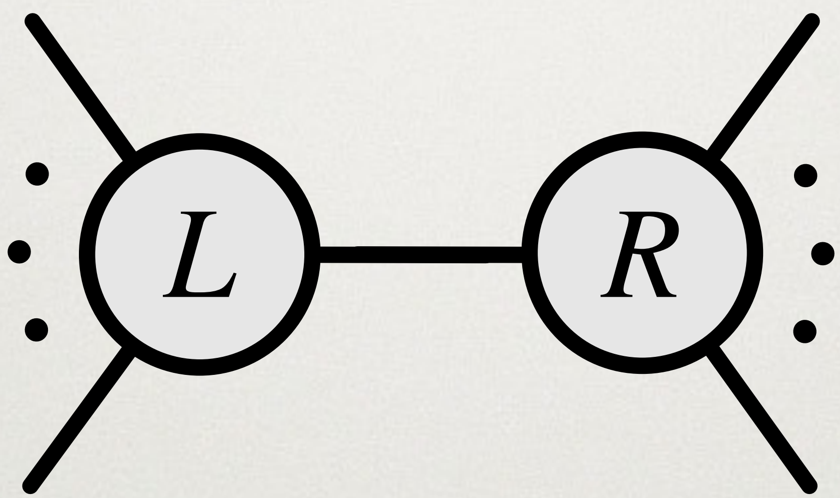

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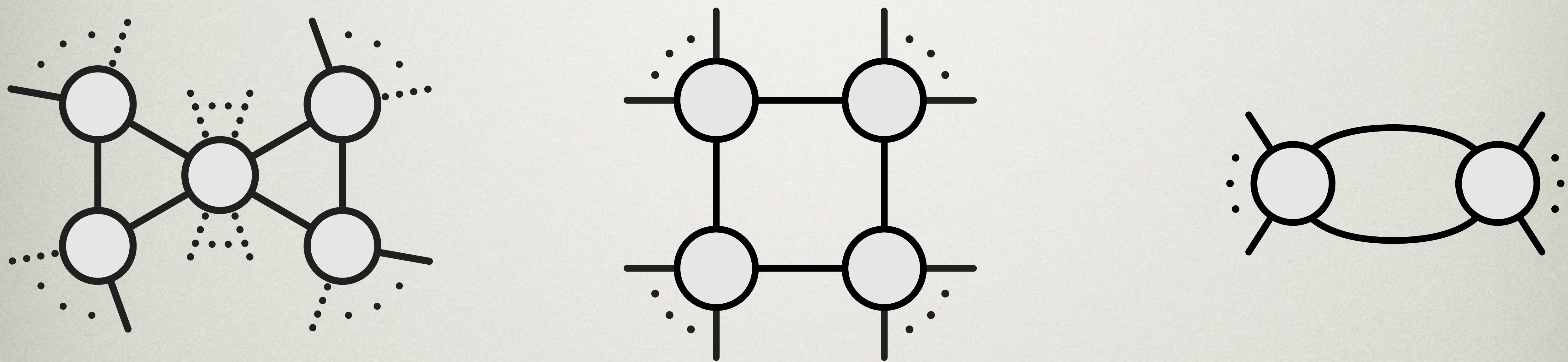

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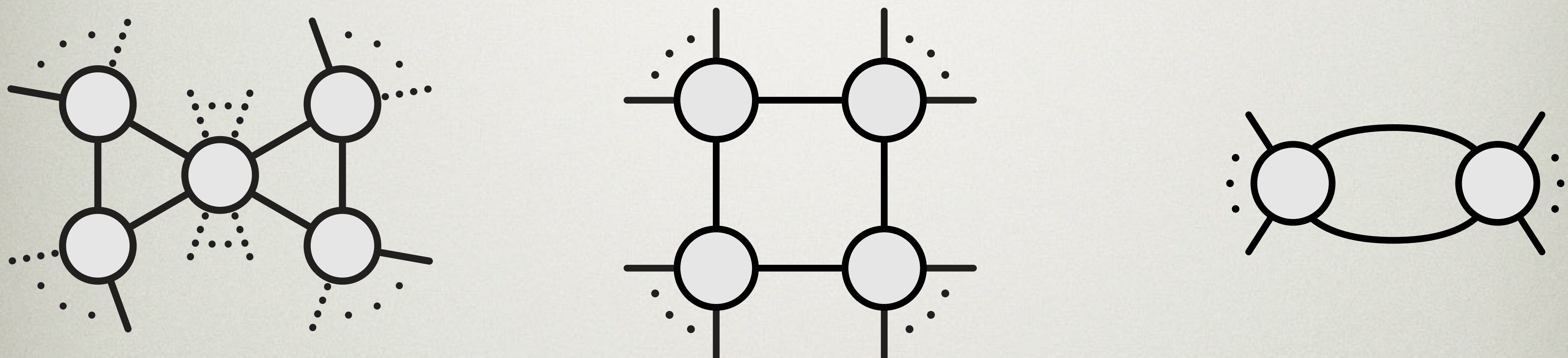

  - defined for all ***all*** quantum field theories—*exclusively* in terms of physical (observable) states
  - can be used to reconstruct ***all*** loop amplitudes

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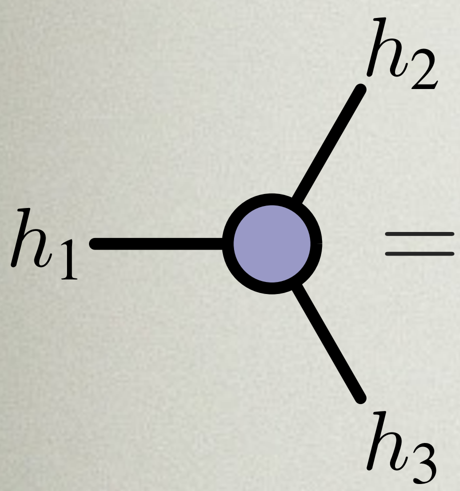
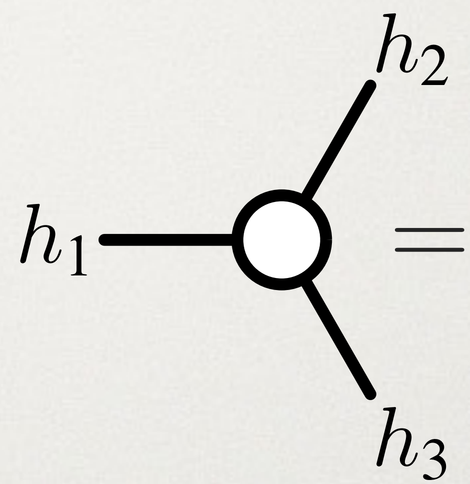
What's Special about Massless, 4d?

- Massless, 3-particle amplitudes in 4 dimensions:



$$= \langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1}$$

$$h_1 + h_2 + h_3 \leq 0$$



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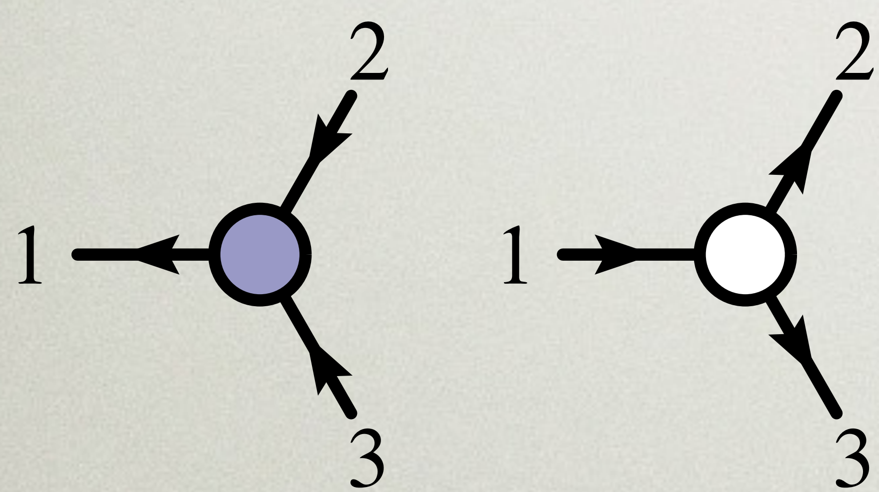
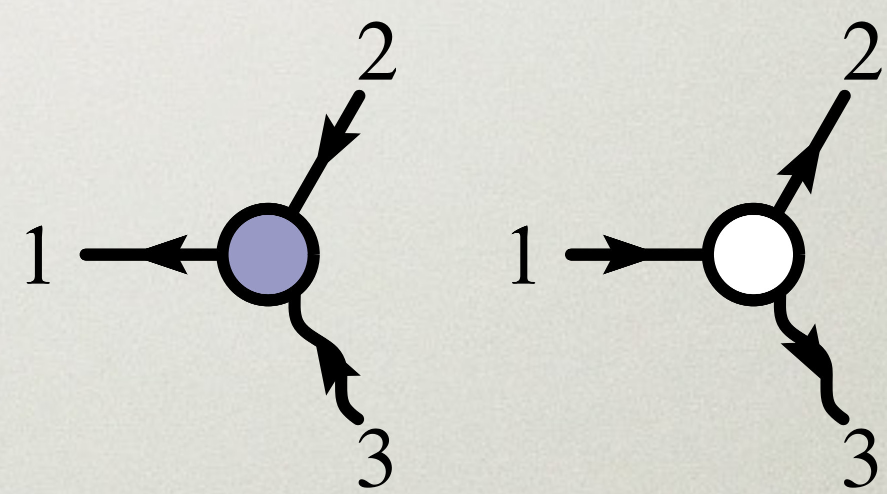


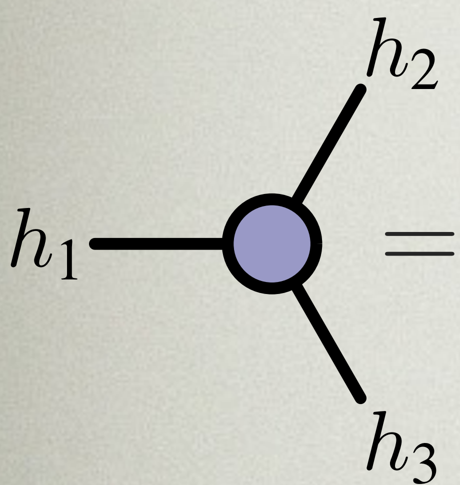
pure Yang-Mills



massless QED

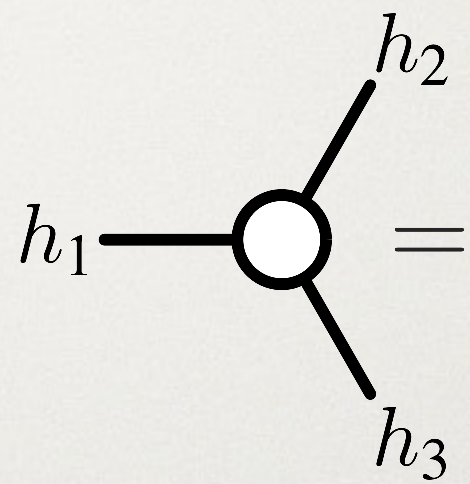
What's Special about Massless, 4d?

- Massless, 3-particle amplitudes in 4 dimensions:



$$= \langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1}$$

$$h_1 + h_2 + h_3 \leq 0$$



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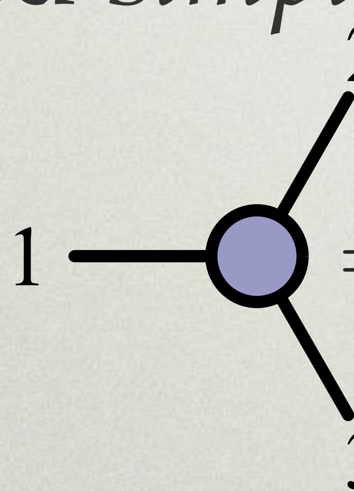
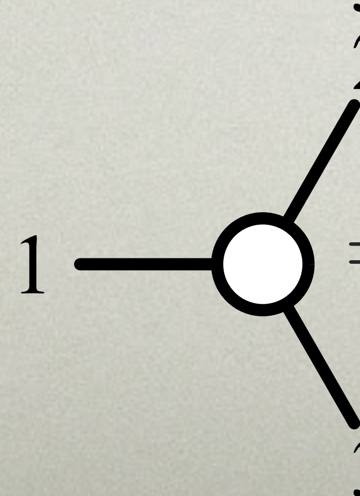
$$h_1 + h_2 + h_3 \geq 0$$

- Enhanced *simplicity* of maximal supersymmetry

[Arkani-Hamed, Cachazo, Kaplan (2008)]



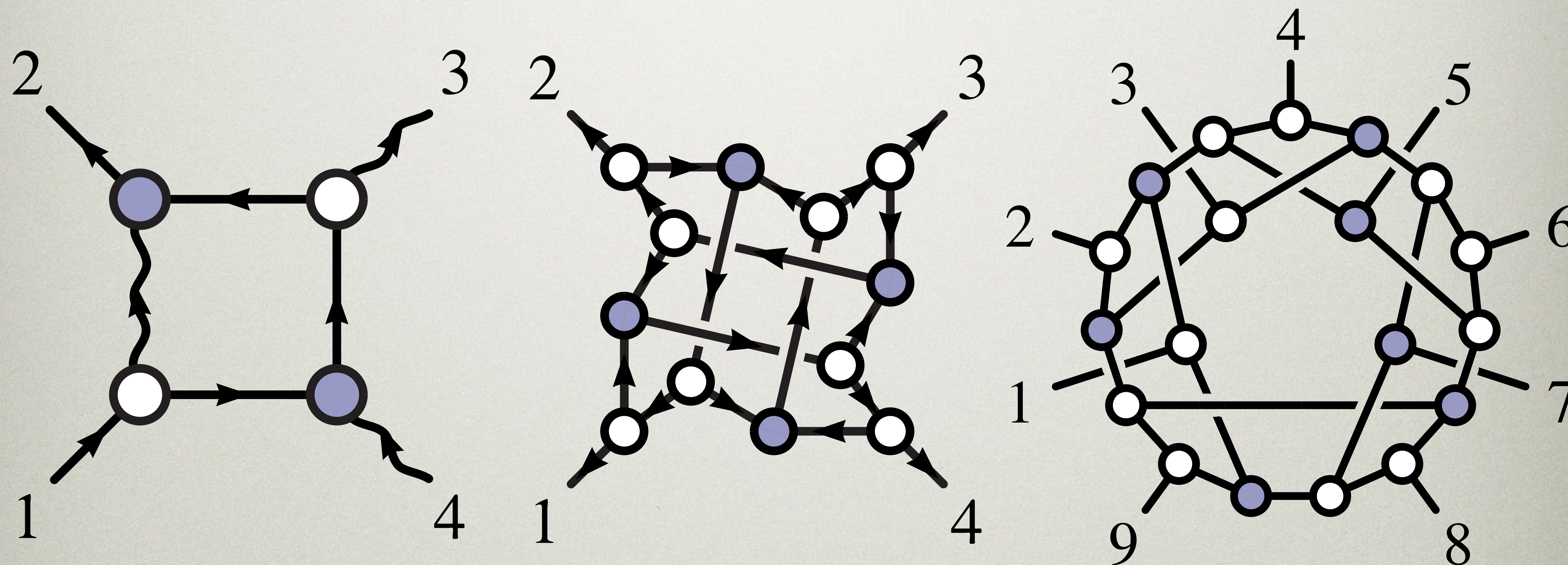
$$= \frac{\delta^{2\times 4}(\lambda\cdot\widetilde{\eta})}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle}\,\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda}) \equiv \mathcal{A}_3^{(2)}$$



$$= \frac{\delta^{1\times 4}(\widetilde{\lambda}^{\perp}\cdot\widetilde{\eta})}{[1\,2]\;[2\,3]\;[3\,1]}\,\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda}) \equiv \mathcal{A}_3^{(1)}$$

Primitive On-Shell Functions

- On-shell functions built from 3-point vertices—edges label states (which dictate the vertices)



The Grassmannian Correspondence

$$f_\Gamma \equiv \prod_i \Big(\sum_{h_i, q_i} \int d^3\mathrm{LIPS}_i \Big) \prod_v \mathcal{A}_v$$

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On-Shell Physics		Grassmannian Geometry
• on-shell diagrams and functions		• $\{$strata $C \in G(k,n)$, volume-form $\Omega_C\}$
• physical symmetries – trivial symmetries (identities)	$\Leftrightarrow$	• volume-preserving diffeomorphisms – cluster coordinate mutations
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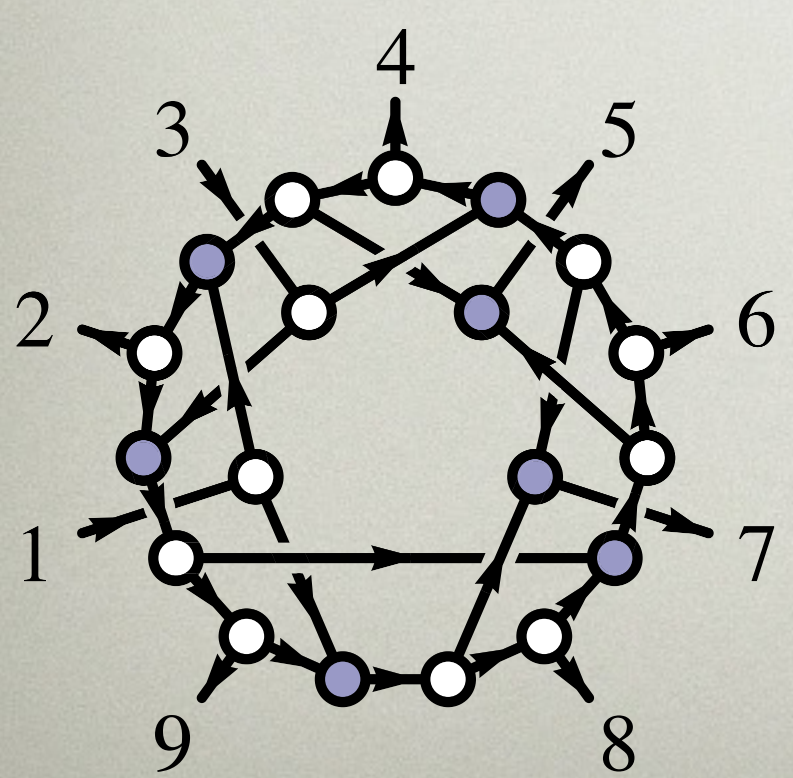

$$C \equiv \begin{pmatrix} \alpha_1 & 1 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & 1 & 0 & \alpha_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_5 & 1 & \alpha_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \alpha_7 & \alpha_8 \\ 0 & 0 & 0 & 0 & 0 & \alpha_9 & 1 & \alpha_{10} & 0 \\ \alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_{12} \\ 0 & \alpha_{13} & \alpha_{14} & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Omega(\vec{\alpha}) \equiv \left(\frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_{14}}{\alpha_{14}} \right) \times (1 + \alpha_2\alpha_4\alpha_{13}(\alpha_8 + \alpha_7\alpha_{12}))^{\mathcal{N}-4}$$

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- ✦ **reducibility** into functions with non-degenerate Ω_C
- ✦ volume-preserving diffeomorphisms correspond to active symmetry transformations:

$$\int \Omega_C \, \delta(C, p, h) \mapsto \int \Omega_{C'} \delta(C', p, h) = \int \Omega_C \, \delta(C', p, h) = \int \Omega_C \, \delta(C, p', h')$$

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***Questions (math/physics)*:**

- how many?
- do these extend to full scattering amplitudes?
- a *functional* basis for amplitude integrands?

New (Asymptotic) Symmetries?

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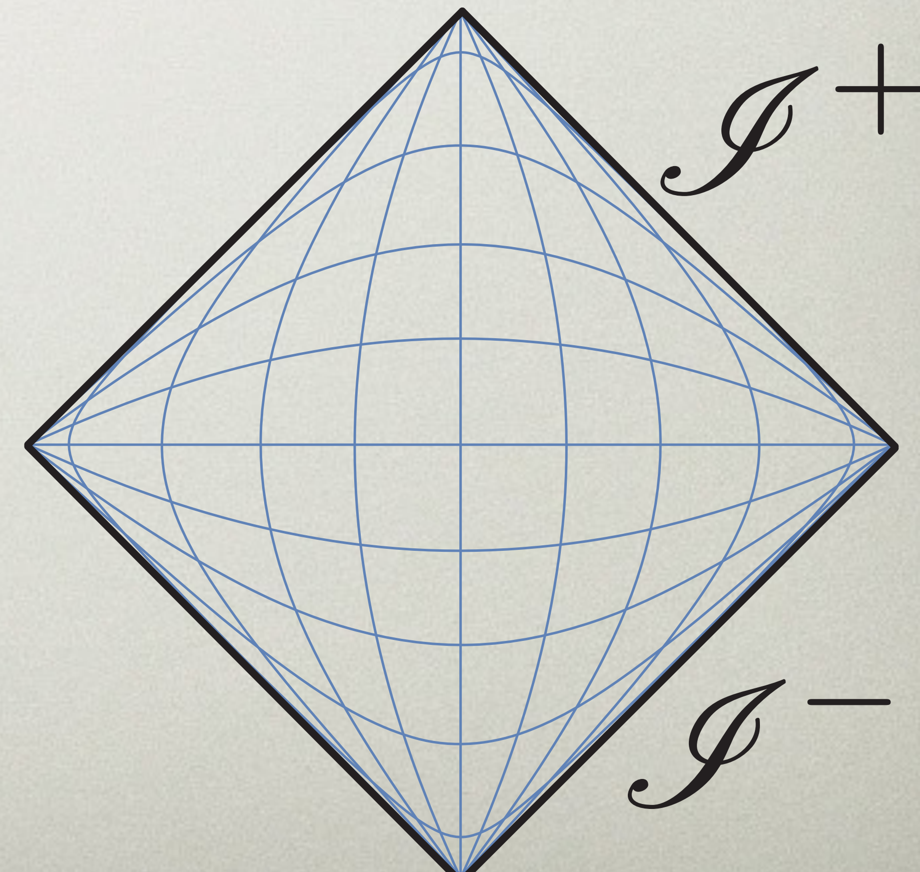
- Some provocative recent claims:
  - the S-matrices of (asymptotically-)flat (4d) theories of massless particles enjoy ***infinite dimensional*** symmetries
  - soft theorems are Ward identities for these symmetries

[Strominger; Strominger *et al*]

[**JB**, Haco, Hawking, Perry (2017)]



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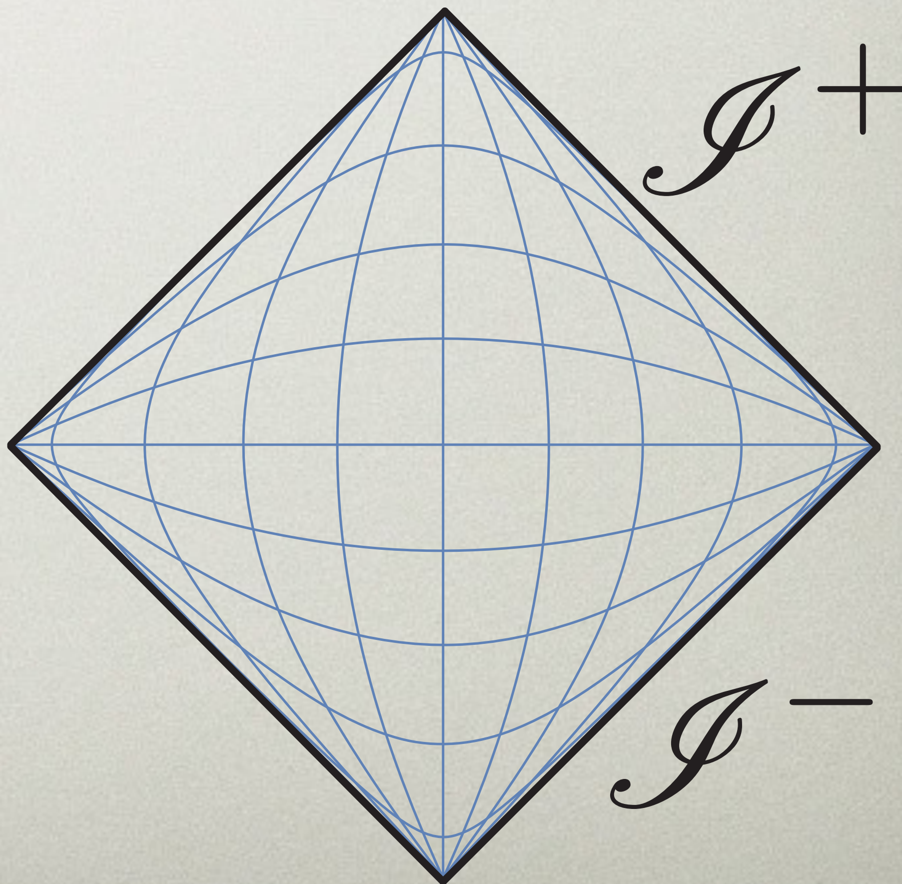

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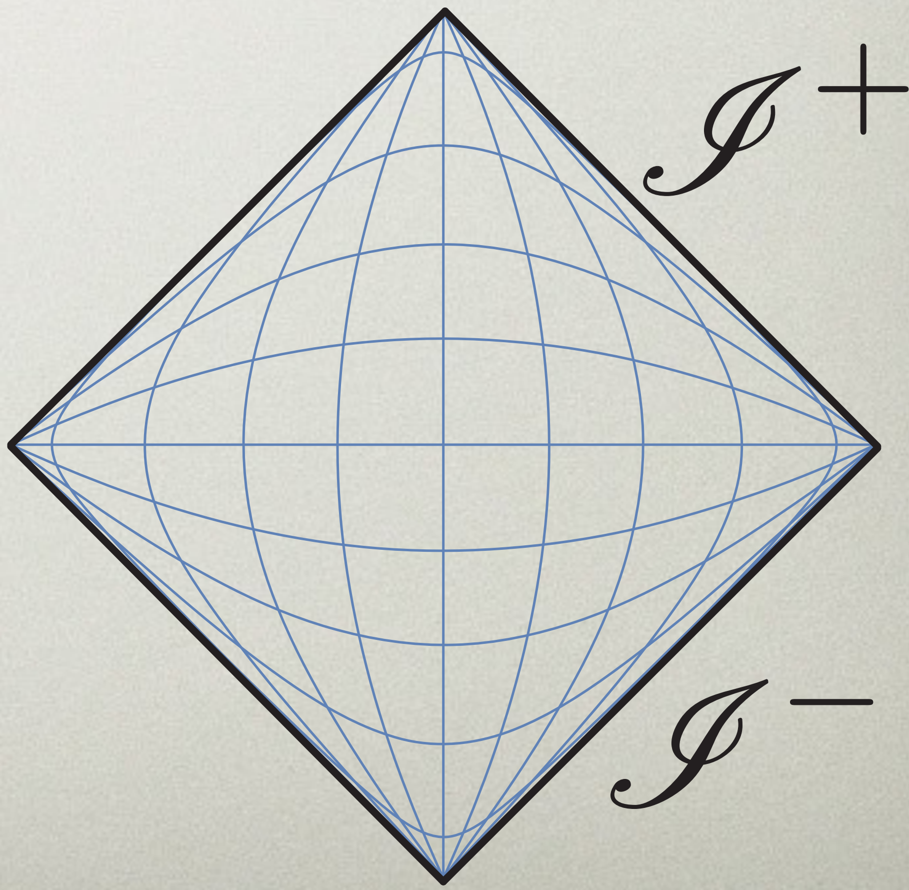
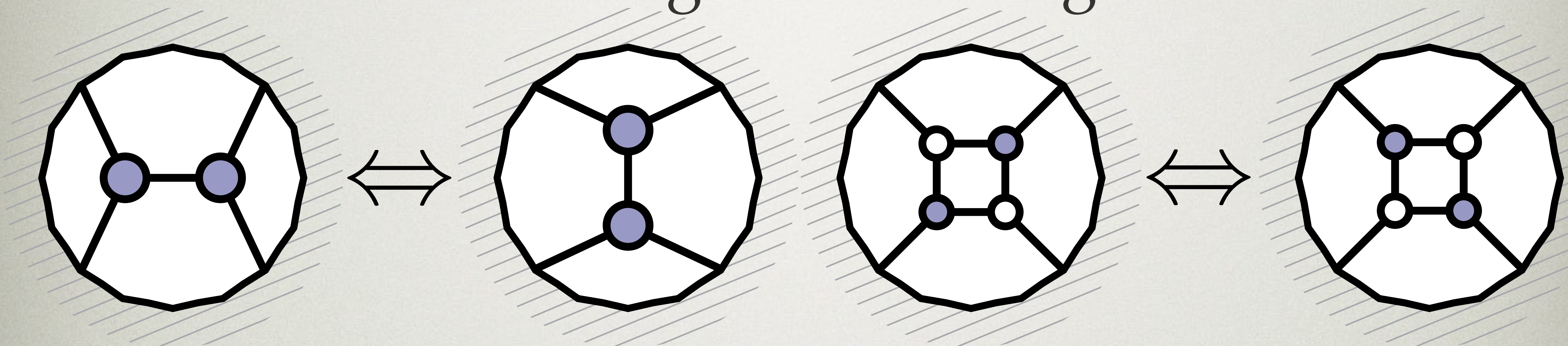
- Connections to the *Yangian*?



Diagrammatic Relations for SYM

- Two identities among on-shell diagrams:



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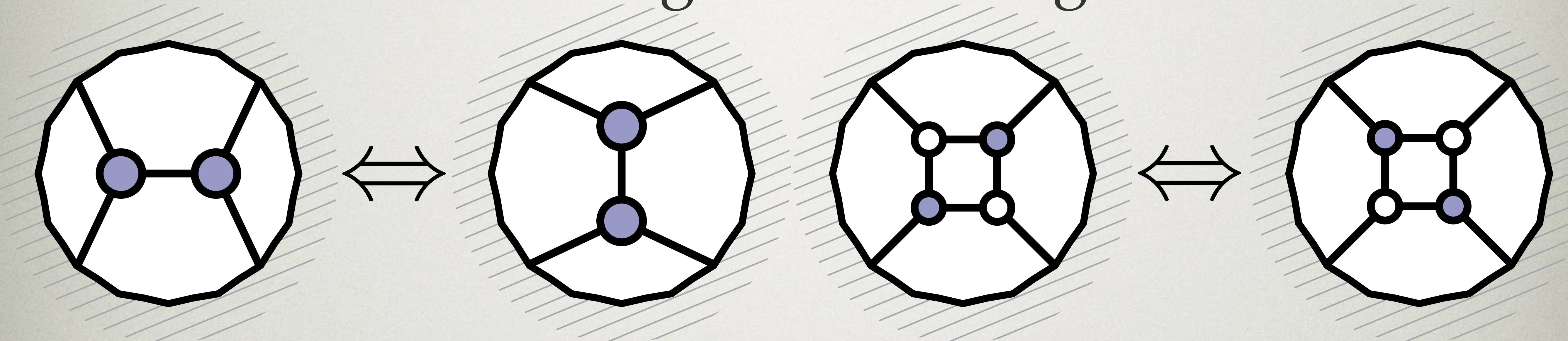
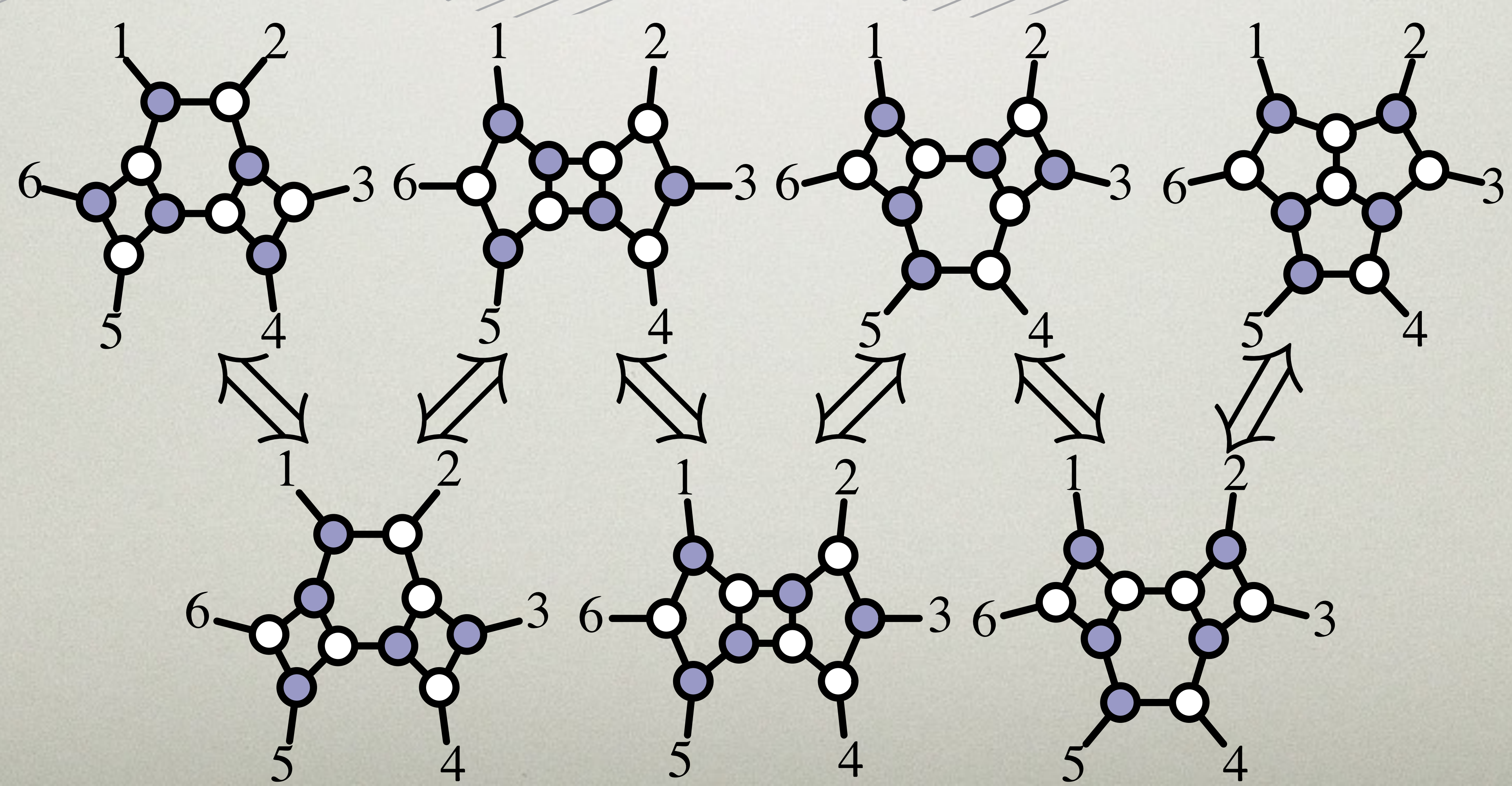
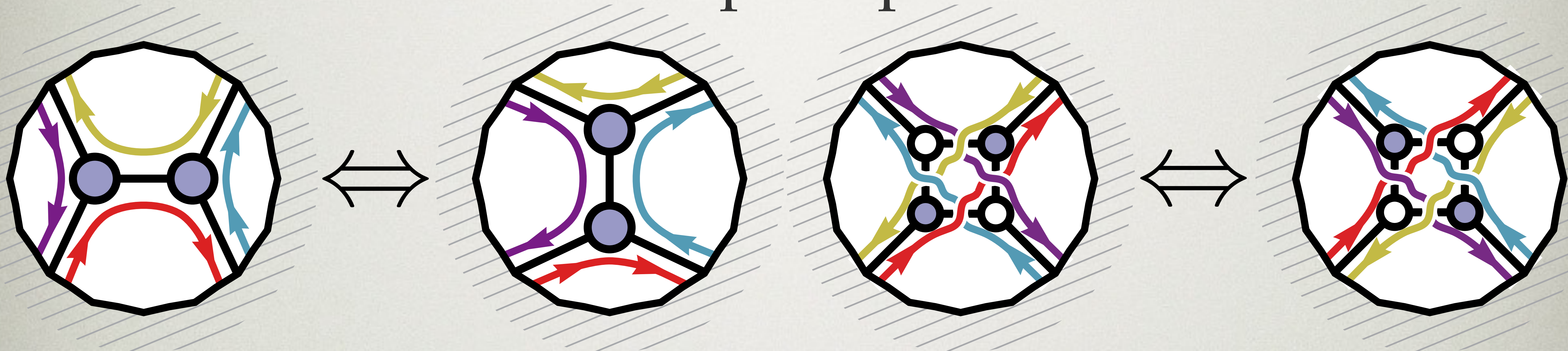


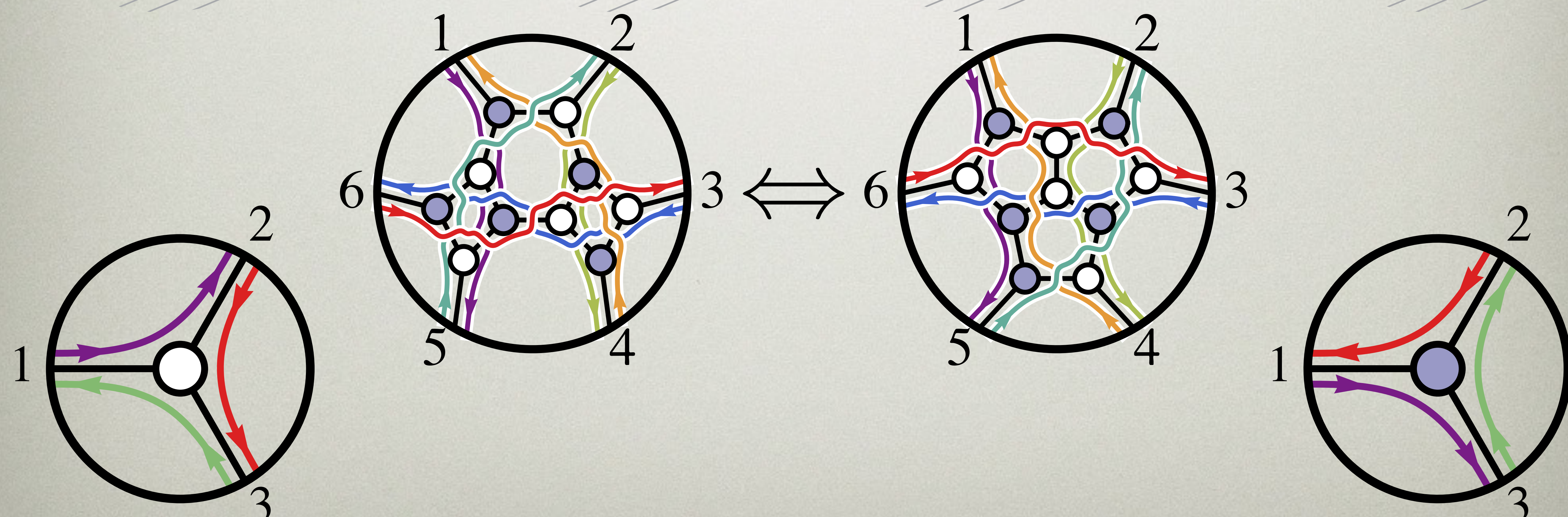


Planar Combinatoric Classification

[Postnikov (2006)]

- Moves leave invariant path-permutation-labels



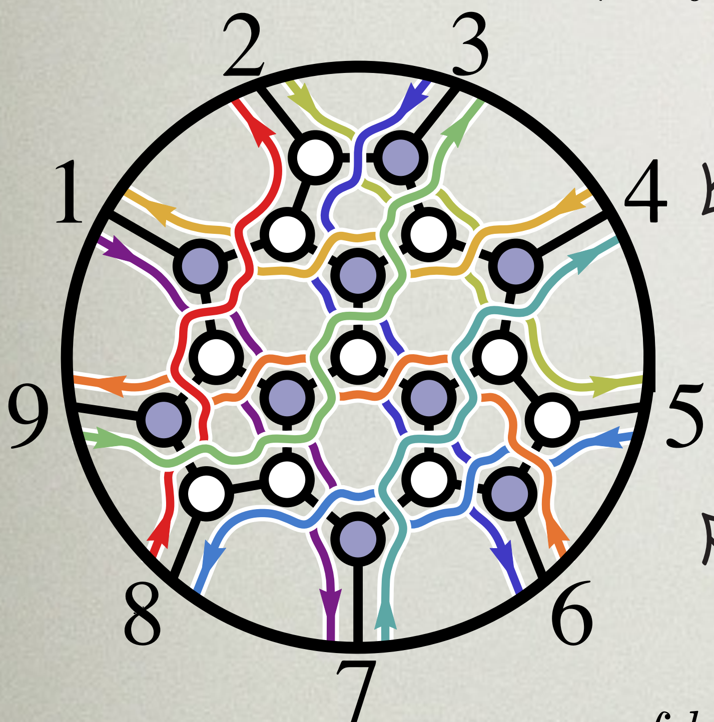


[Arkani-Hamed, **JB**, Cachazo, Goncharov, Postnikov, Trnka (2012)]

Web of Dualities for Planar SYM

$$C(\alpha)\equiv\begin{pmatrix}1 & \alpha_8 & (\alpha_5+\alpha_{14}\alpha_8) & \alpha_5\alpha_{11} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & \alpha_{10} & (\alpha_{10}\alpha_{13}+\alpha_4) & \alpha_4\alpha_7 & 0 & 0\\ -\alpha_3\alpha_9 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & (\alpha_3+\alpha_{12}\alpha_6)\\ -\alpha_9 & 0 & \alpha_1 & \alpha_1\alpha_{11} & 0 & -\alpha_1\alpha_2 & -\alpha_1\alpha_2\alpha_7 & 0 & 1\end{pmatrix}\in G(4,9)$$

$\Updownarrow$

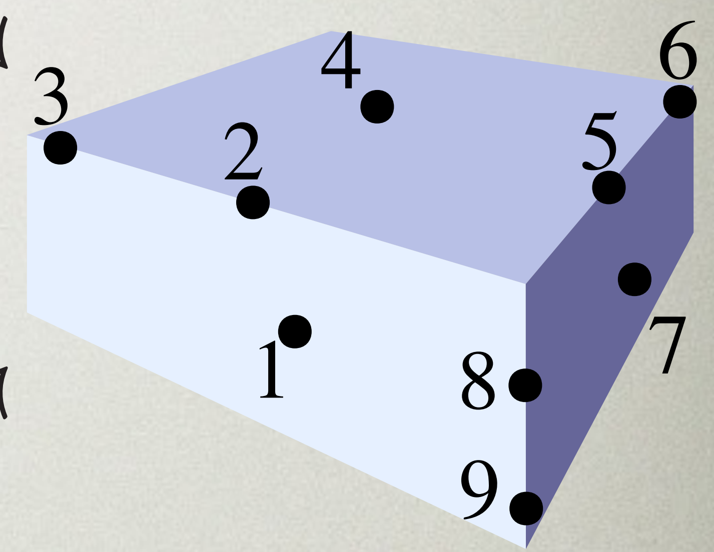


$\Longleftrightarrow$

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow\\ 7 & 5 & 6 & 1 & 8 & 9 & 4 & 2 & 3\end{pmatrix}$$

$$\sigma\equiv\{7,\ 5,\ 6,\ 10,\ 8,\ 9,\ 13,\ 11,\ 12\}$$

$\Longleftrightarrow$



$\Updownarrow$

$$f_\sigma\equiv\int\frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_{14}}{\alpha_{14}}\,\delta^{k\times4}\big(C(\alpha)\cdot\widetilde{\eta}\big)\delta^{k\times2}\big(C(\alpha)\cdot\widetilde{\lambda}\big)\delta^{2\times(n-k)}\big(\lambda\cdot C(\alpha)^{\perp}\big)$$

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- Recently published, authoritative description of the foundations of this story, with a full treatment for planar, maximally supersymmetric Yang-Mills

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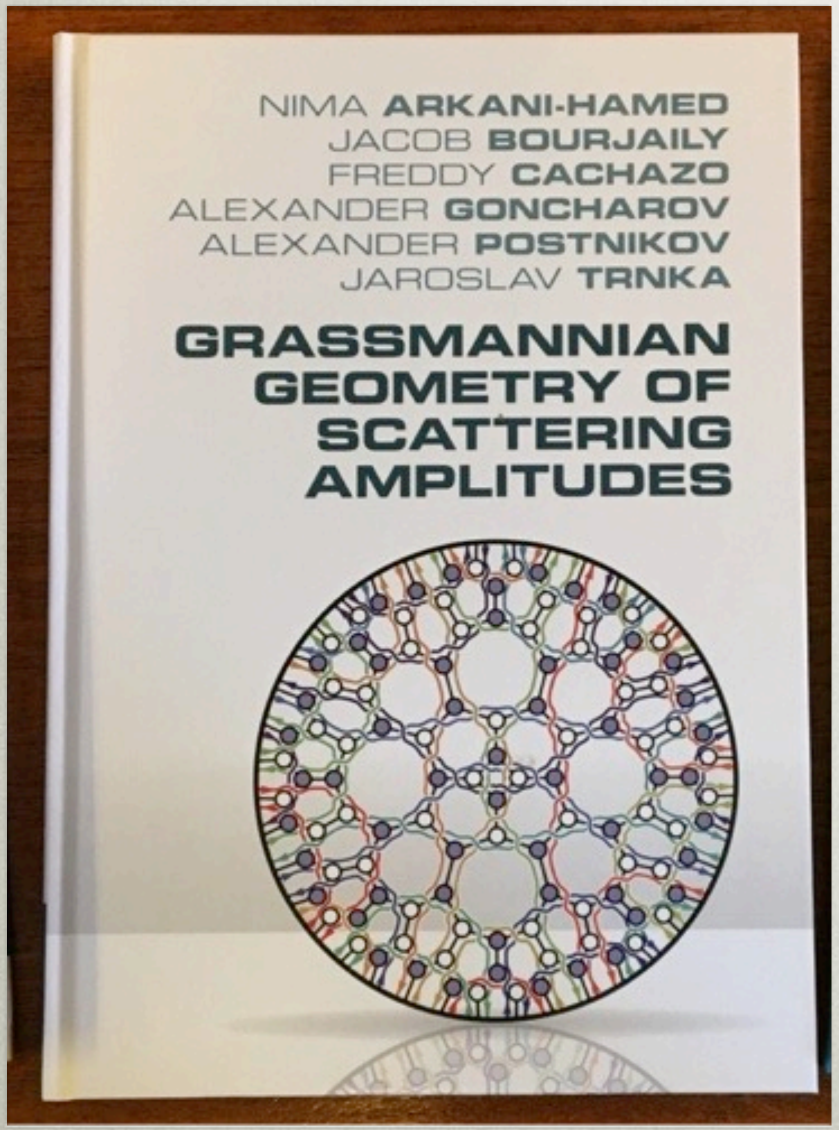

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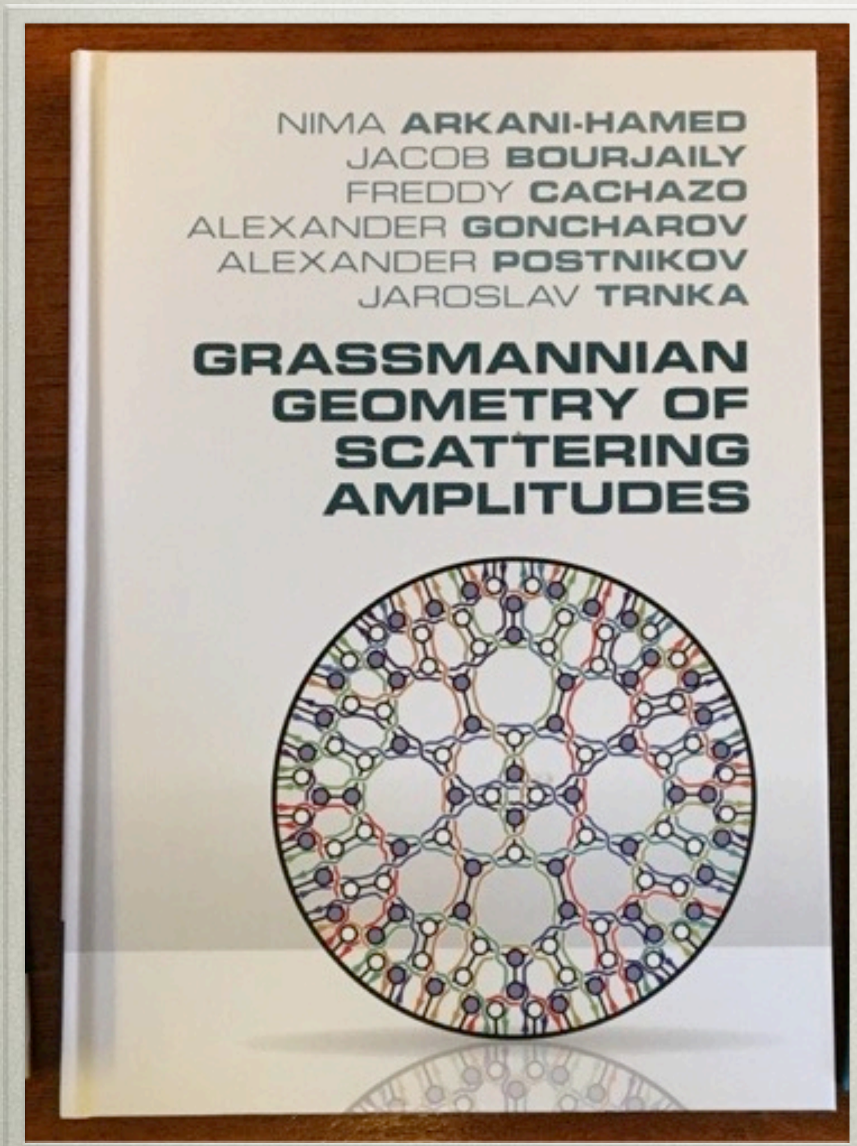

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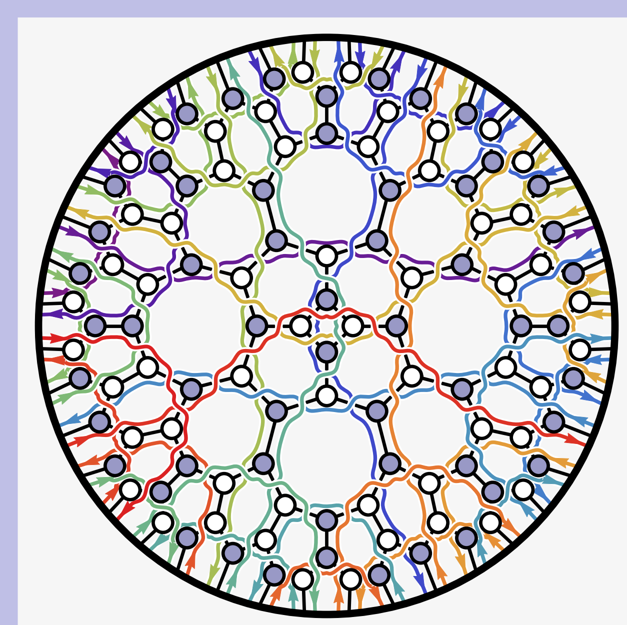

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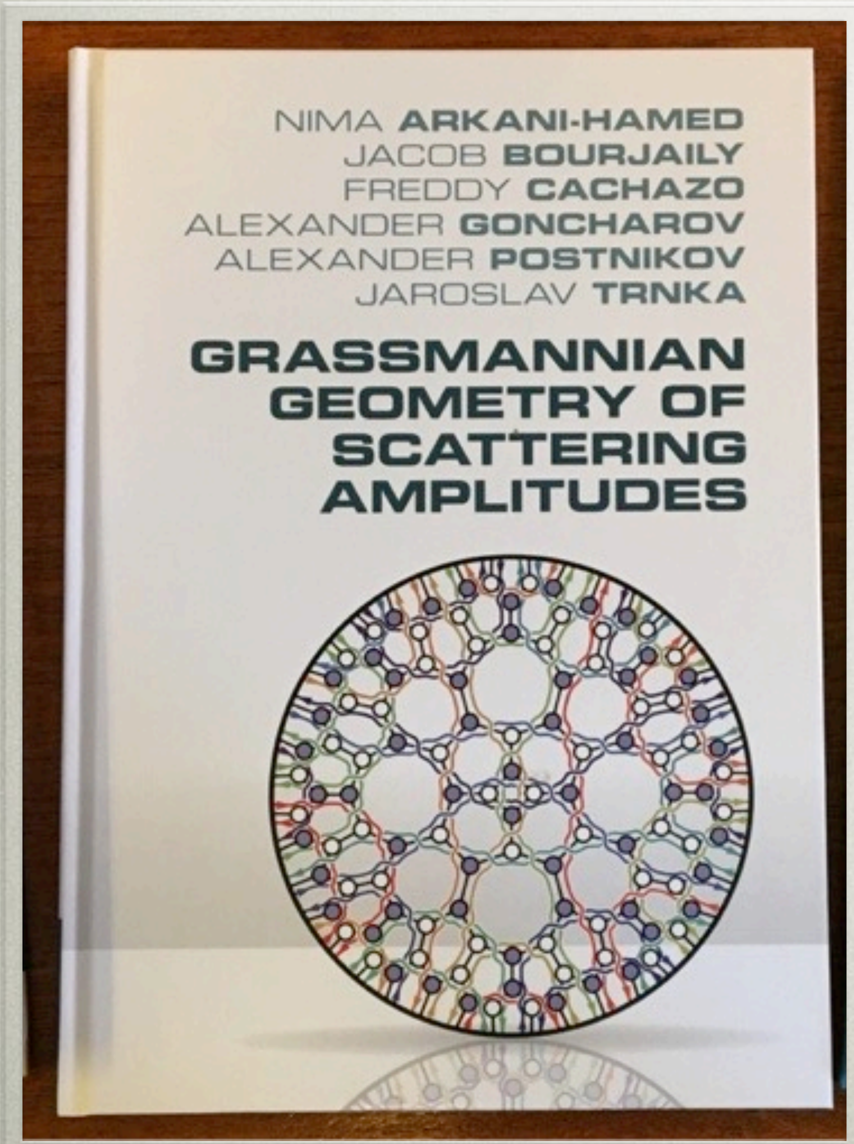

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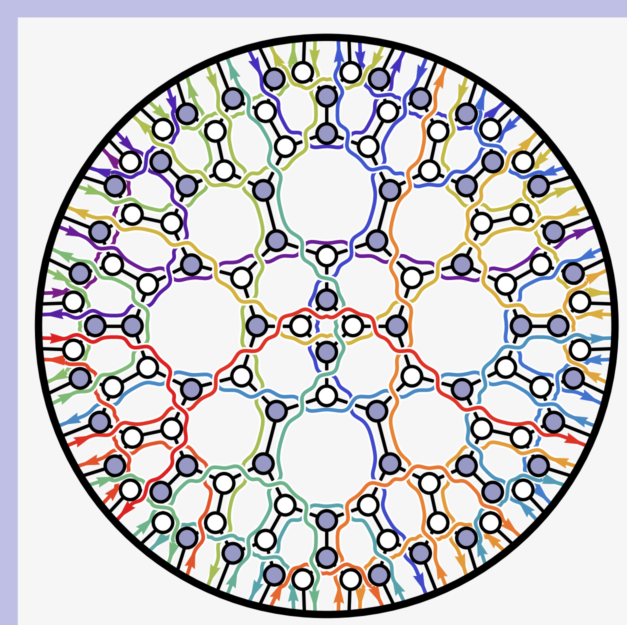

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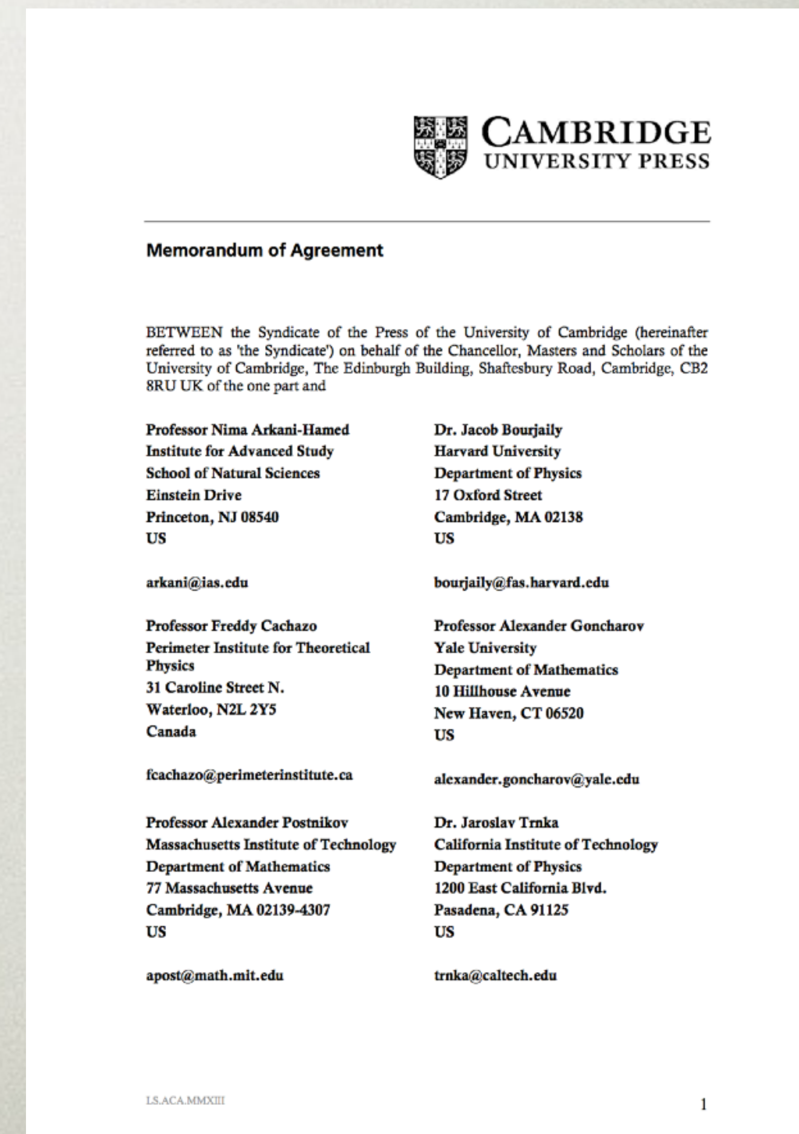

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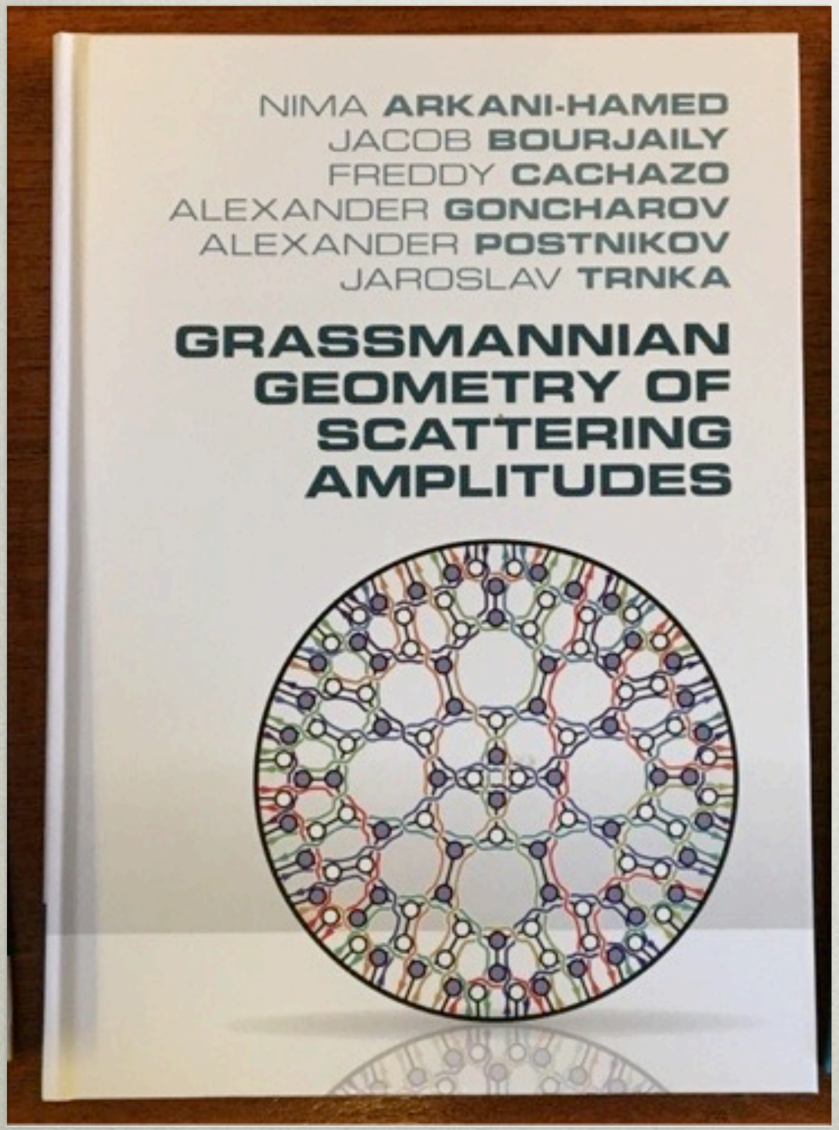

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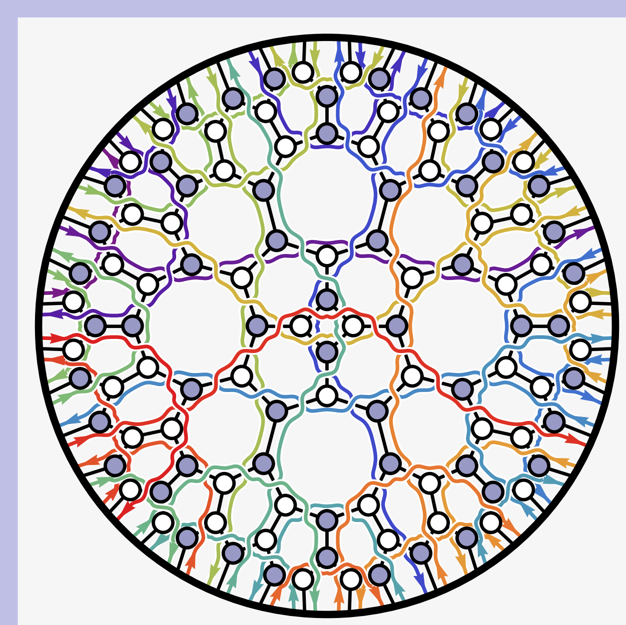

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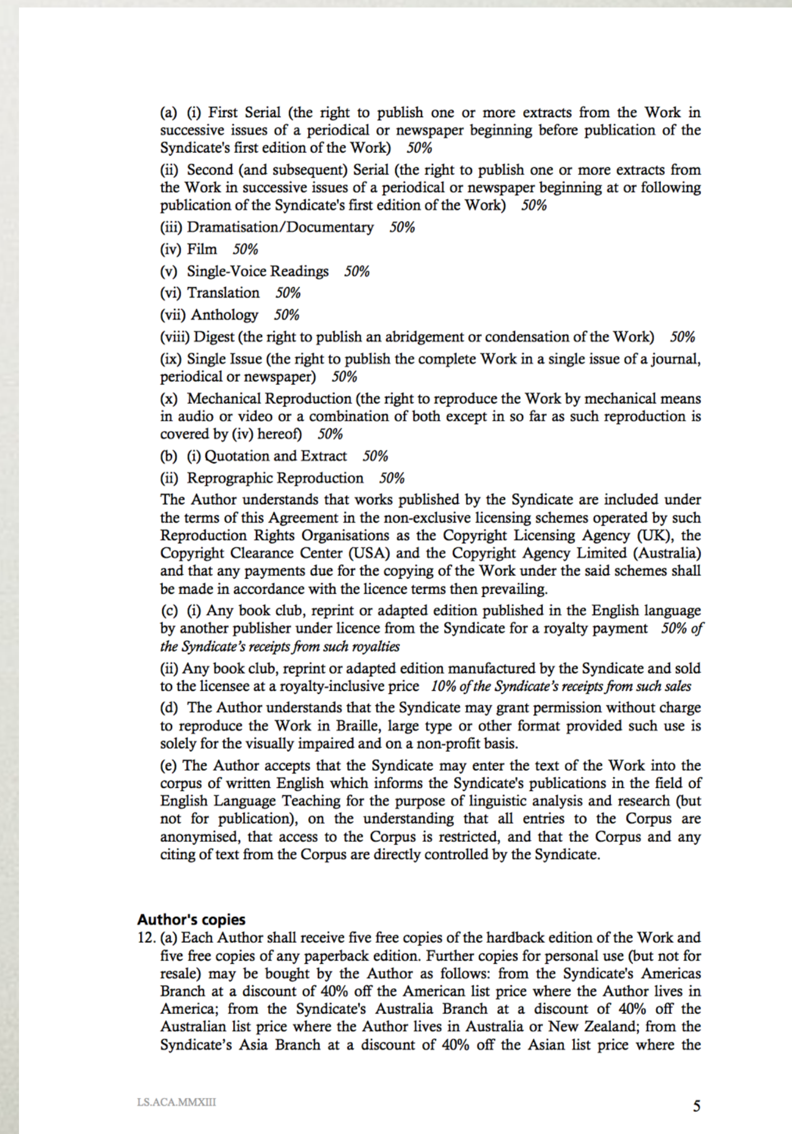

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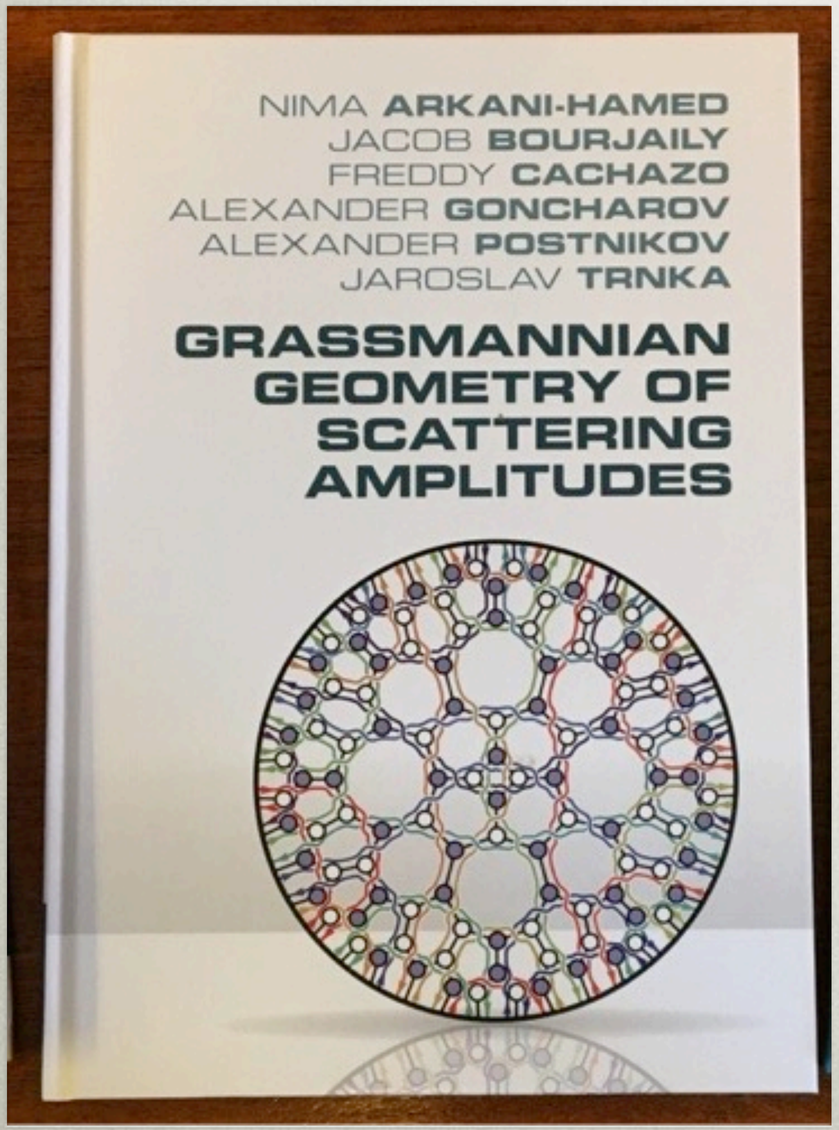

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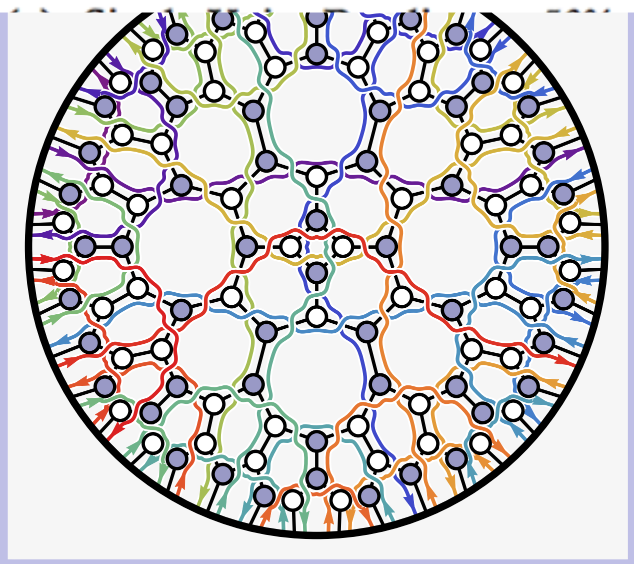

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Jacob L. Bourjaily, 2012

[JB (2012)]

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Explorations Beyond Planar SYM

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- Implications for physics far beyond amplitudes
- Implications for diverse areas of mathematics—graph theory, combinatorics, algebraic geometry, …

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- For k=2 (MHV) on-shell functions, planar tree amplitudes (Parke-Taylor) form a complete basis:

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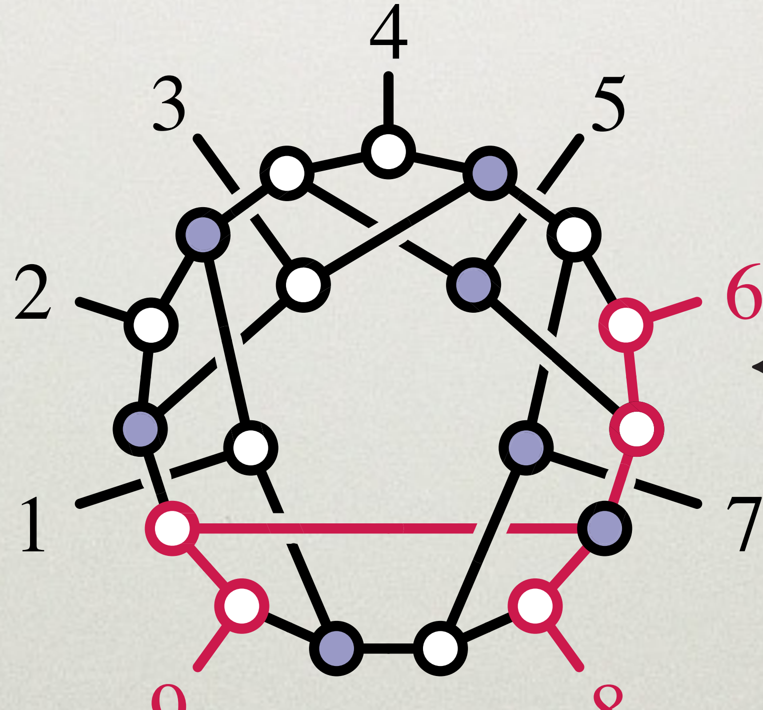
[Arkani-Hamed, **JB**, *et al*. (2014)]



$$\Leftrightarrow \left\{\begin{pmatrix}1\,2\,4\\1\,8\,9\\2\,9\,3\\3\,6\,4\\4\,6\,5\\6\,8\,7\\6\,8\,9\end{pmatrix}\right\}$$

$$\frac{\left(\langle 91\rangle\langle 32\rangle\langle 46\rangle-\langle 16\rangle\langle 43\rangle\langle 29\rangle\right)^2}{\langle 12\rangle\langle 24\rangle\langle 41\rangle\langle 18\rangle\langle 91\rangle\langle 29\rangle\langle 93\rangle\langle 32\rangle\langle 36\rangle\langle 43\rangle\langle 65\rangle\langle 54\rangle\langle 87\rangle\langle 76\rangle\langle 69\rangle}$$

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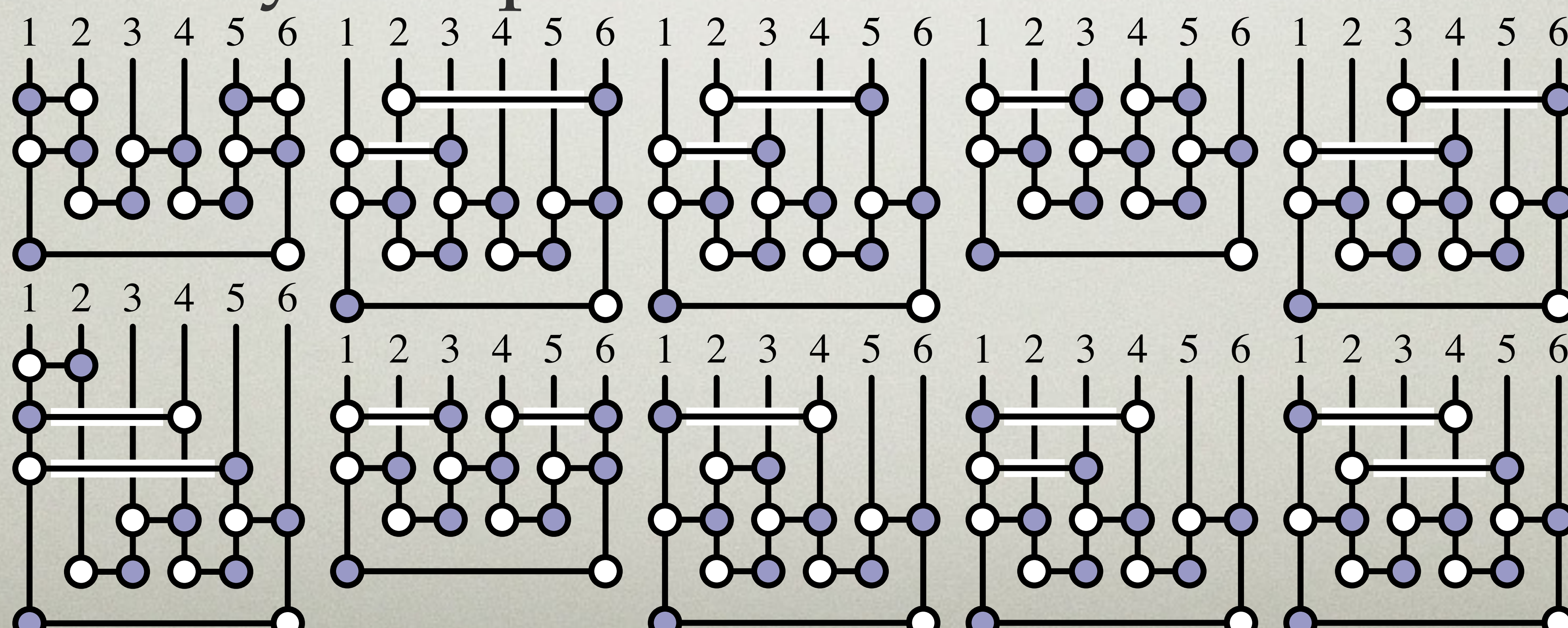

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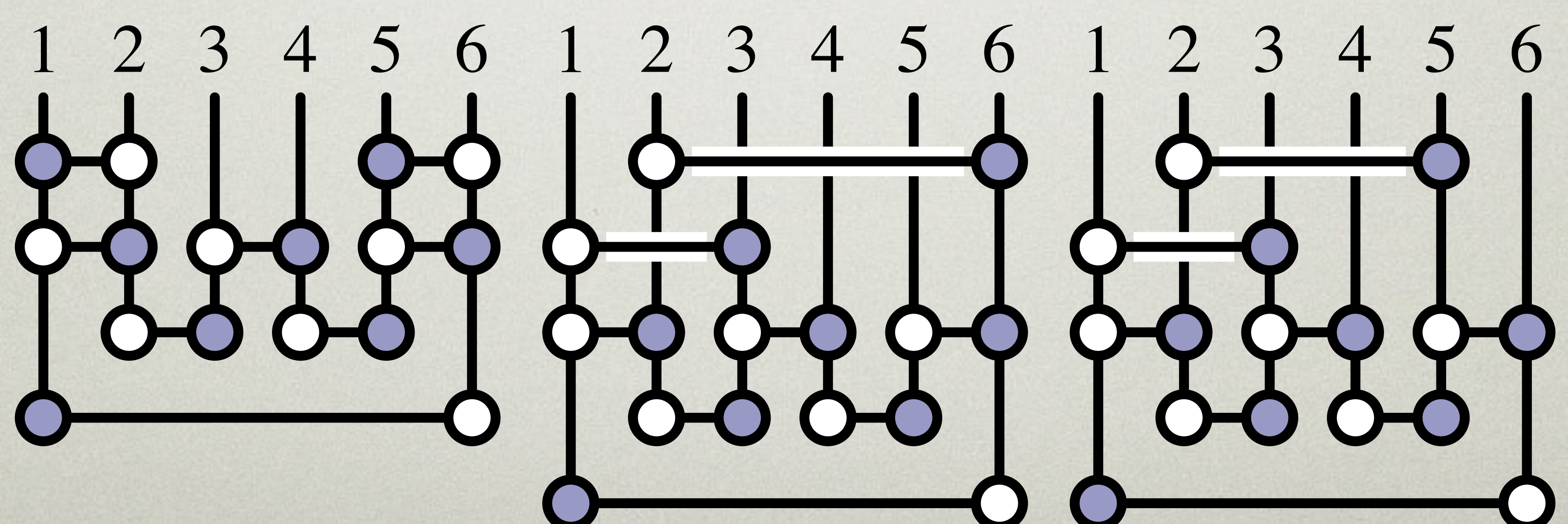

Prescriptive Approaches to Perturbation Theory (prior to loop integration)

(Re)construction of Loop Integrands

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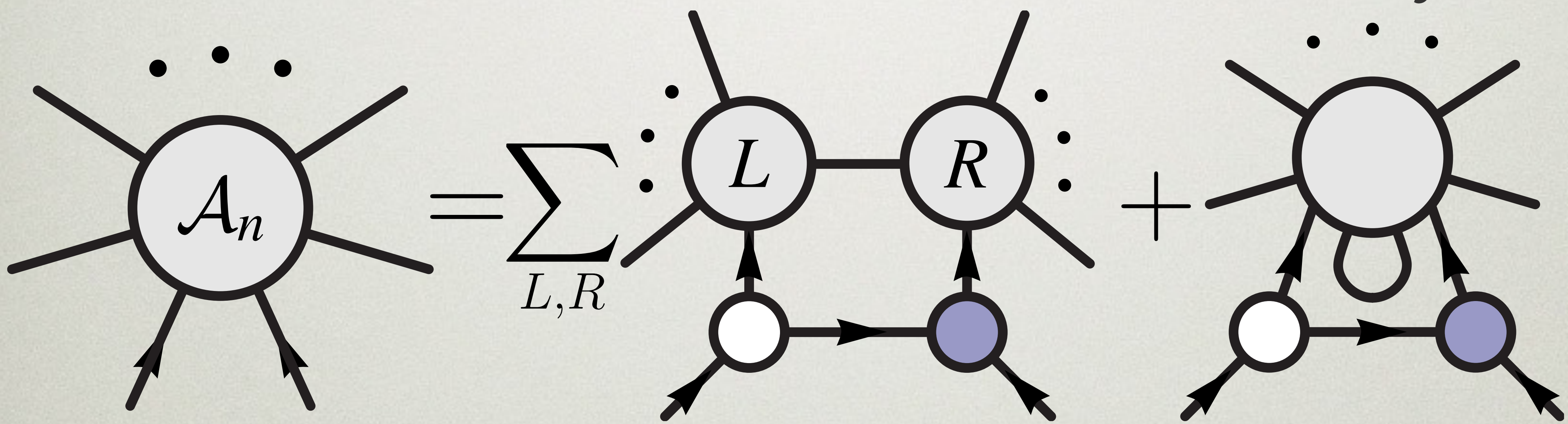

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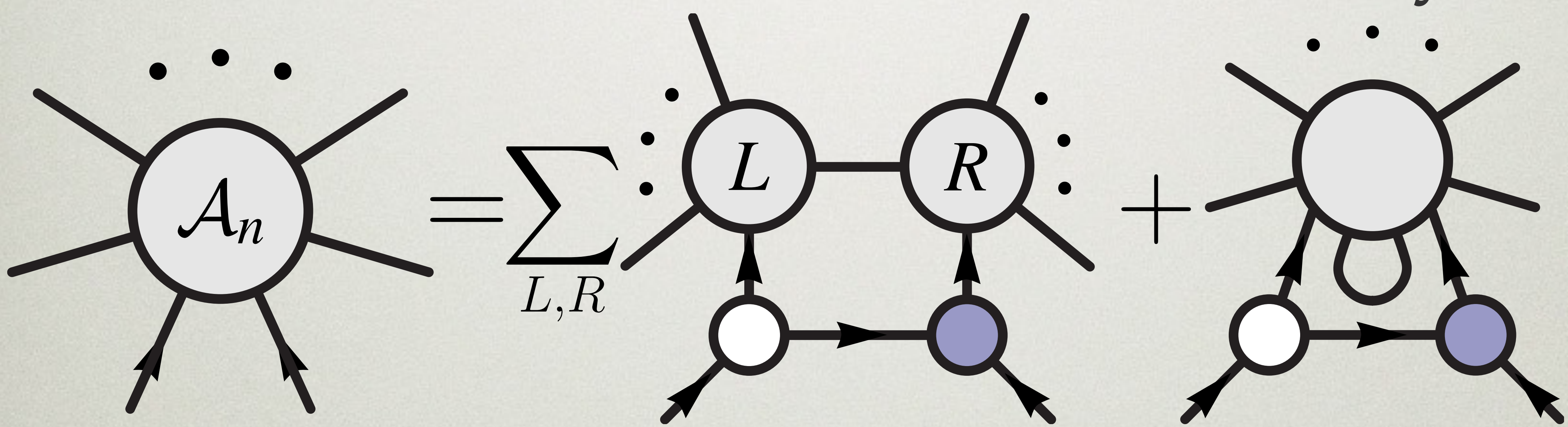

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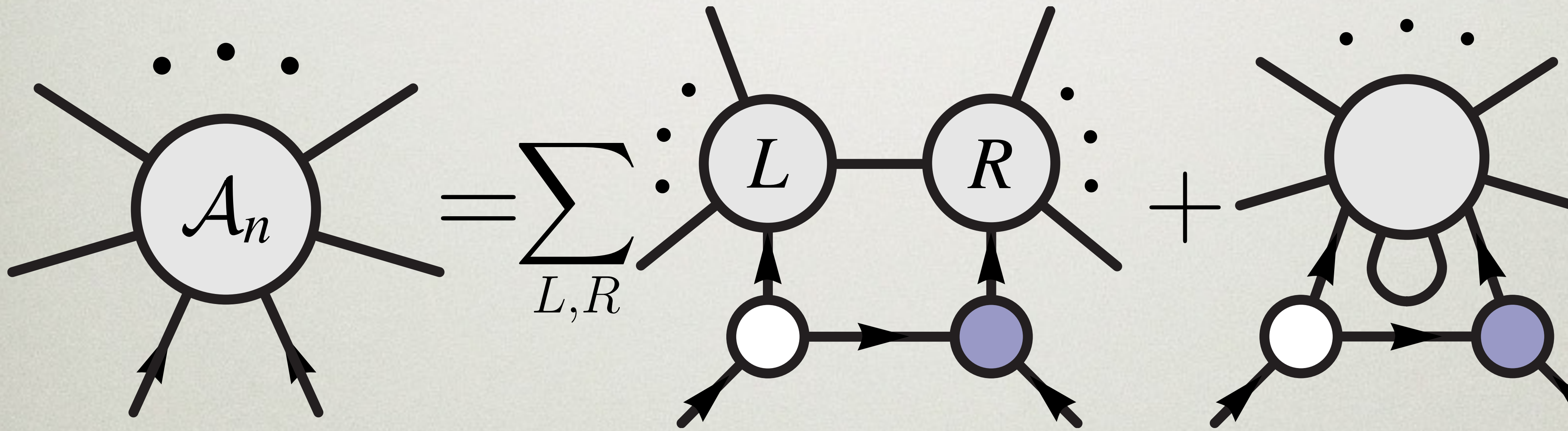

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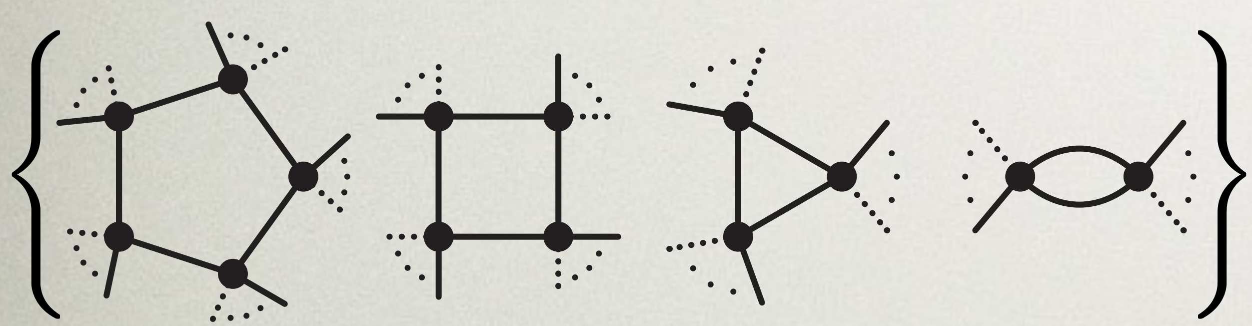
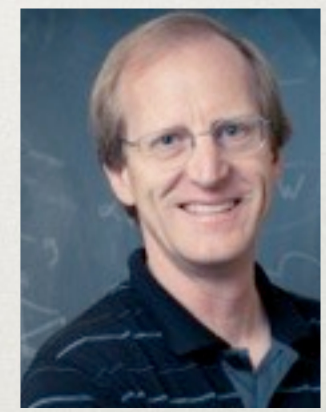
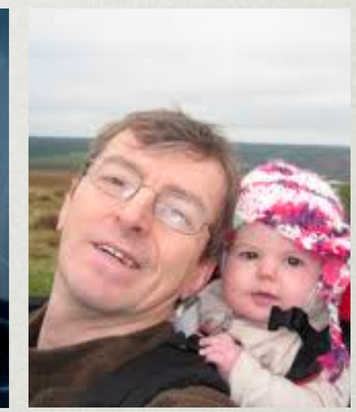
- complications:
  - spurious propagators, mixture of components, …
- someday may prove *ideal*; usefulness is moot today

The Unitarity-Based Approach

- Pick an arbitrary (but complete) basis of Feynman integrals, and use **cuts** to determine coefficients (given as *on-shell functions*)




[Bern, Dixon, Dunbar, Kosower]

- Extremely general and powerful, with important applications at the LHC—*e.g.*, BlackHat @ NLO

[Berger, Bern, *et al.*]

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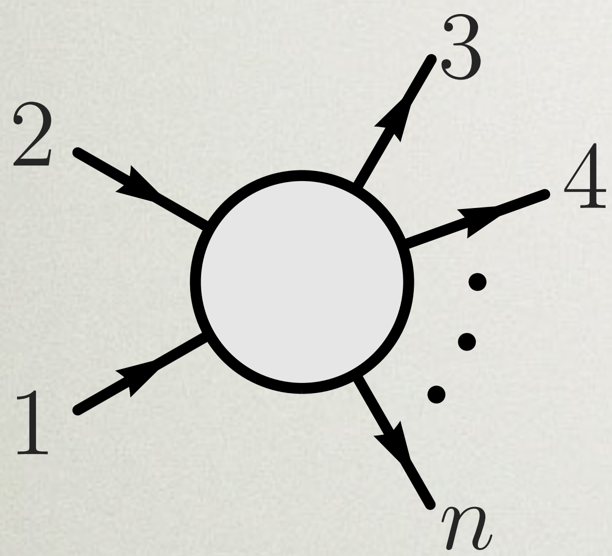

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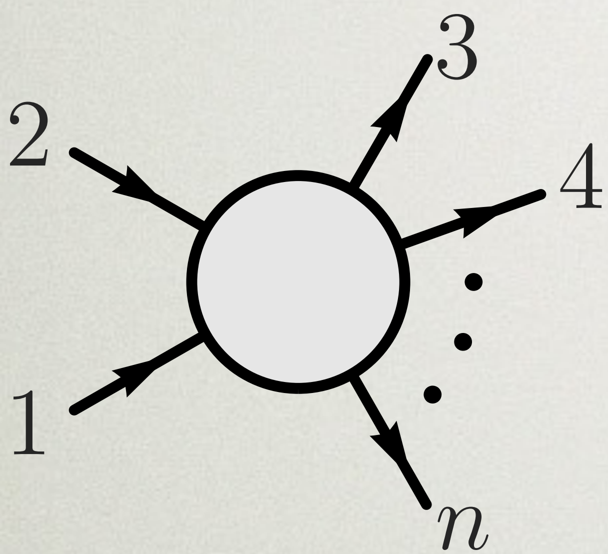

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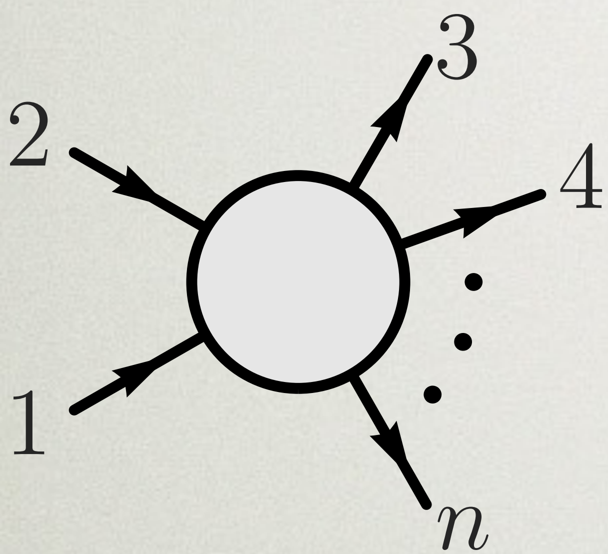

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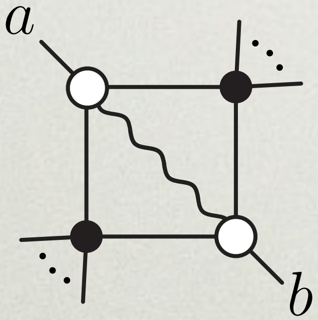
$$\left\{ 1 + \sum_{a<b} \right.$$  $$\left. + \ldots \right\}$$

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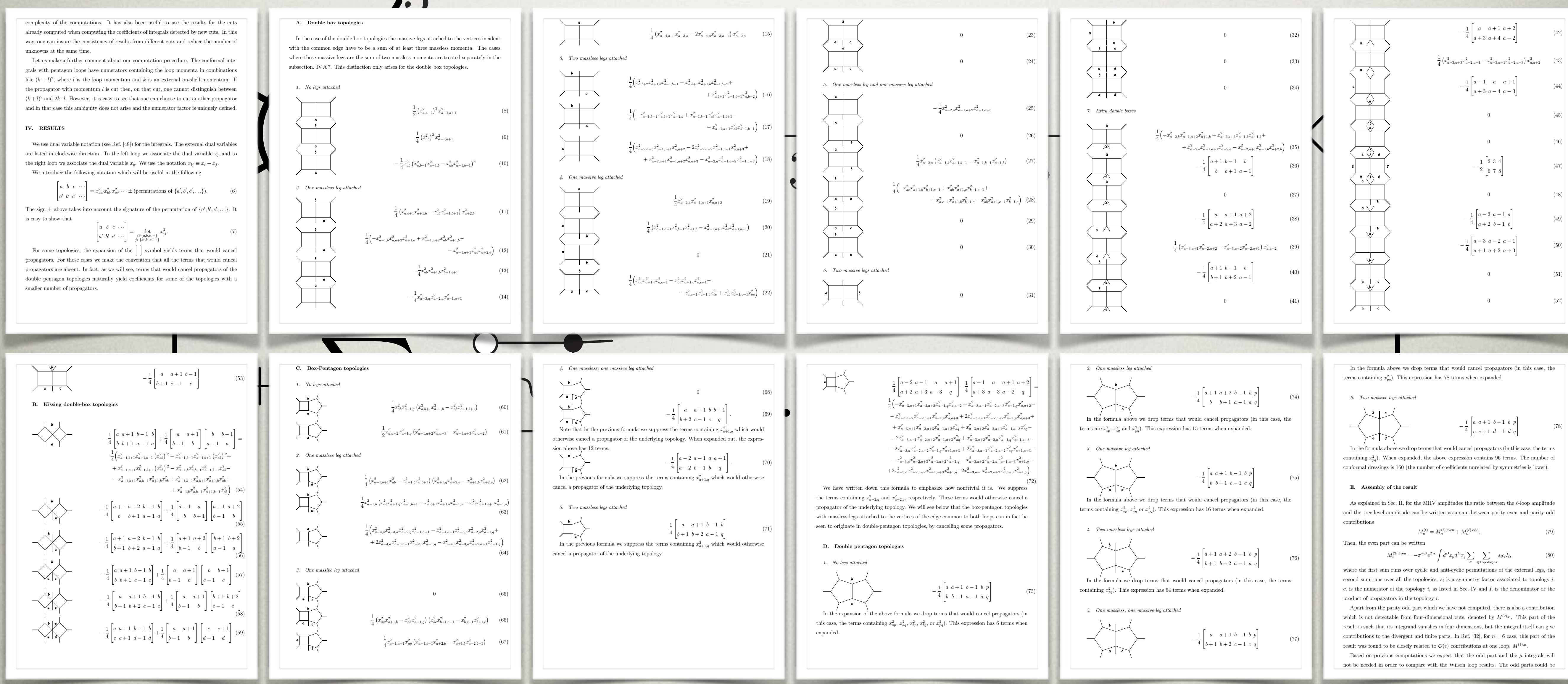
[Vergu (2009)]



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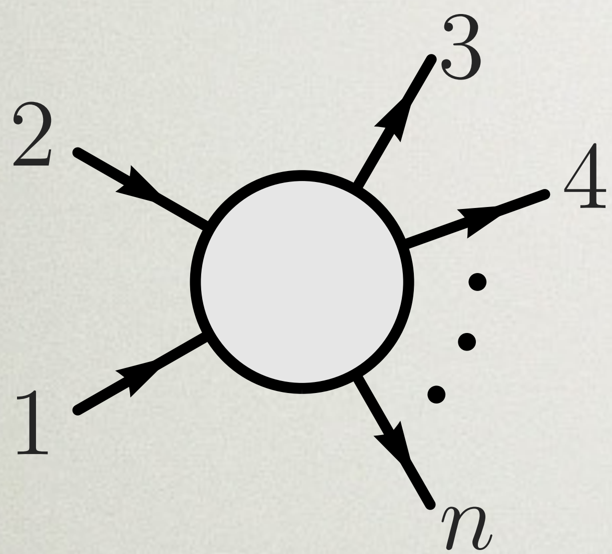

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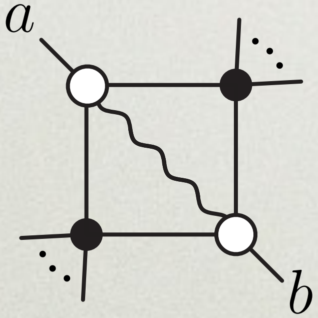

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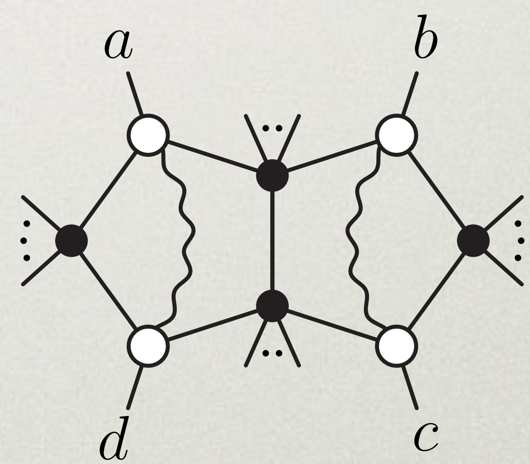

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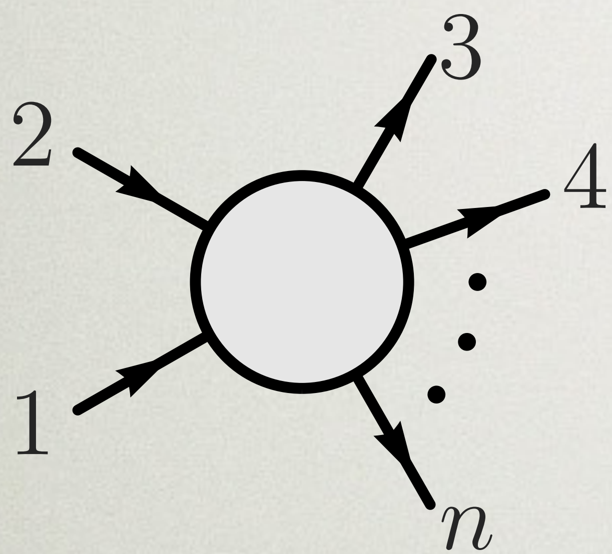

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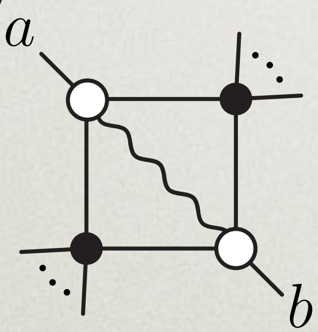

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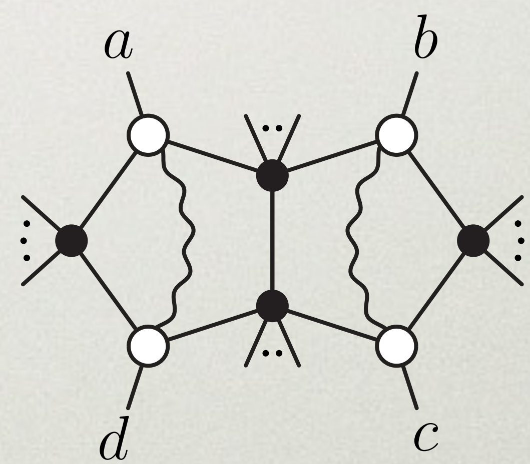

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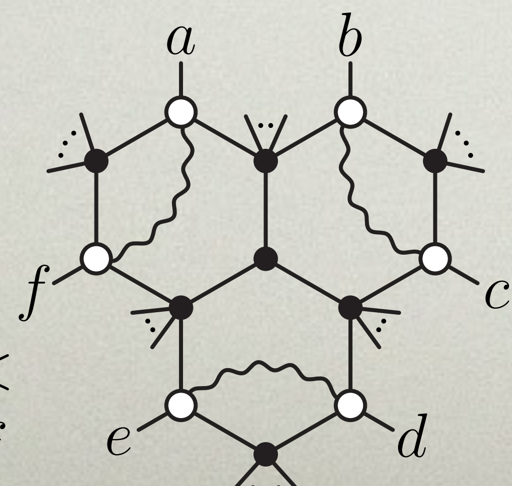

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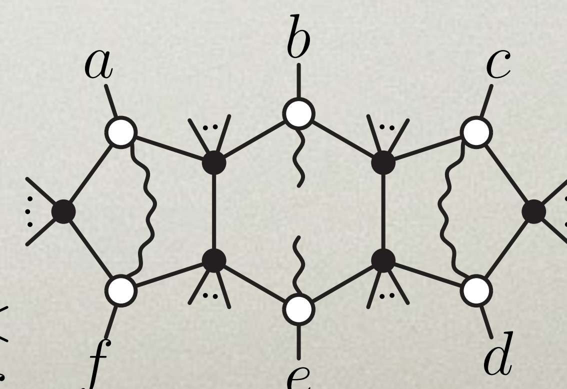

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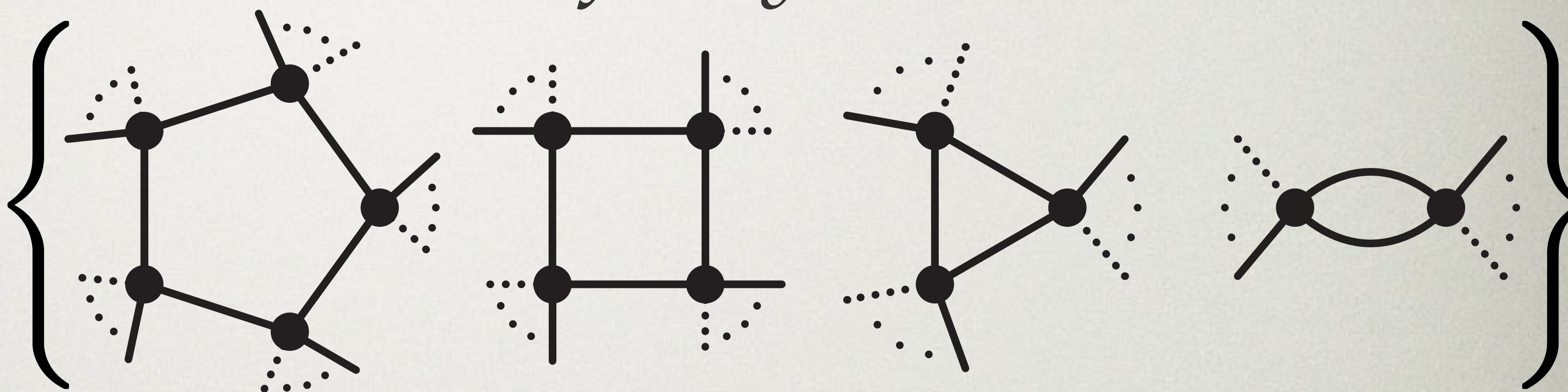

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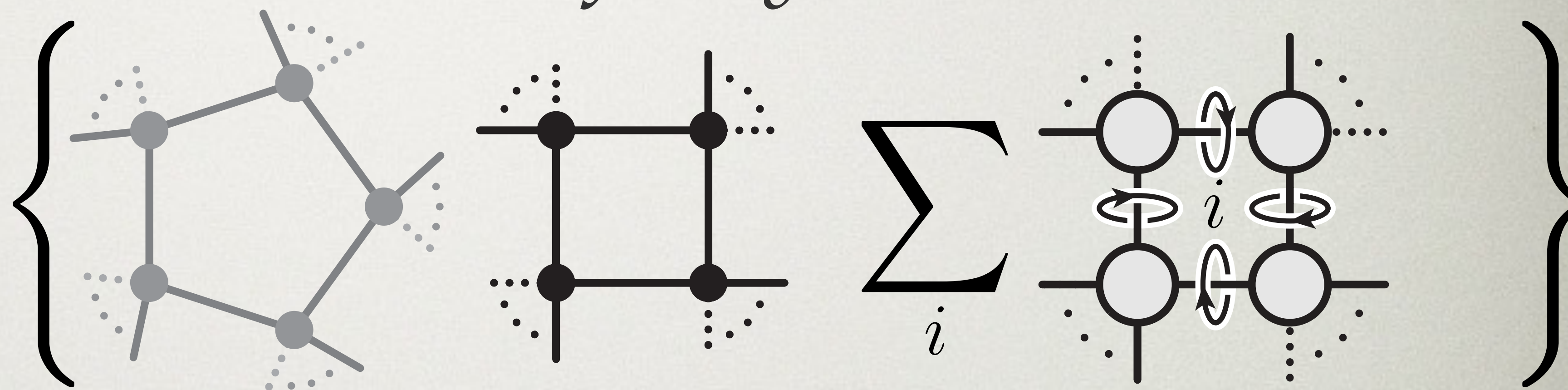

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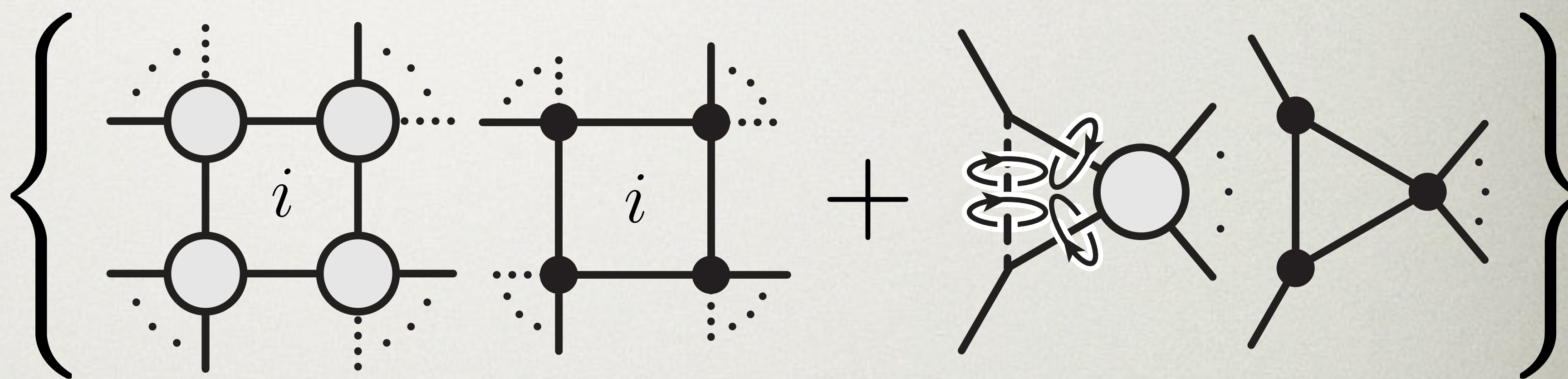

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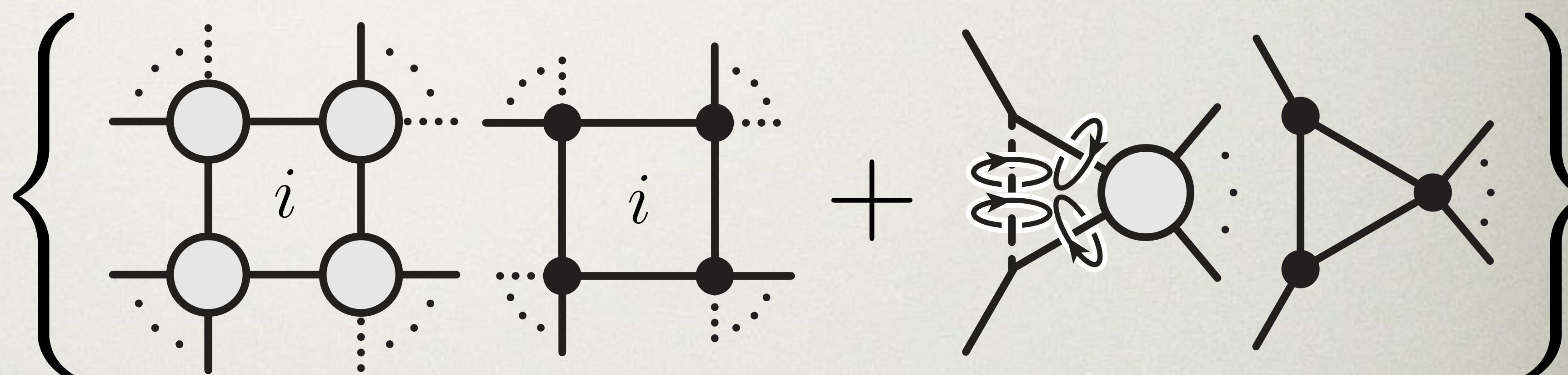

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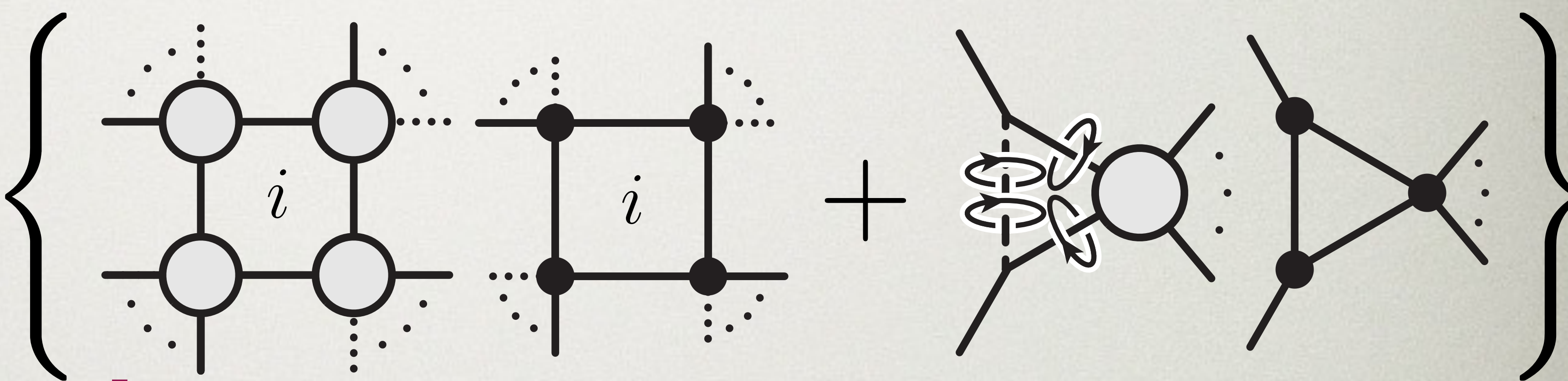

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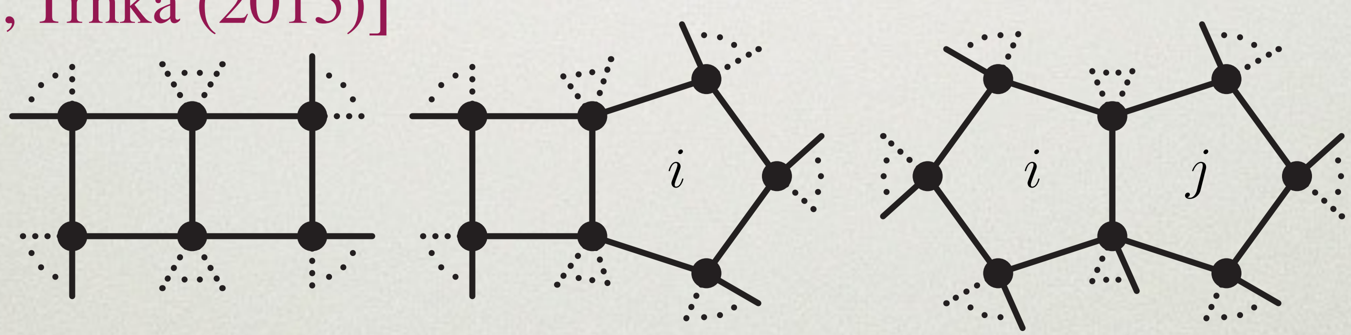

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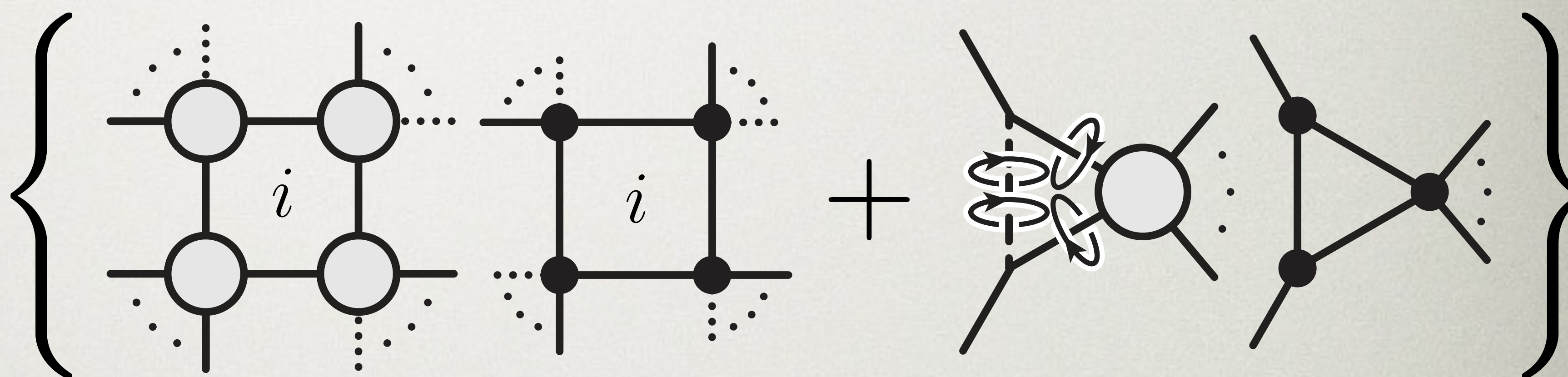

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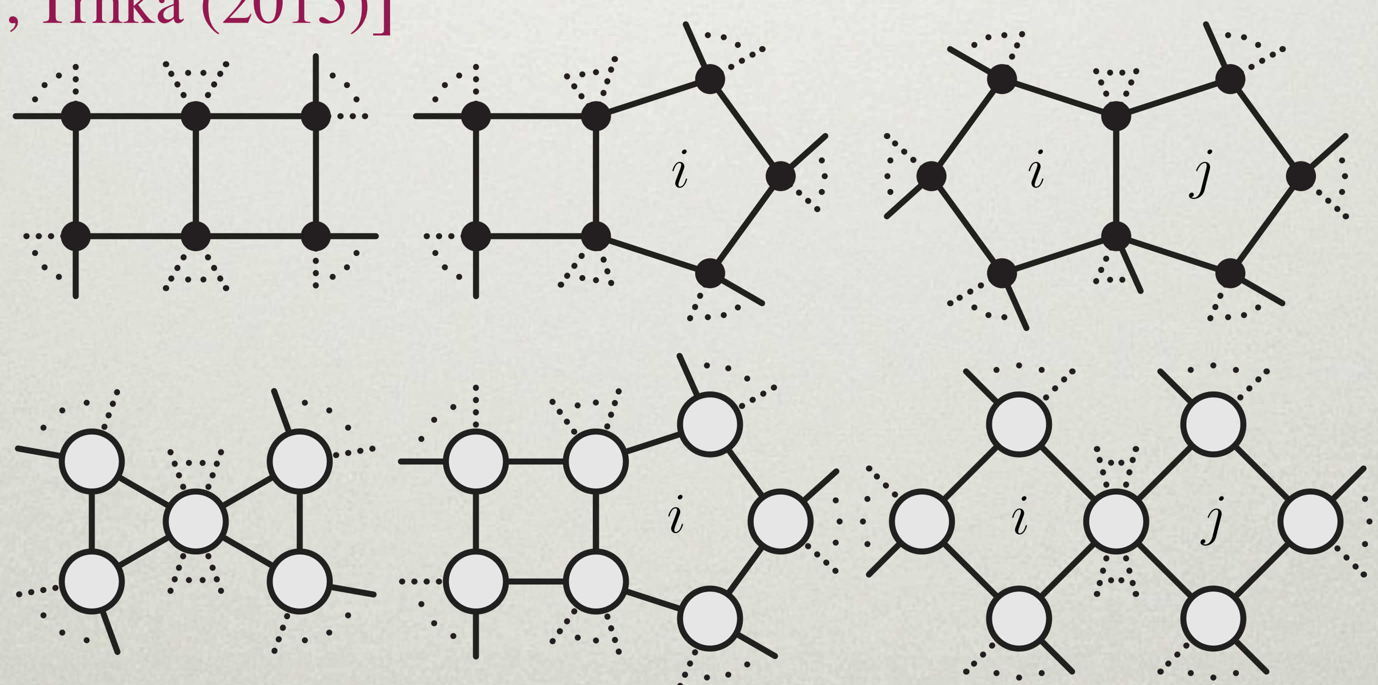

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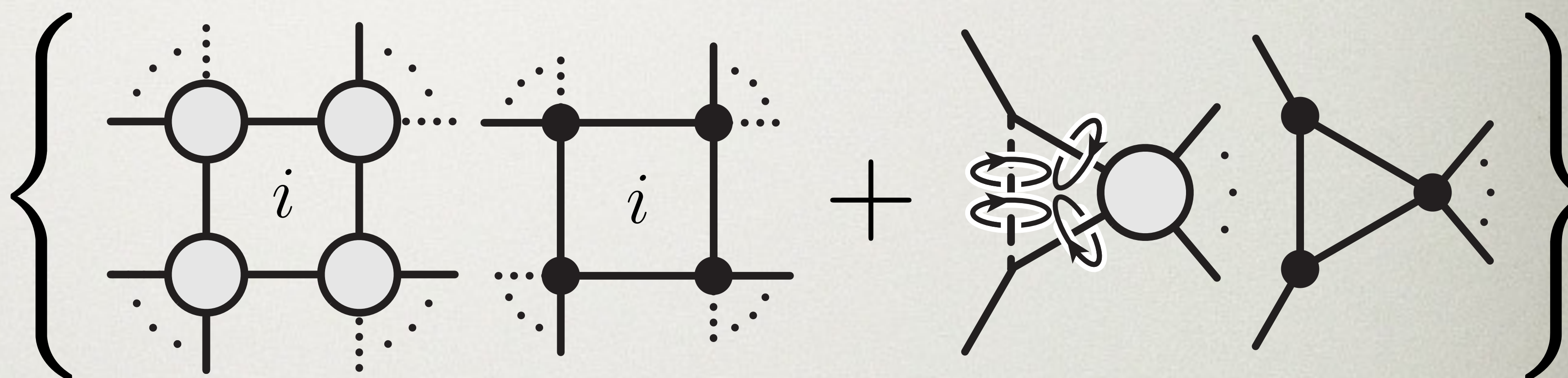

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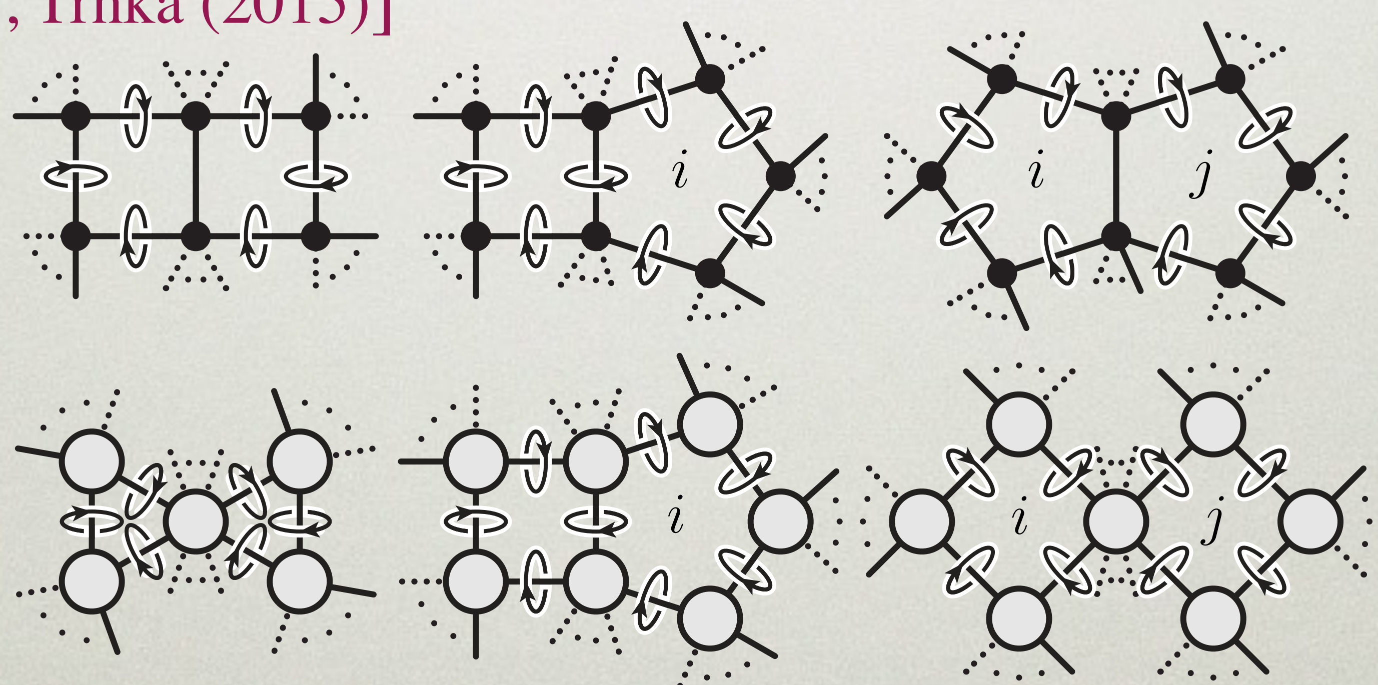

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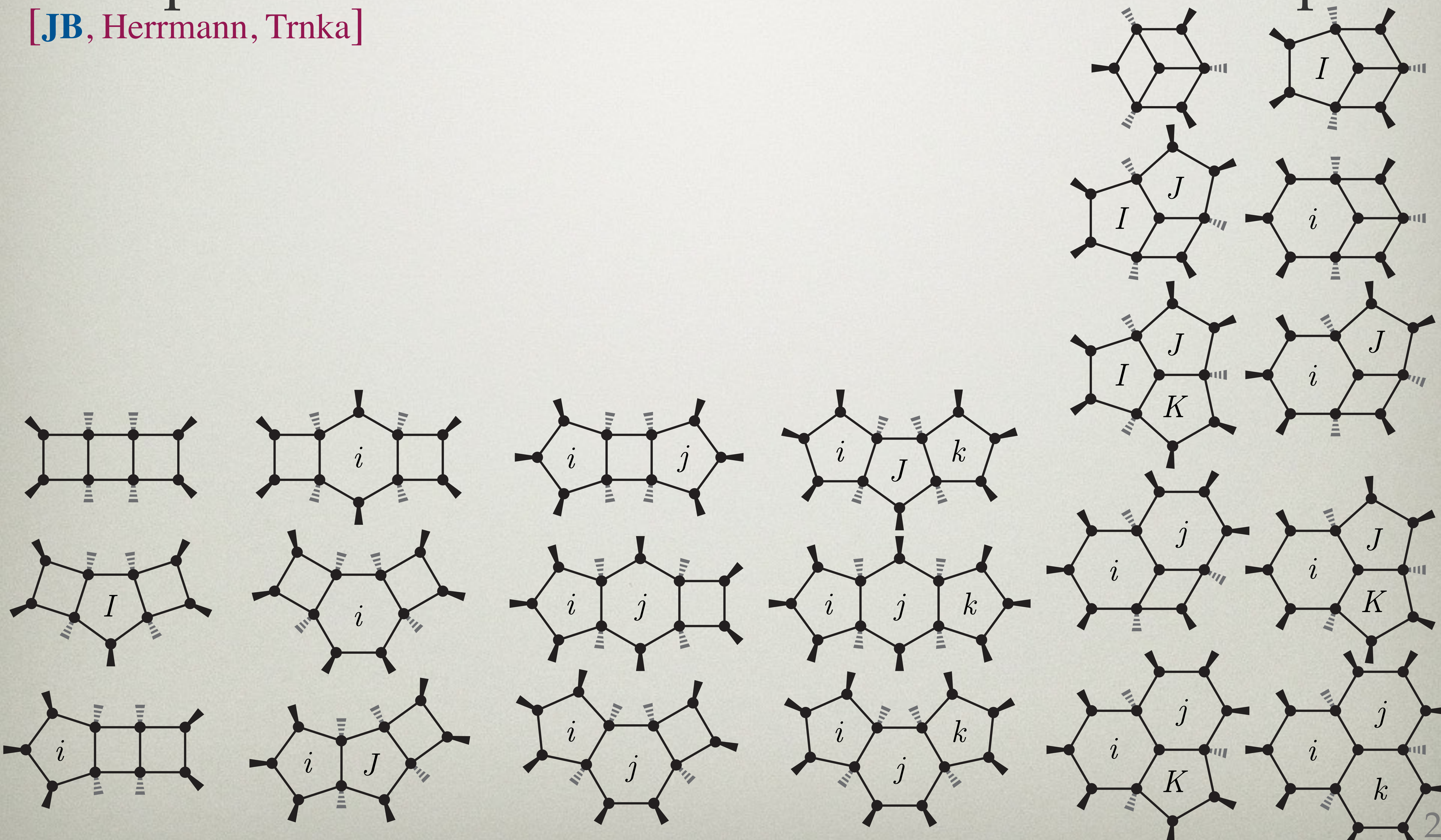

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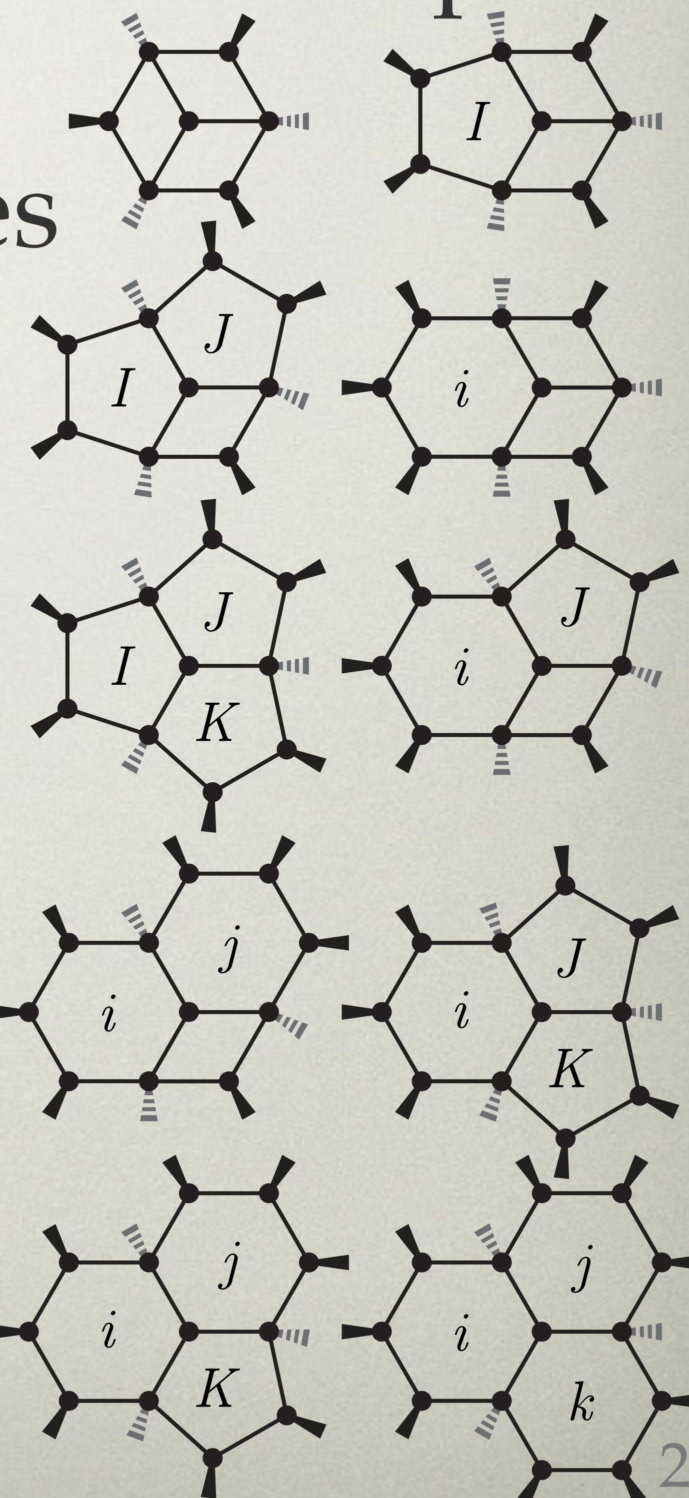

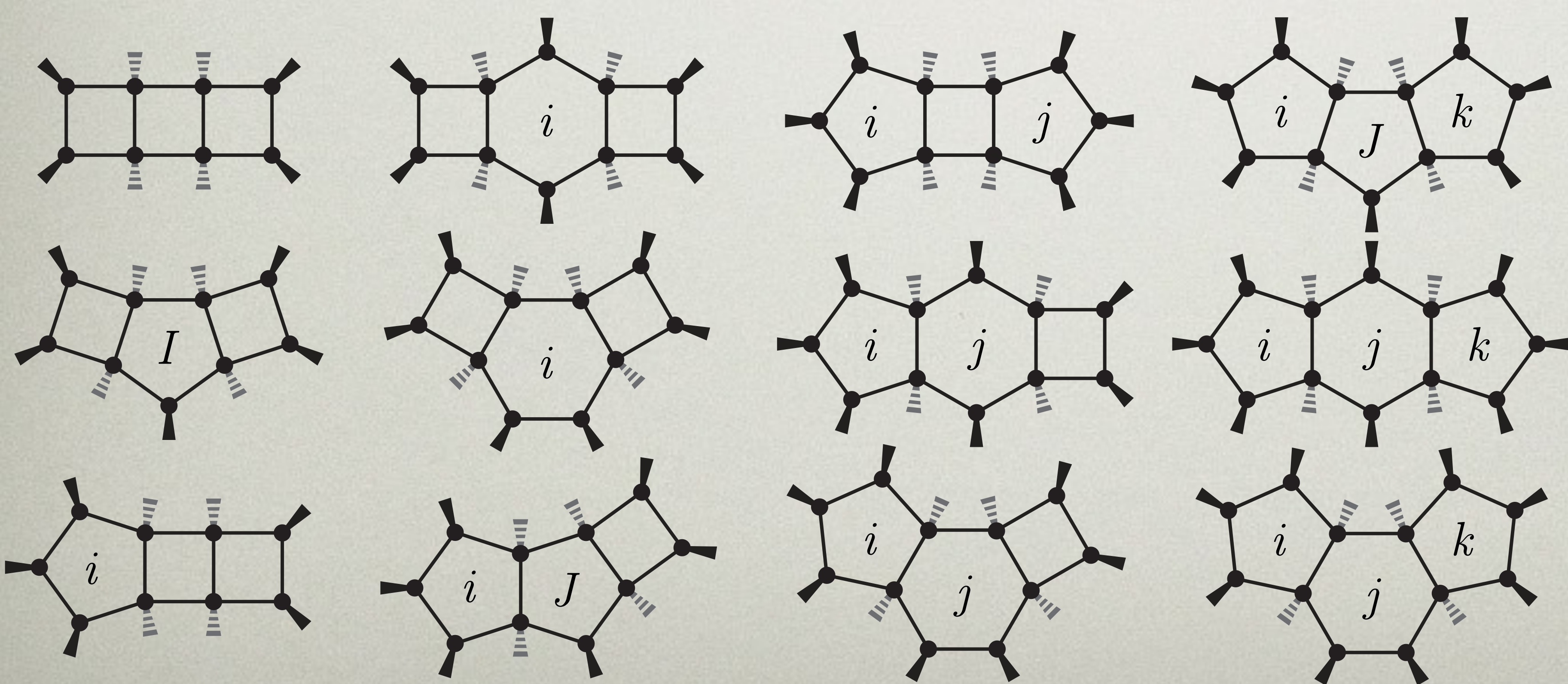

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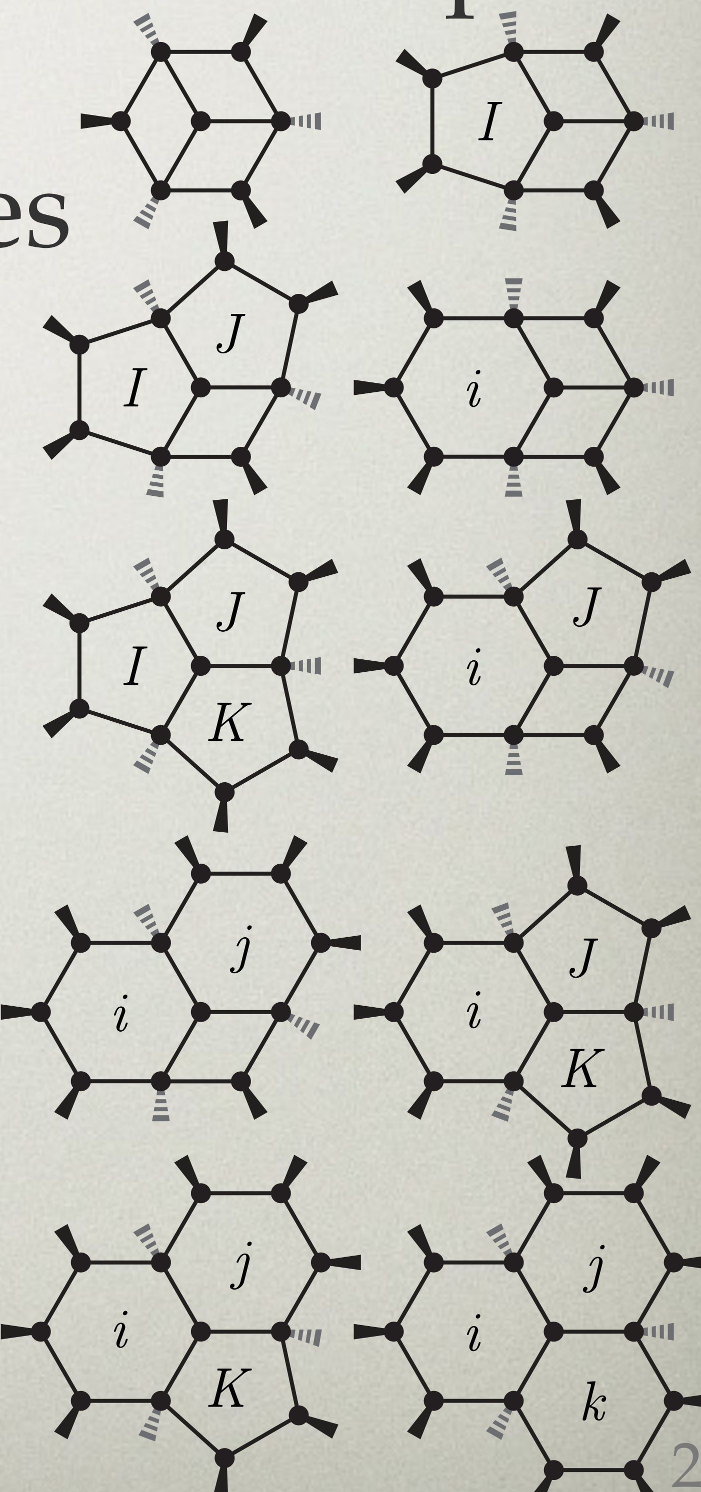
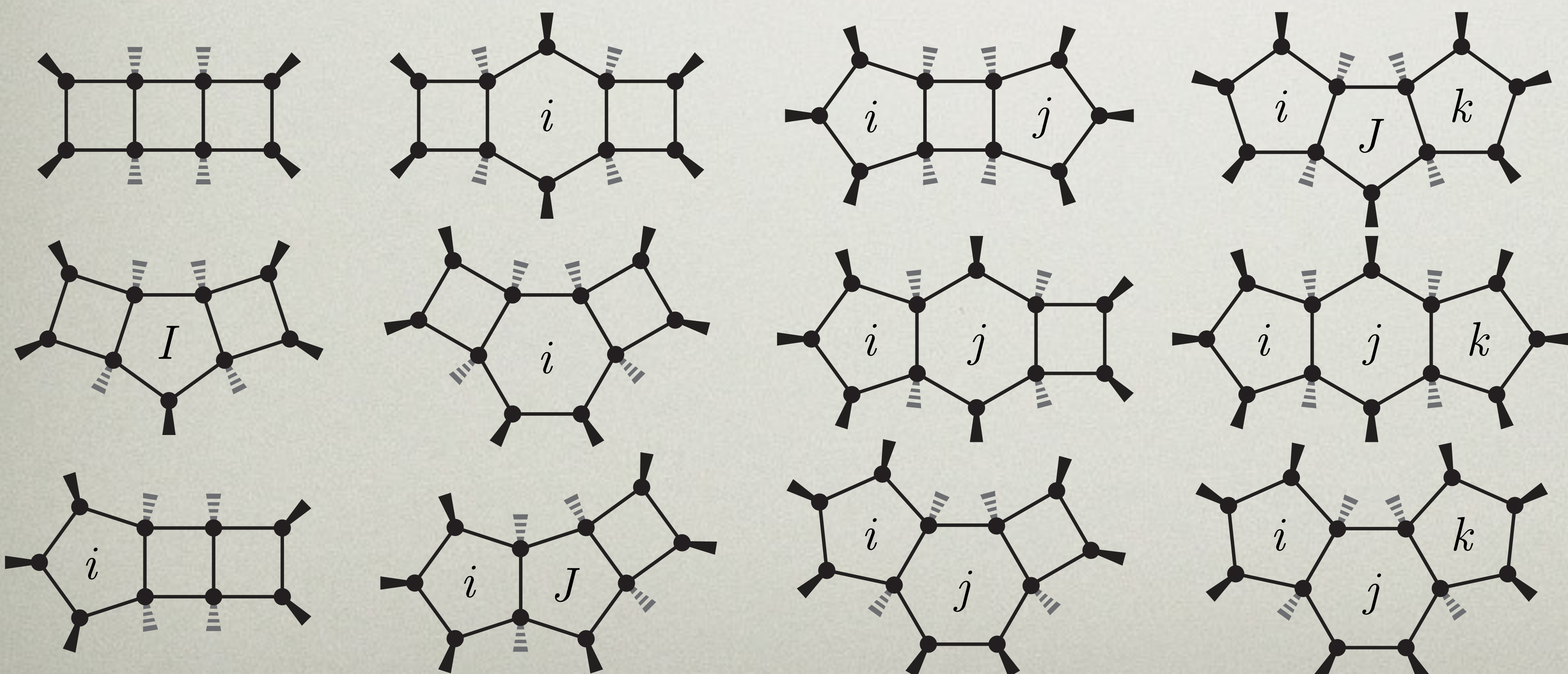
- Construction of N=8 integrands?





Probing the Limits of Our Reach

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[*The Parking Spot Escalation* (29 Nov. 2012)]

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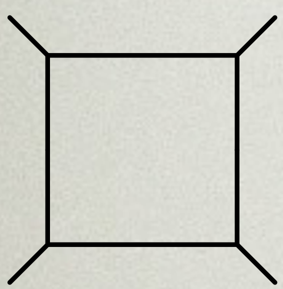

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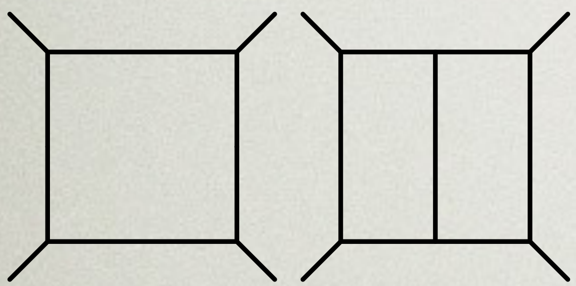
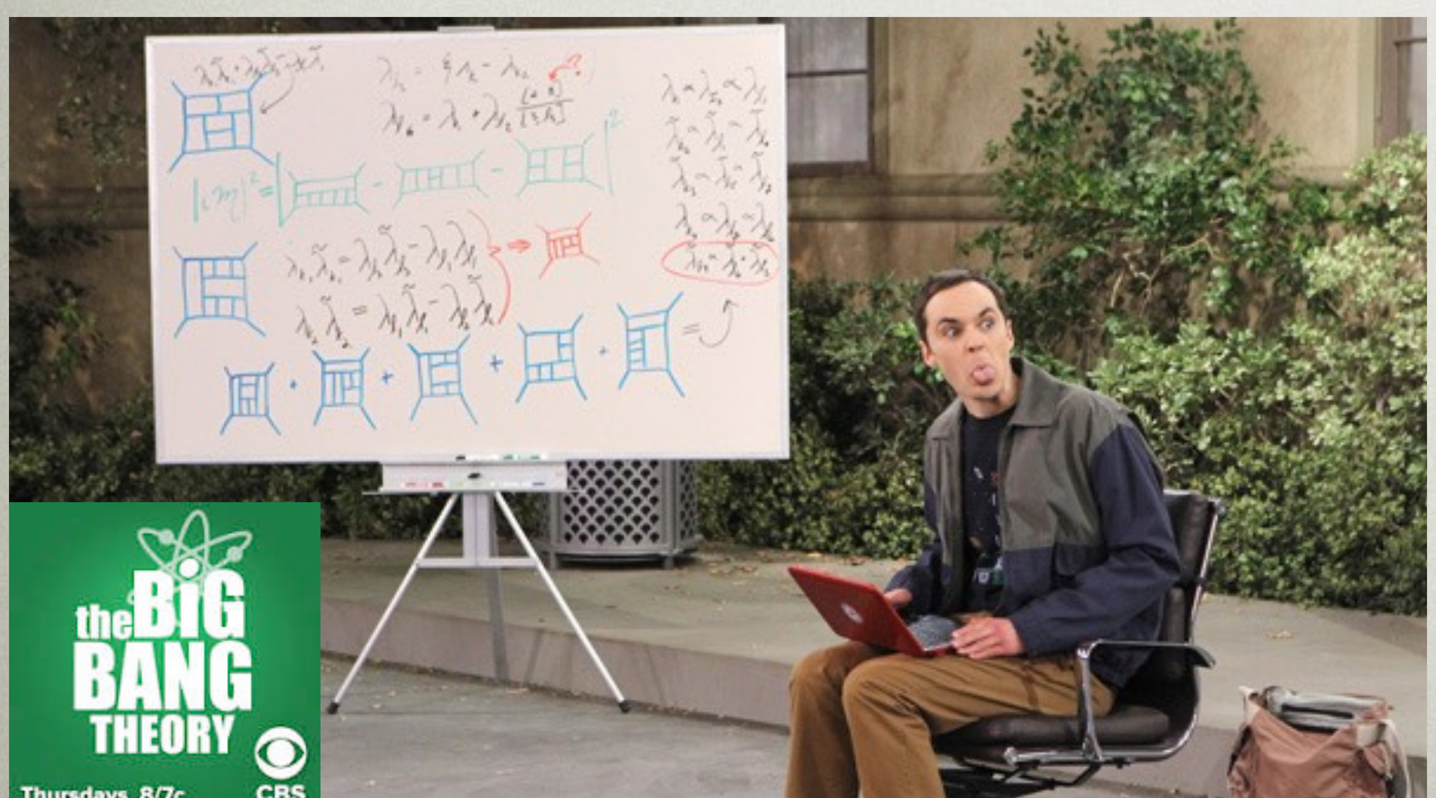
1997 [Bern, Rozowsky, Yan]





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generalized unitarity

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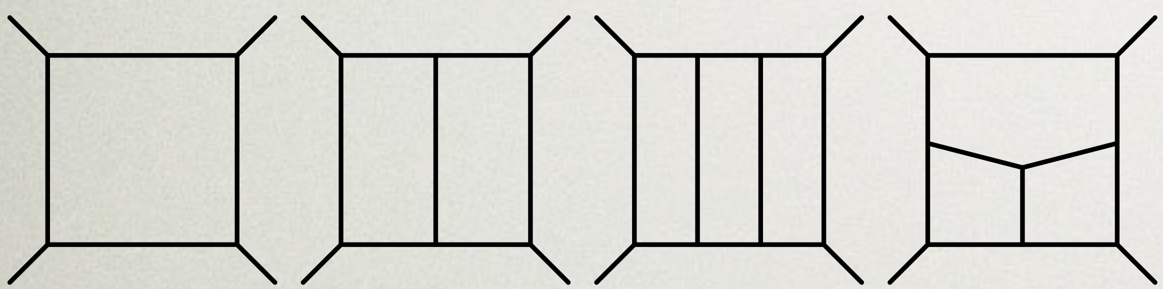
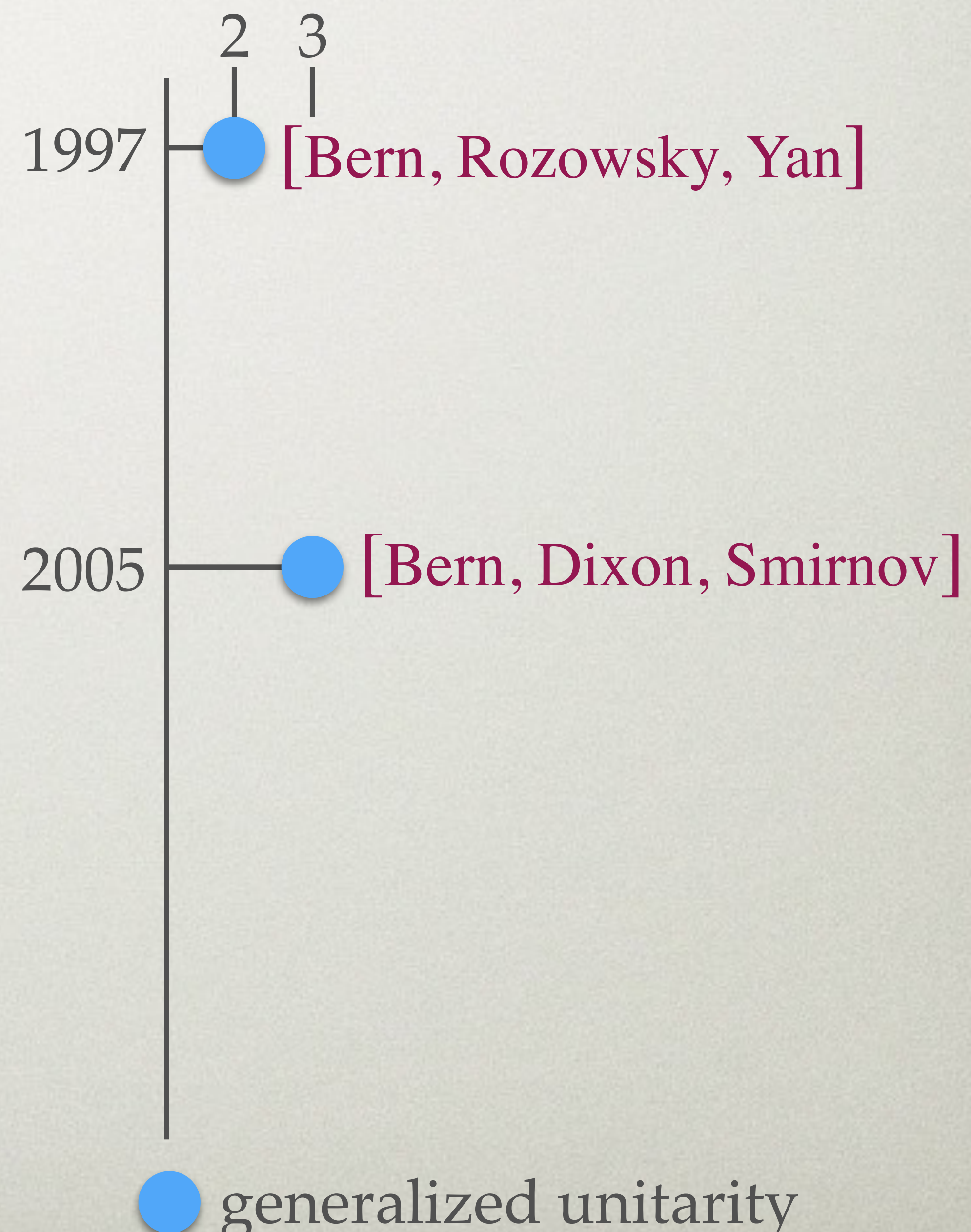


[*The Parking Spot Escalation* (29 Nov. 2012)]



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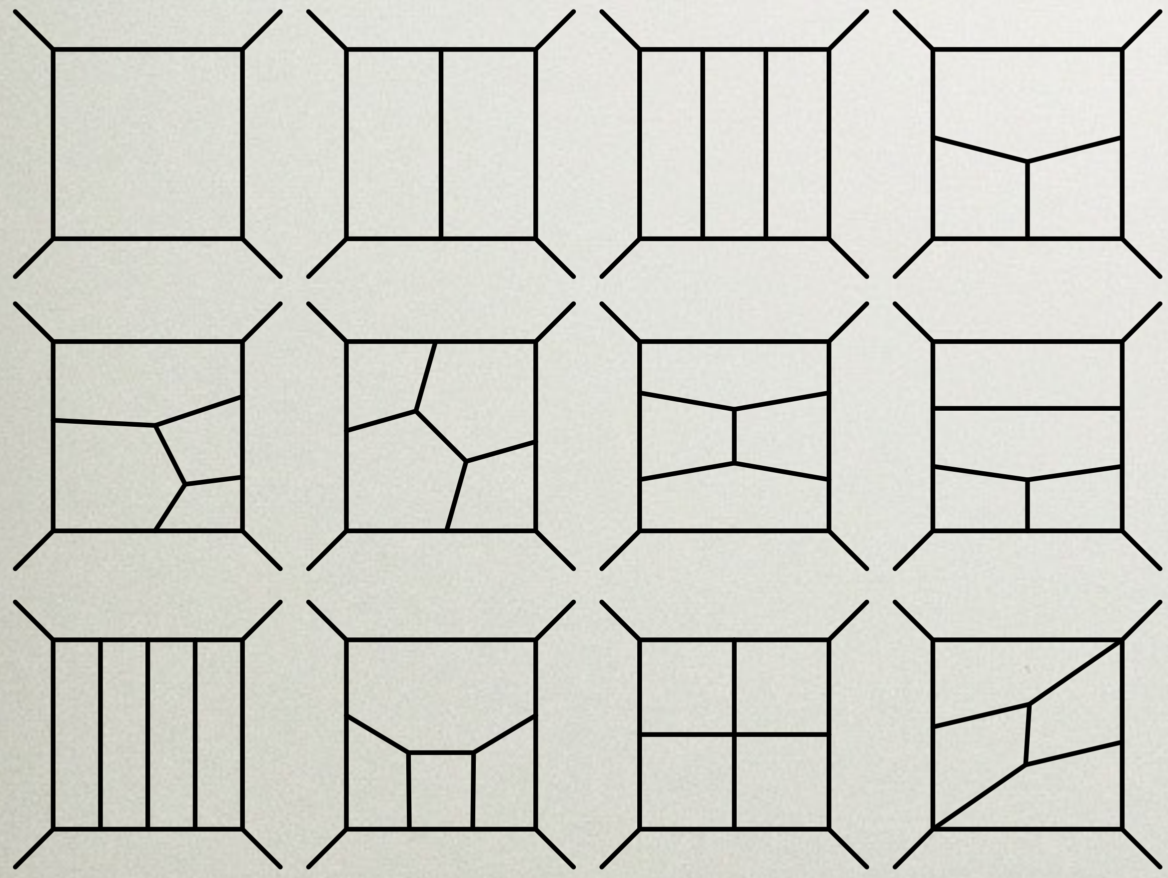

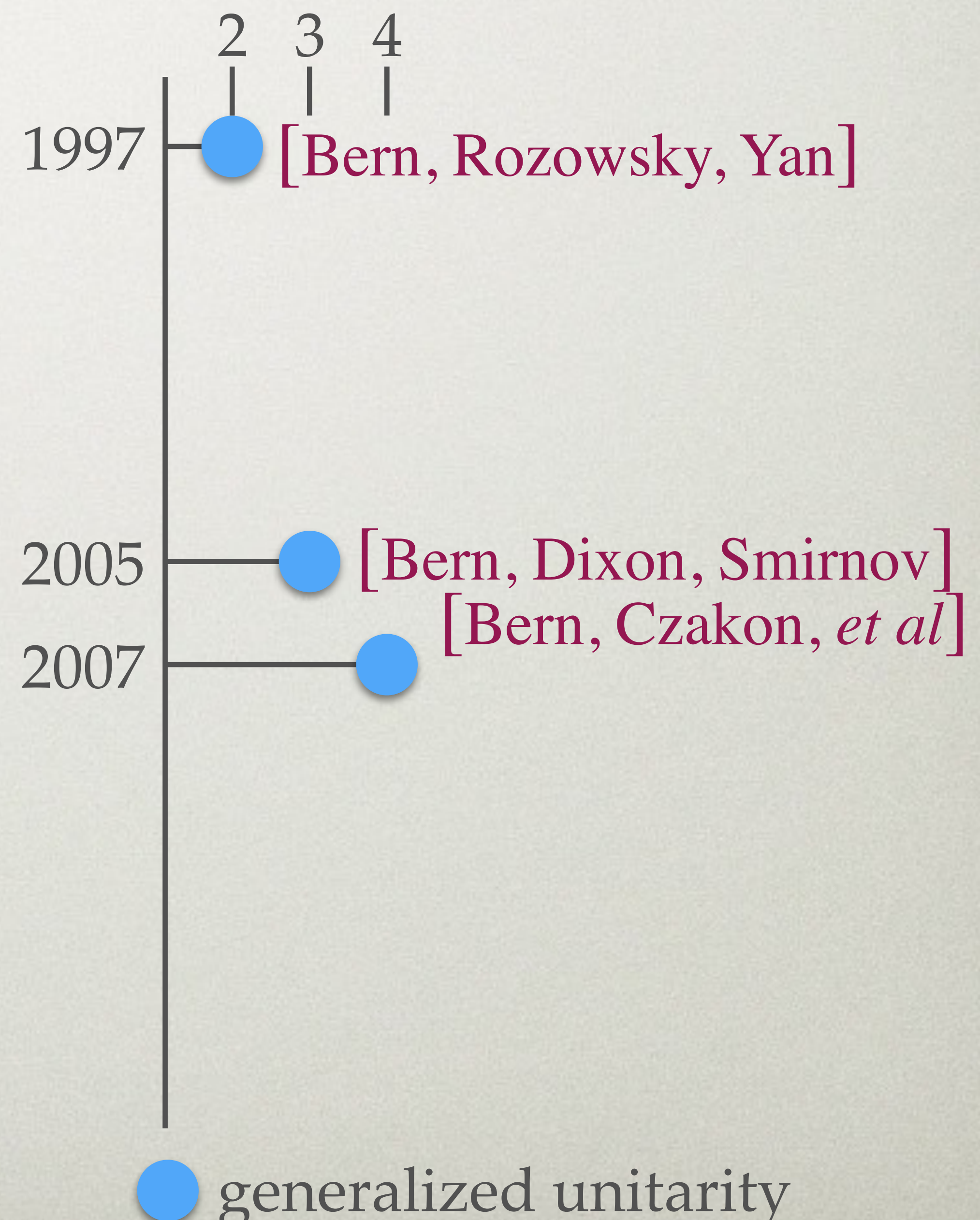

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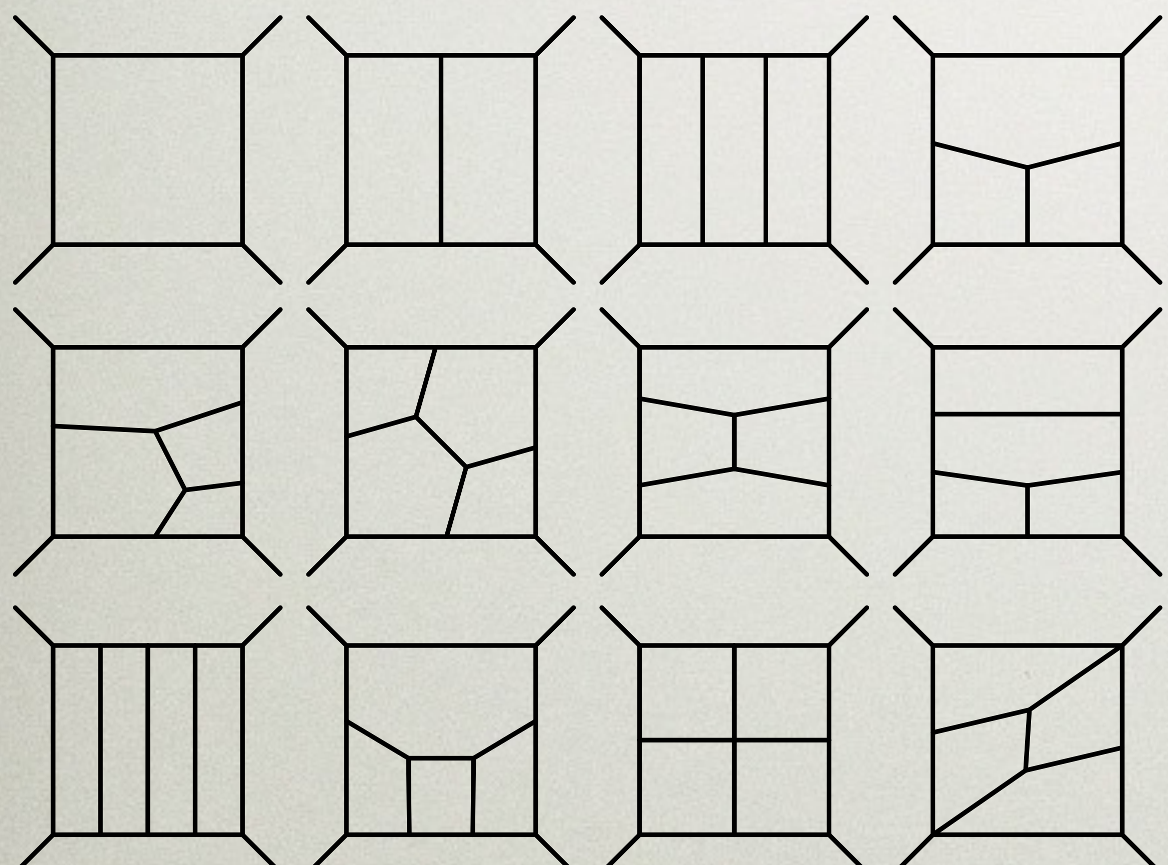

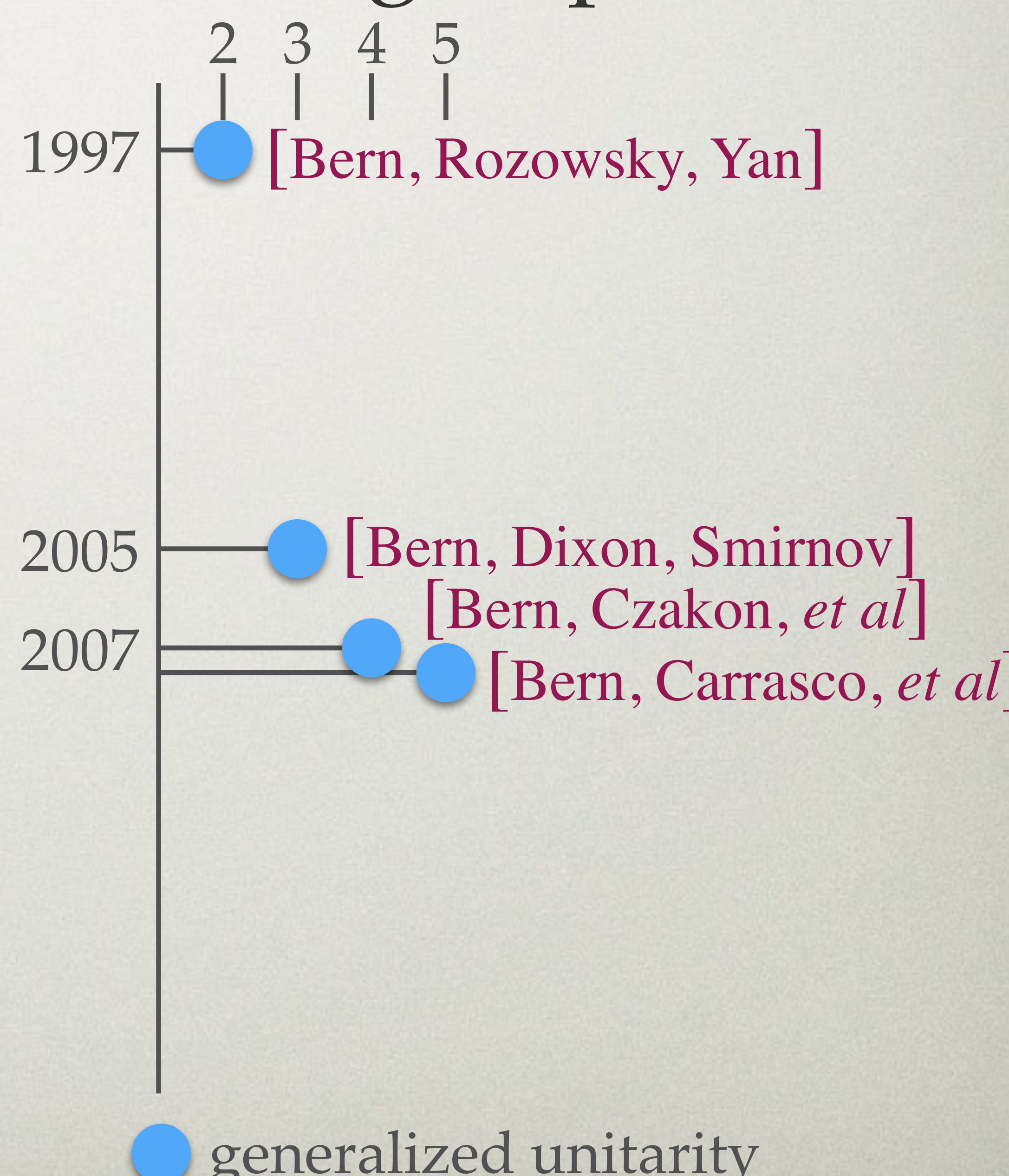

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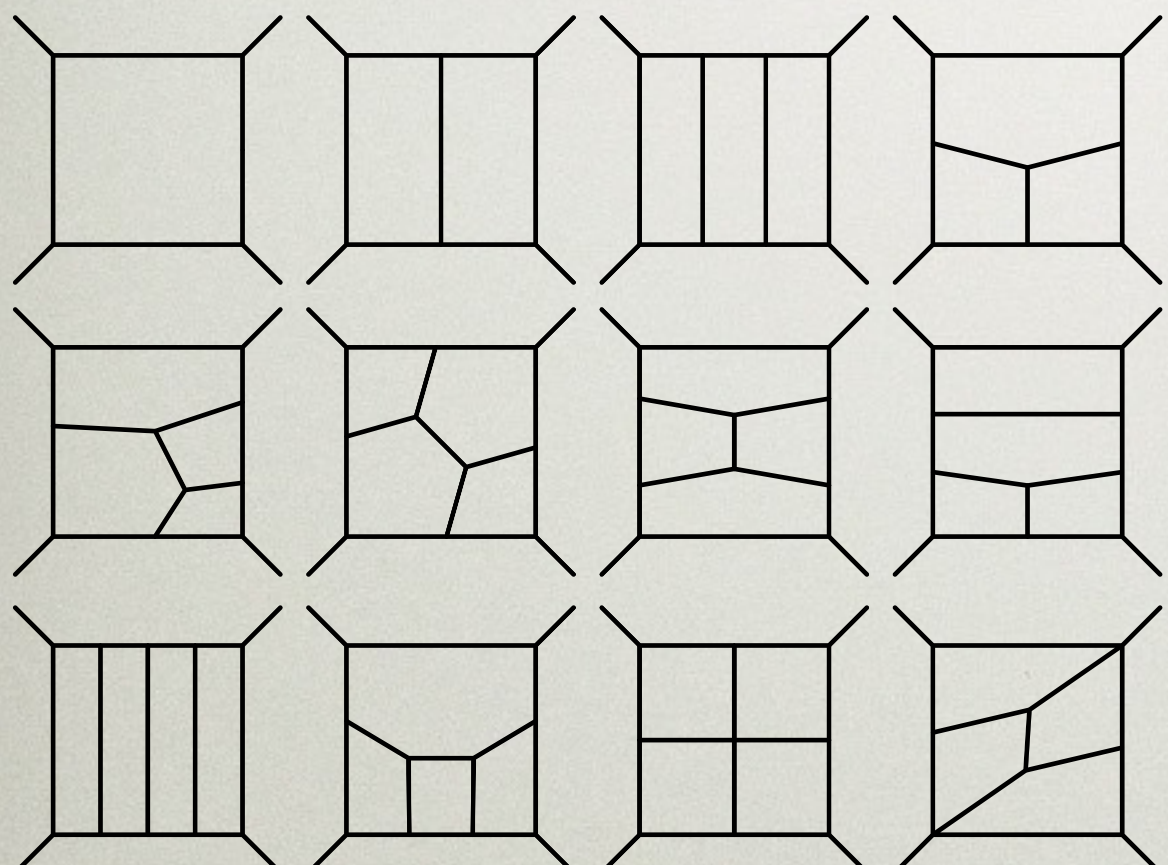

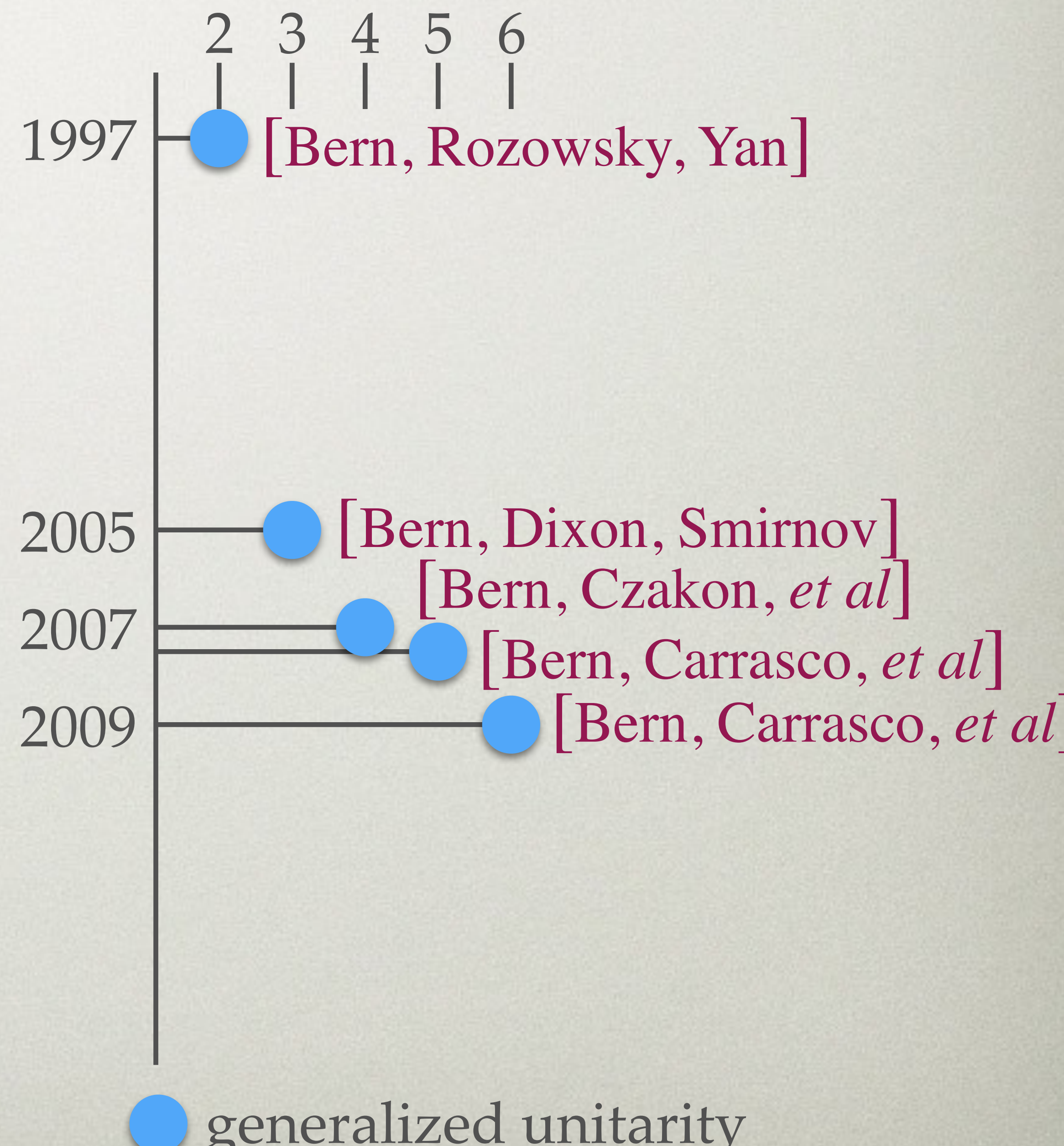

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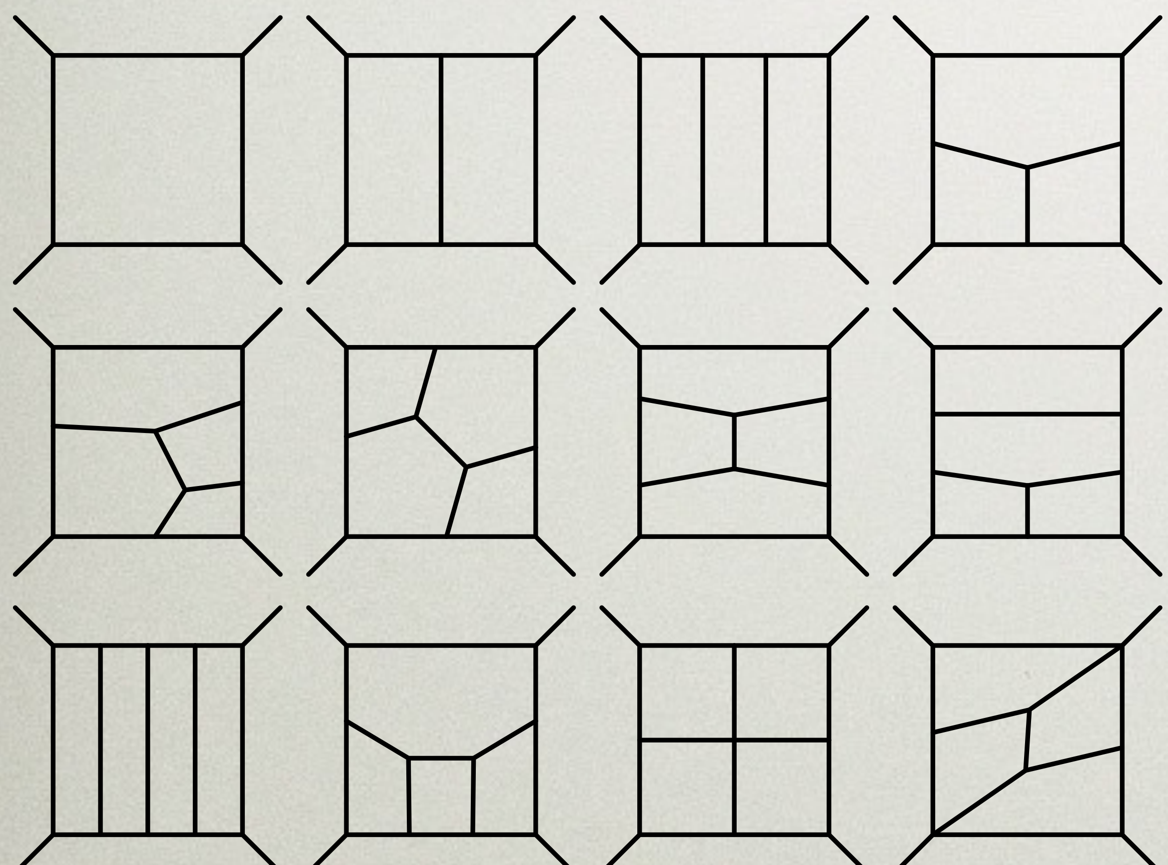

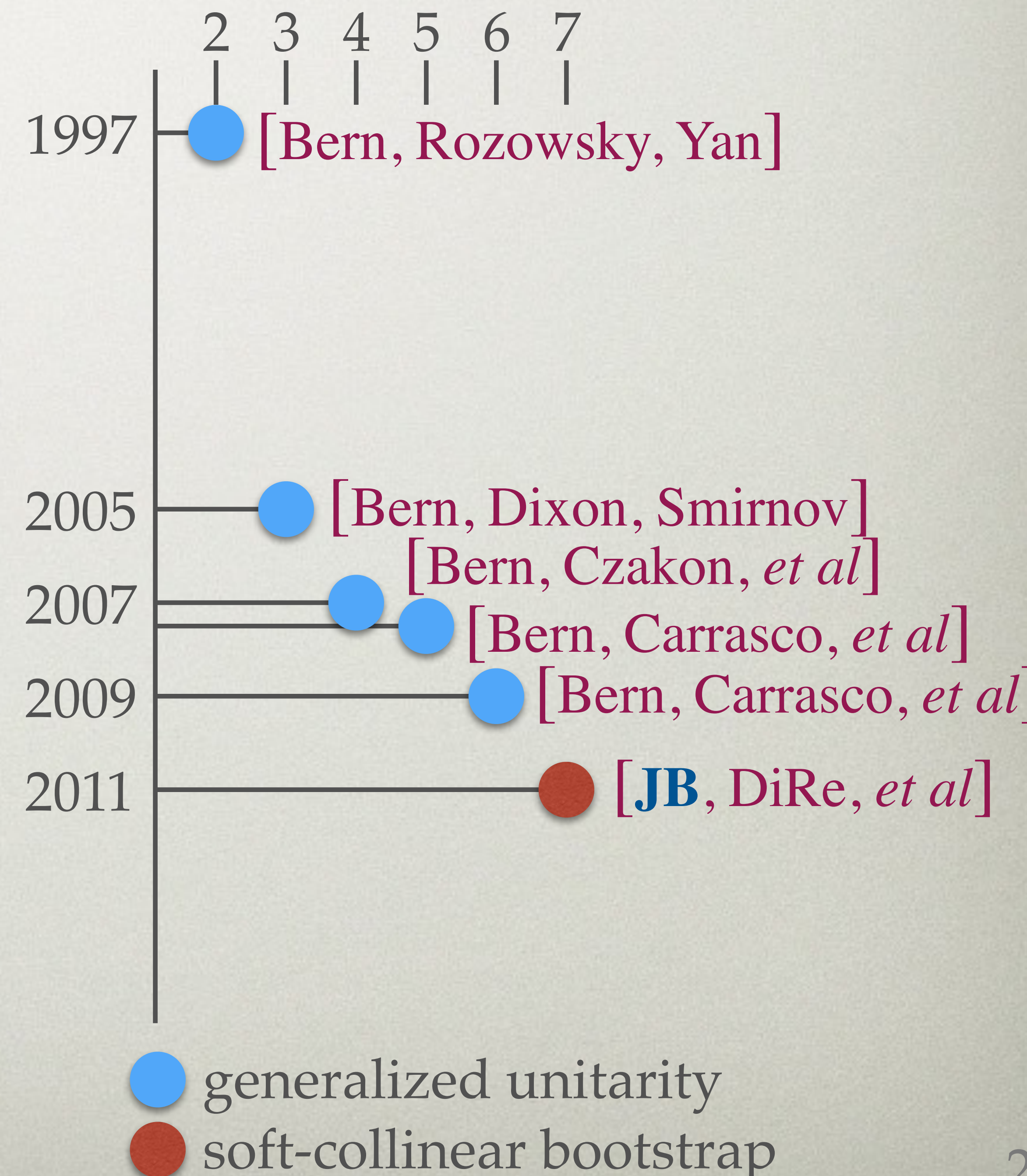

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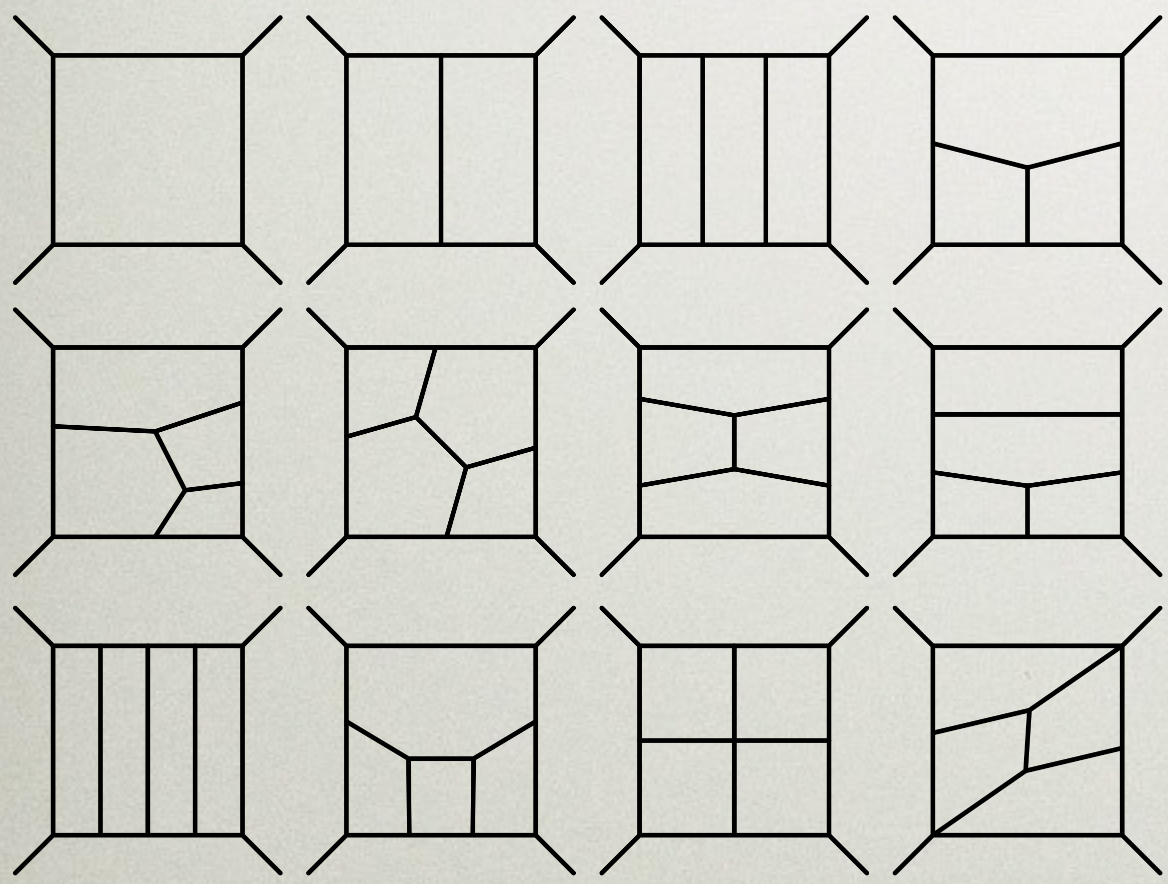

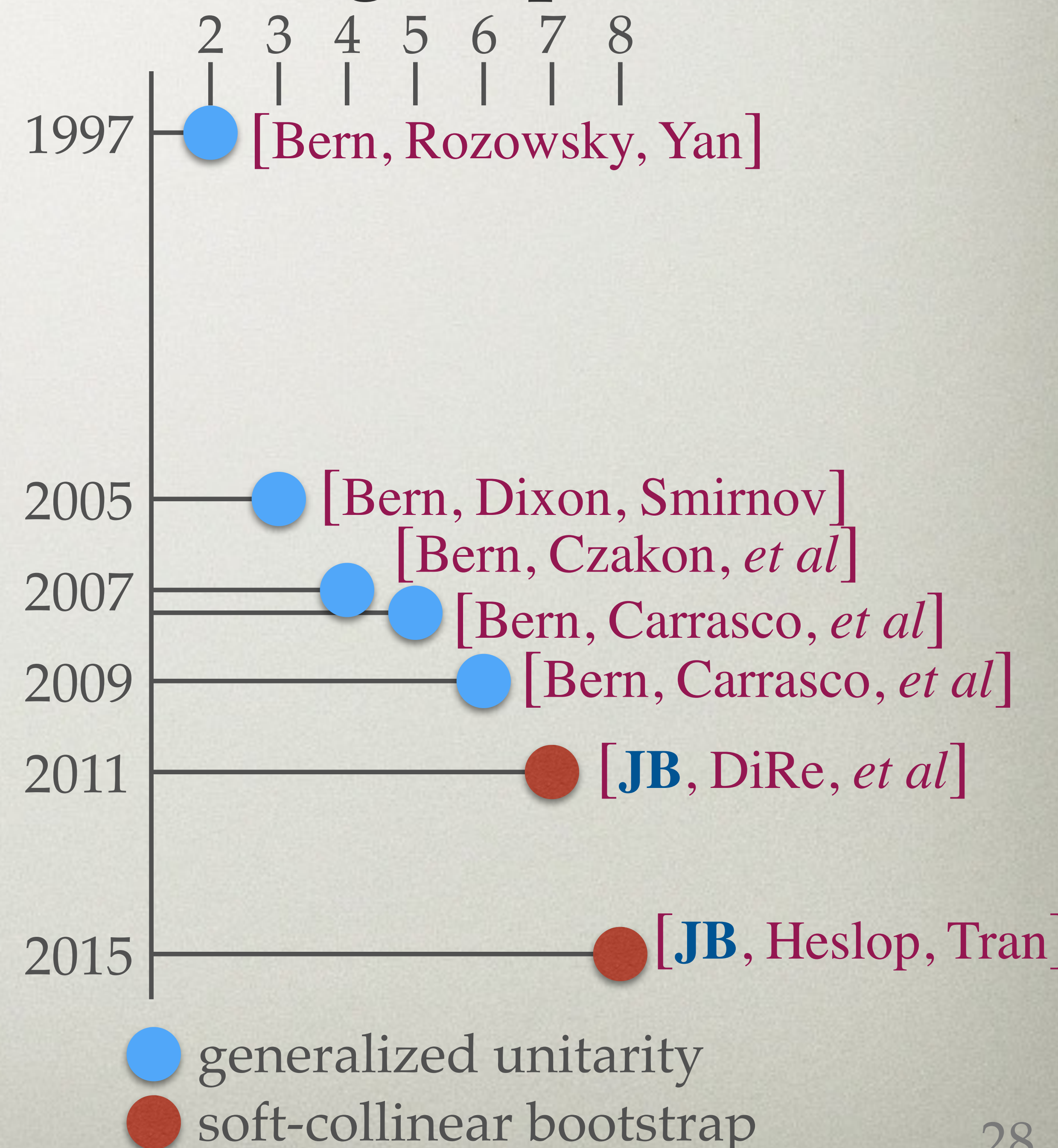

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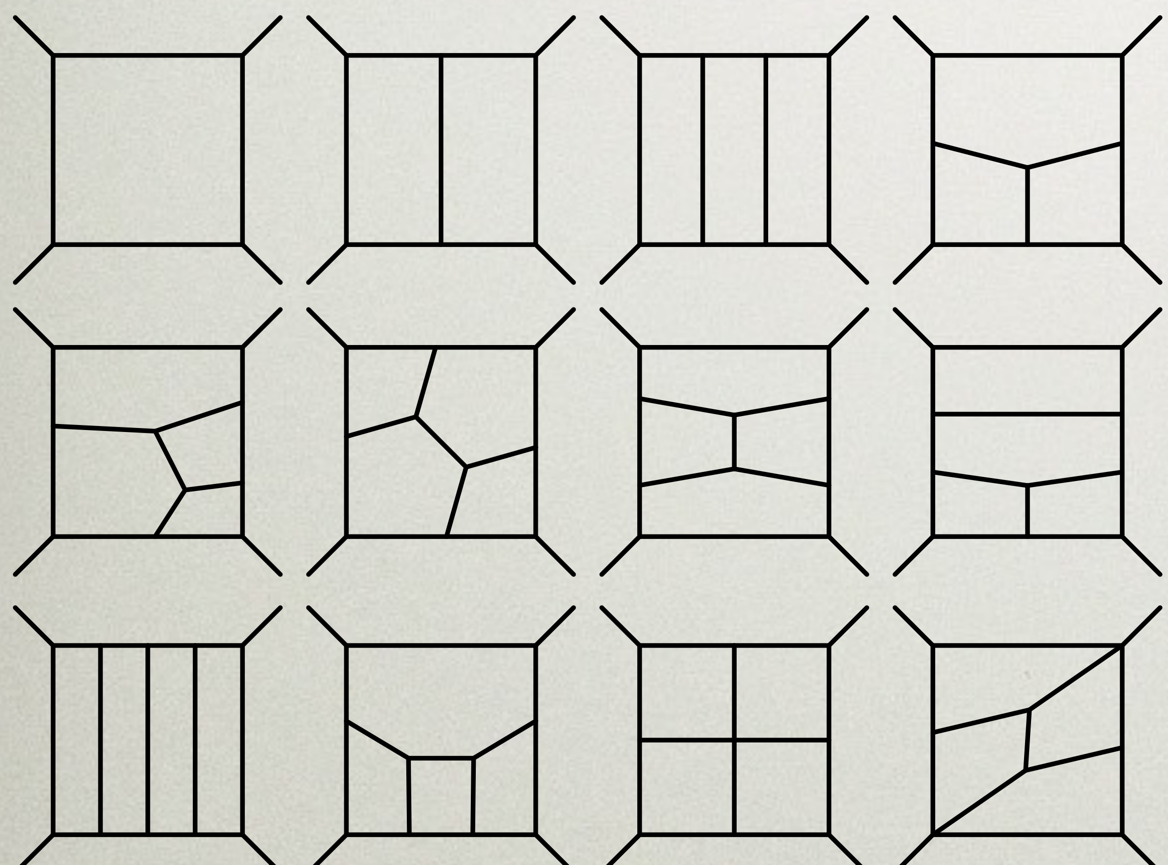

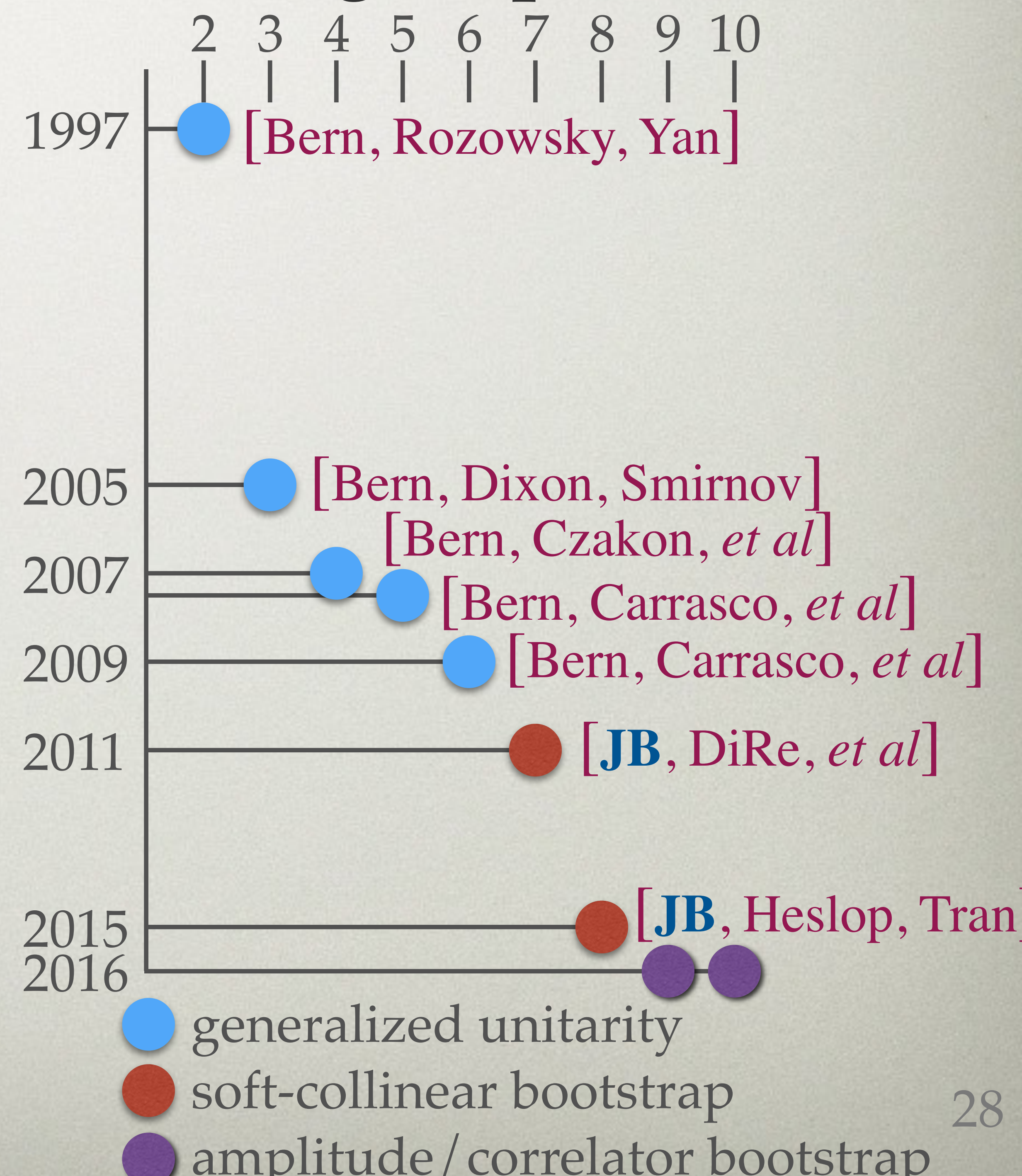

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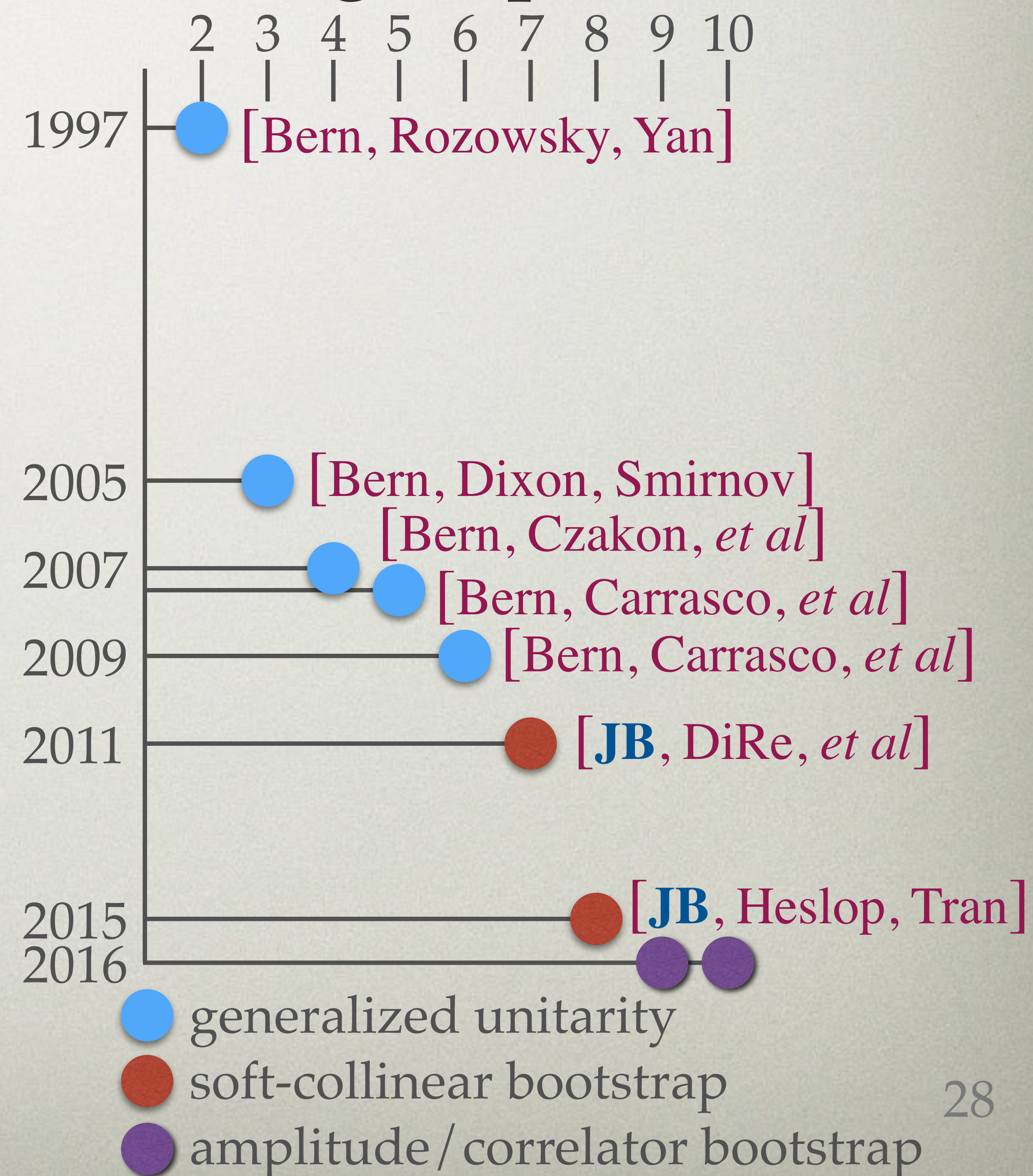
loop	#Feynman	#DCI ints	#*f*-graphs
1	940	1	1
2	47,380	1	1
3	4,448,500	2	1
4	6.723×10^{8}	8	3
5	1.483×10^{11}	34	7
6	4.484×10^{13}	229	26
7	1.780×10^{16}	1,873	127
8	8.969×10^{18}	19,949	1,060
9	5.592×10^{21}	247,856	10,525
10	4.226×10^{24}	3,586,145	136,433



Novel Approaches, Novel Insights

- The **soft-collinear bootstrap**: the four-point amplitude is *uniquely* fixed by the criterion that its logarithm is at most log-squared divergent: [**JB**, DiRe, *et al*]

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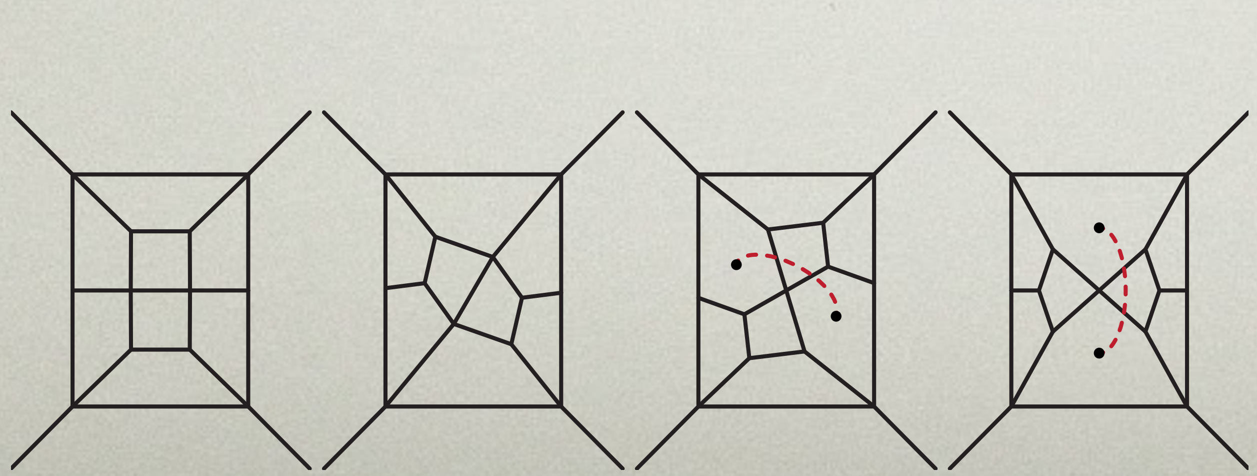

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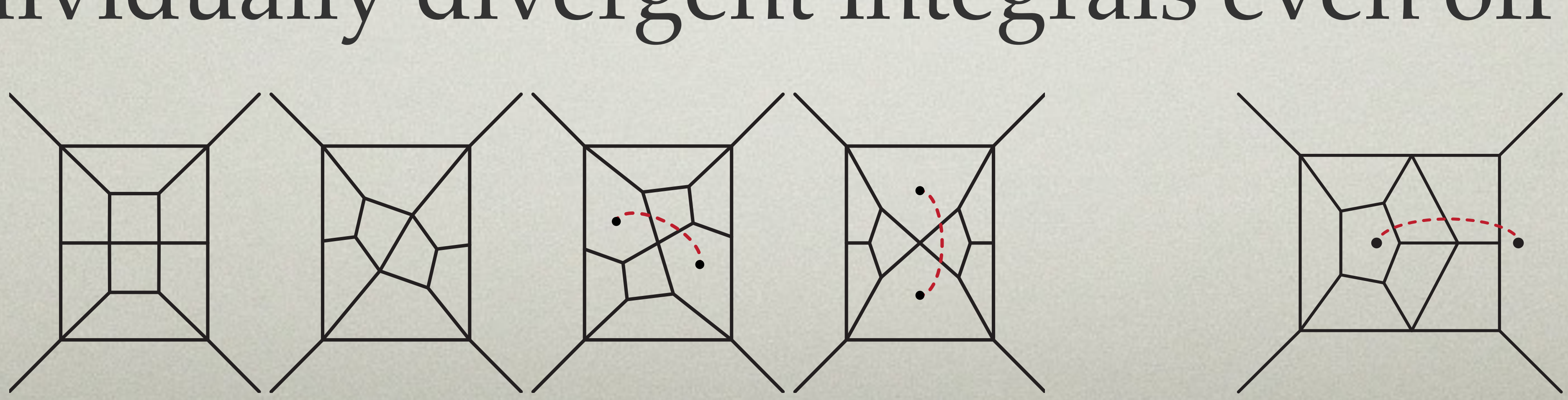

Amplitude/Correlator Duality

- There is a lossless translation between the *off-shell* correlator and the *on-shell* scattering amplitude!

$$\mathcal{F}^{(\ell)} \equiv \frac{1}{2}\frac{1}{\xi^{(4)}}\frac{\mathcal{G}_4^{(\ell)}(x_i)}{\mathcal{G}_4^{(0)}(x_i)} \qquad \xi^{(n)} \equiv \prod_{a=1}^{n} p_a^2 (p_a + p_{a+1})^2$$

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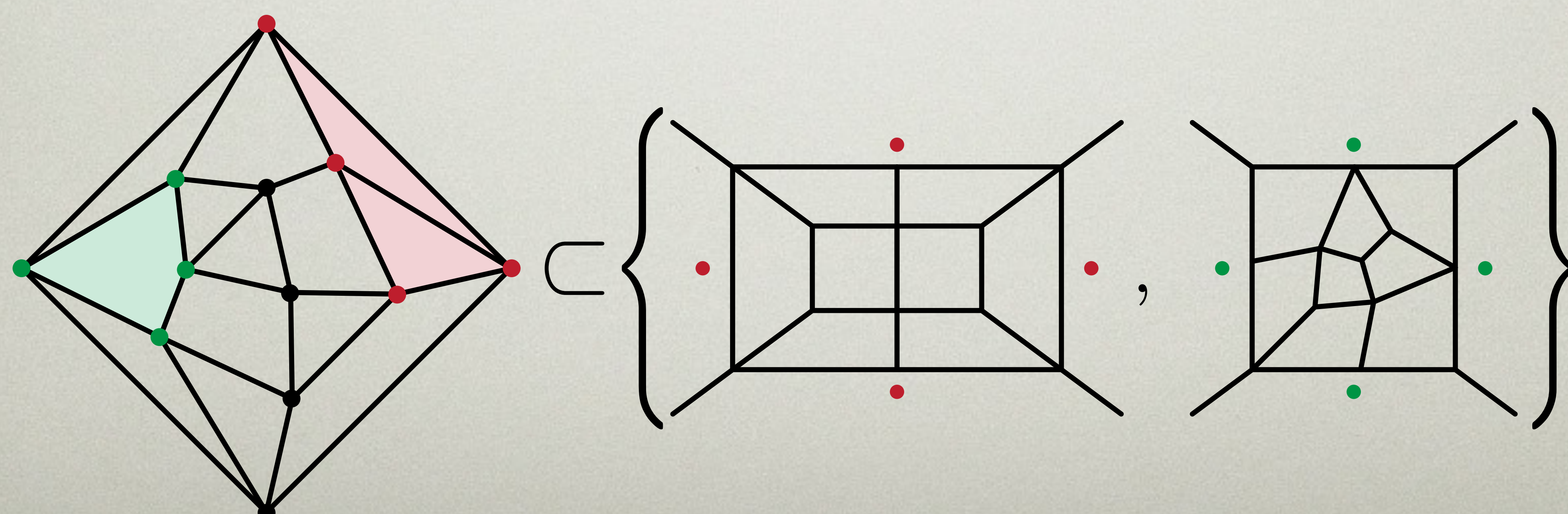

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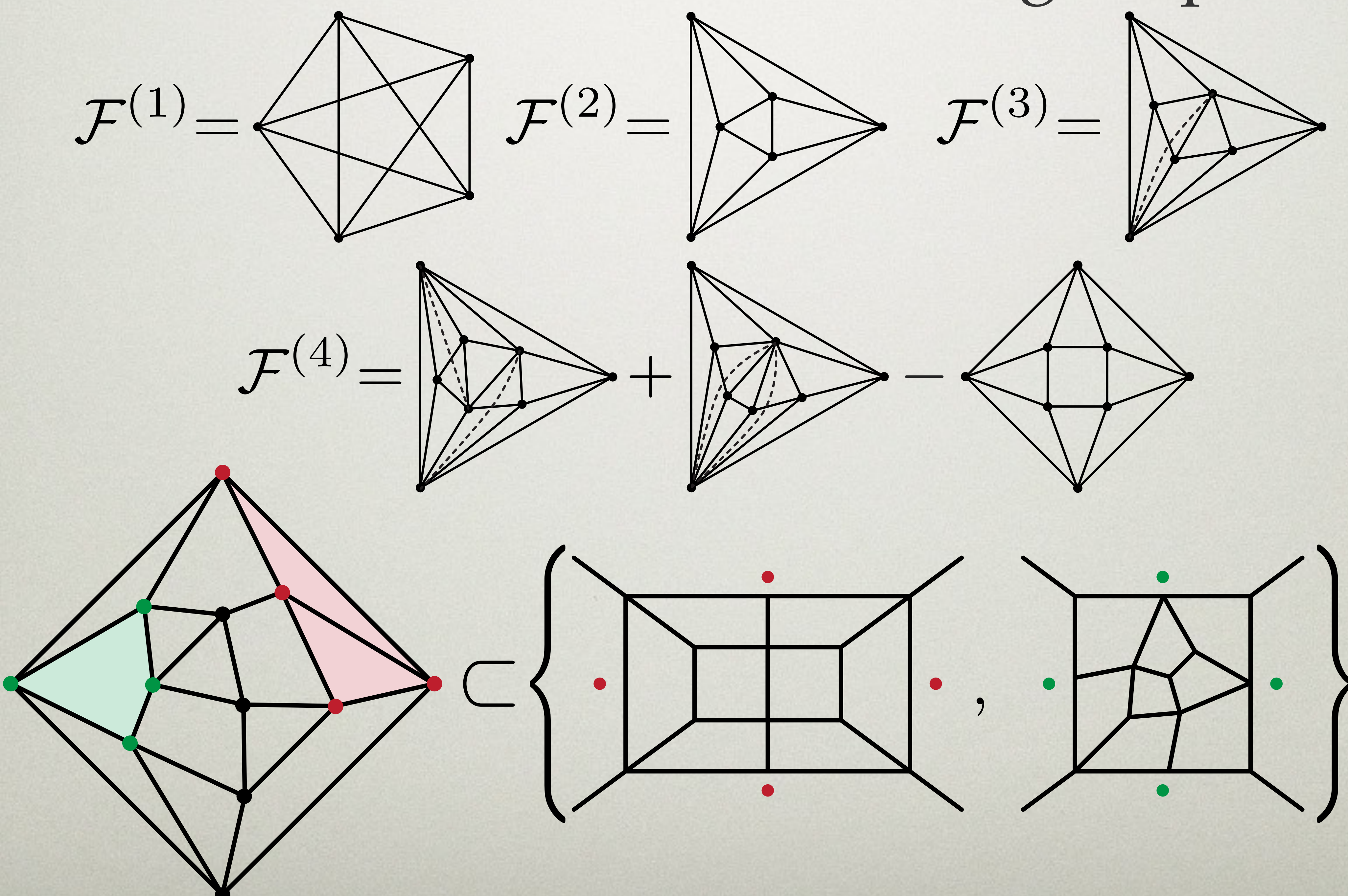

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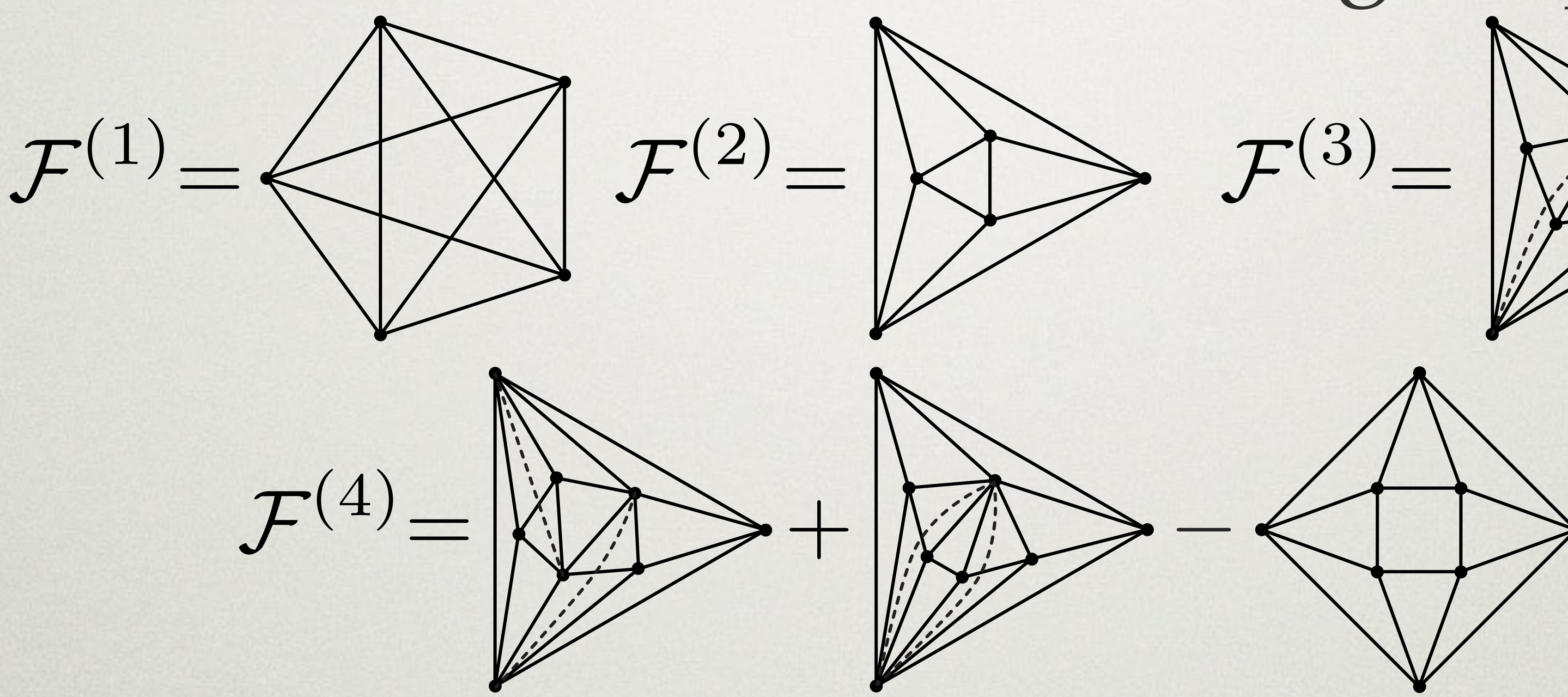

- Importantly, the *four*-point correlator contains (complete?) information of *all n*-point amplitudes!

$$\lim_{\substack{n\text{-point}\\ \text{light-like}}}\left(\xi^{(n)}\mathcal{F}^{(\ell)}\right)=\frac{1}{2}\sum_{k=0}^{n-4}\mathcal{A}_n^k\mathcal{A}_n^{n-k-4}/(\mathcal{A}_n^{n-4,(0)})$$

Simplicity Surviving Loop Integration and Future Directions

Can Simplicity Survive Integration?

- Loop integration remains a serious challenge for preserving simplicity of observable quantities
  - finite observables given in terms of divergent quantities requiring regularization (*is this necessary*?)
  - most regularization schemes *severely* break symmetries **known to exist** for finite observables
  - most versitile integration *techniques* spoil symmetries along the way
- The *traditional* toolbox for loop integration can be theoretically opaque/computationally intractable

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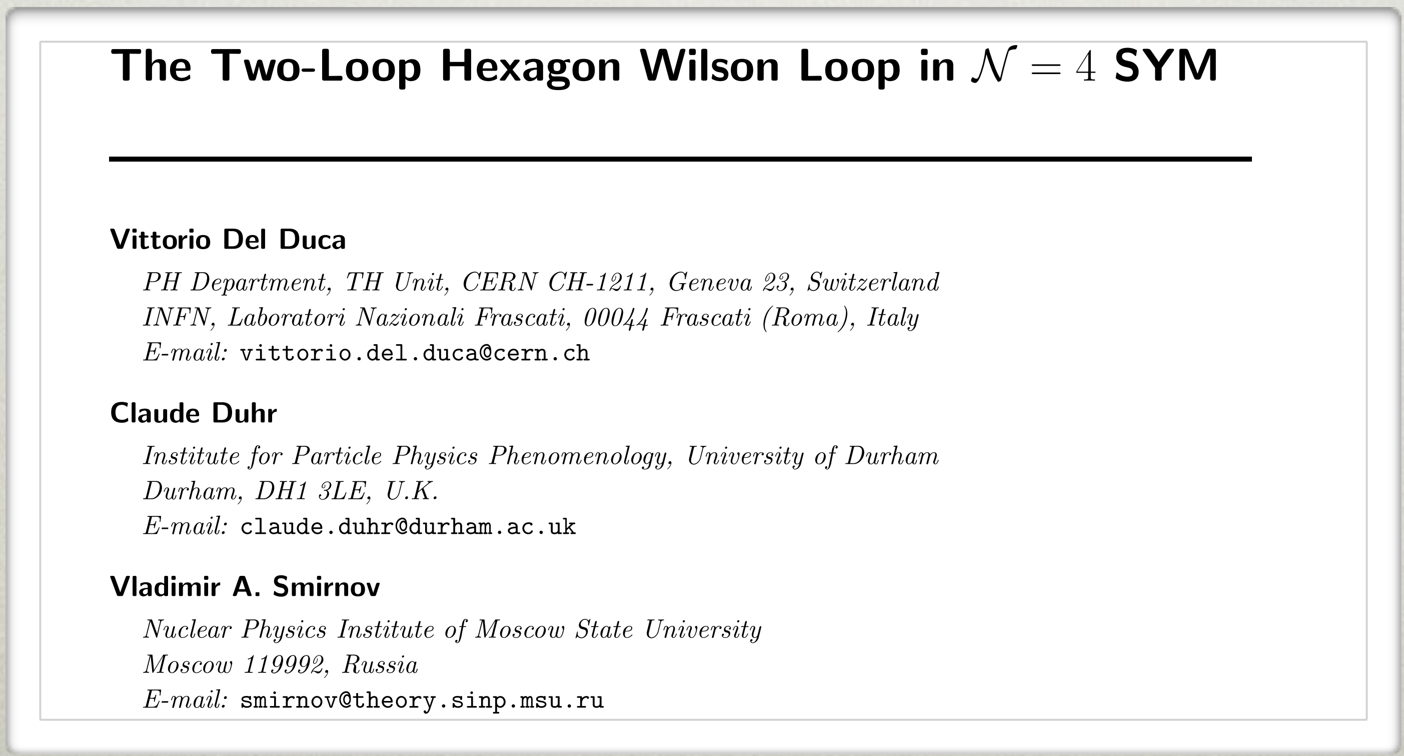

The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

Vittorio Del Duca
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INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy
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Vladimir A. Smirnov
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[Del Duca, Duhr, Smirnov (2010)]

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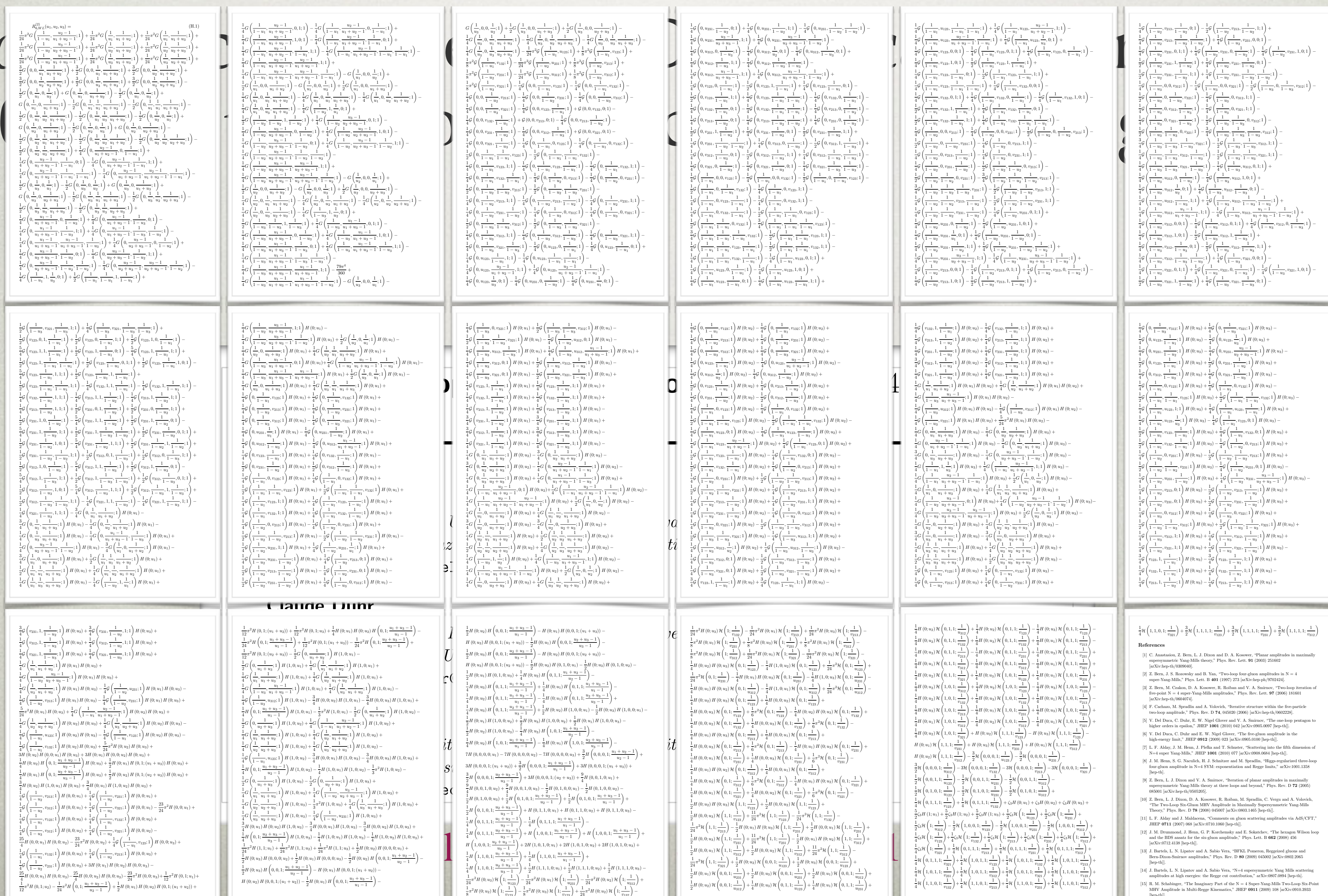

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Classical Polylogarithms for Amplitudes and Wilson Loops

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[1] *Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA*
[2] *Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA*

We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.

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- (Nearly) *finite* integrals for *finite* observables

  [**JB**, Dixon, Dulat, Panzer, (*in prep*)]

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