The Surprising Simplicity of Scattering Amplitudes

Jacob Bourjaily University of Copenhagen

Annual UK Theory Christmas Meeting IPPP, Durham University



Organization and Outline

 Spiritus Movens: the surprising simplicity of QFT Basic Building Blocks: on-shell functions tree amplitudes (and graphs of trees) the Grassmannian duality (massless, 4d) Constructing Loop Amplitude Integrands on-shell, all-loop recursion relations generalized and prescriptive unitarity amplitude/correlator bootstrap Loop Integration & Future Directions

Surprising Simplicities of Quantum Field Theory

Traditional Description of QFT

 Quantum Field Theory: the marriage of (special) relativity with quantum mechanics

 Theories (can be) specified by Lagrangians—or equivalently, by Feynman rules for virtual particles

$$\mathcal{L} \equiv -\frac{1}{4} \sum_{i} (F^a_{i\mu\nu})^2 + \sum_{I} \overline{\psi}_J (i D) \psi_J$$

 Predicted probability (*amplitudes*) from path integrals (over virtual 'histories'):



 $\int \mathcal{D}A \,\mathcal{D}\overline{\psi} \,\mathcal{D}\psi \,e^{i\int d^4x\mathcal{L}}$



 Predictions (often) made perturbatively, according to the loop expansion:

5

 Predictions (often) made perturbatively, according to the loop expansion:

 Predictions (often) made perturbatively, according to the loop expansion:

[Dirac (1933)]

 Predictions (often) made perturbatively, according to the loop expansion:

) =) + ()

[Dirac (1933)] [Feynman; Schwinger; Tomanaga (1947)]

 Predictions (often) made perturbatively, according to the loop expansion:

[Dirac (1933)] [Feynman; Schwinger; Tomanaga (1947)] [Petermann (1957)]

 Predictions (often) made perturbatively, according to the loop expansion:

[Dirac (1933)] [Feynman; Schwinger; Tomanaga (1947)] [Petermann (1957)] [Kinoshita (1990)]



= 2.00231930435801...
 $g_e^{\exp} = 2.00231930436146...$

[Feynman; Schwinger; Tomanaga (1947)] [Petermann (1957)] [Kinoshita (1990)]



= 2.00231930435801... [Feynman; Schwinger; Tomanaga (1947)] $g_e^{\exp} = 2.00231930436146...$ [Feynman; Schwinger; Tomanaga (1947)] [Petermann (1957)] [Kinoshita (1990)]

the most precisely tested idea in all of science!

Explosions of Complexity

 While ultimately correct, the Feynman expansion renders all but the simplest predictions those involving the fewest particles, at the

lowest orders of perturbation computationally intractable or theoretically inscrutable





By Zvi Bern, Lance J. Dixon and David A. Kosower

Bern, Dixon, Kosower, Scientific American (2012)



6

Background amplitudes crucial for e.g. colliders



Background amplitudes crucial for e.g. colliders



Once considered computationally intractable

Background amplitudes crucial for e.g. colliders

Supercollider physics

E. Eichten

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

I. Hinchliffe

Lawrence Berkeley Laboratory, Berkeley, California 94720

K. Lane

The Ohio State University, Columbus, Ohio 43210

C. Quigg

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

Eichten *et al.* summarize the motivation for exploring the 1-TeV $(=10^{12} \text{ eV})$ energy scale in elementary particle interactions and explore the capabilities of proton-(anti)proton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.

[*Rev.Mod.Phys.* **56** (1984)]



Once considered computationally intractable

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two—four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

Background amplitudes crucial for e.g. colliders



Once considered computationally intractable

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two—>four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

Background amplitudes crucial for e.g. colliders

220 diagrams—thousands of terms

Once considered computationally intractable

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two—>four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

Background amplitudes crucial for e.g. colliders

THE CROSS SECTION FOR FOUR-GLUON PRODUCTION BY GLUON-GLUON FUSION

Stephen J. PARKE and T.R. TAYLOR

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510 USA

Received 13 September 1985

The cross section for two-gluon to four-gluon scattering is given in a form suitable for fast numerical calculations.

[Nucl.Phys. **B269** (1985)]

Once considered computationally intractable

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two \rightarrow four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

Background amplitudes crucial for e.g. colliders



412	S.J. Parke, T.R. Taylor / Four gluon production
gluons. The cross sect four gluons with mom	tion for the scattering of two gluons with momenta p_1, p_2 into a netta p_3, p_4, p_5, p_6 is obtained from eq. (5) by setting $I = 2$ and
replacing the moment	$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, b_{N_1} - a_{N_2} - a_{n_5} - a_{n_5}$

As the result of the computation of two hundred and forty Feyn we obtain

 $A_{\binom{0}{2}}(p_1, p_2, p_3, p_4, p_5, p_6)$ = $(\mathfrak{D}^{\dagger}, \mathfrak{D}^{\dagger}_{\rho}, \mathfrak{D}^{\dagger}_{\sigma}, \mathfrak{D}^{\dagger}_{\tau})^{(0)}_{(2)} \cdot \begin{pmatrix} K_{\rho} & K \\ K_{\sigma} & K_{\tau} \end{pmatrix}$

where D, D, D, and D, are 11-com p_1, p_2, p_3, p_4, p_5 and p_6 , and K, K_{ρ}, K_{σ} and K, are constant 11×11 symmetric matrices. The vectors $\mathcal{D}_{\rho}, \mathcal{D}_{\sigma}$ and \mathcal{D}_{τ} are obtained from the vector \mathcal{D} by the permutation $(p_2 \leftrightarrow p_3), (p_5 \leftrightarrow p_6)$ and $(p_2 \leftrightarrow p_3, p_5 \leftrightarrow p_6)$, respectively, of the more in \mathcal{D} . The individual components of the vector \mathcal{D} represent the sum tents of the vector D represent the sums of all contribu tional to the appropriately chosen eleven basis color factors. The ices K, which are the suitable sums over the color indices of products of th color bases, contain two independent structures, proportional to $N^4(N^2-1)$ and $N^2(N^2-1)$, respectively (N is the number of colors, N = 3 for QCD):

 $K = \frac{1}{2}g^8 N^4 (N^2 - 1)K^{(4)} + \frac{1}{2}g^8 N^2 (N^2 - 1)K^{(2)}$

Here g denotes the gauge coupling constant. The matrices $K^{(4)}$ and $K^{(1)}$ are given in table 1. The vector \tilde{B} is related to the thirty-three diagrams $D^0(I = 1 - 33)$ for two-gluon to four-scalar scattering, elseven diagrams $D^0(I = 1 - 16)$ for two-scalar to four-scalar scattering, in the following way:

 $\mathcal{D}_{0} \approx \frac{2s_{14}}{\sqrt{|s_{15}s_{45}(s_{16}s_{46})|s_{21}s_{56}}} \{t_{123}^{2}C^{G} \cdot D_{0}^{G} - 4s_{14}t_{123}E(p_{5} + p_{6}, p_{6})C^{F} \cdot D_{0}^{F}$ $-2s_{14}G(p_5+p_6,p_5+p_6)C^{\rm S}\cdot D_0^{\rm S}\},$

 $\mathcal{D}_2 = \frac{s_{56}}{s} C^{\rm G} \cdot D_2^{\rm G}$,

where the constant matrices $C^{O}(11 \times 33)$, $C^{P}(11 \times 11)$ and $C^{S}(11 \times 16)$ are given in table 2. The Lorentz invariants s_{ij} and t_{ijk} are defined as $s_{ij} = (p_i + p_j)^2$, $t_{ijk} = (p_i + p_j + p_k)^2$ and the complex functions E and G are given by $E(p_{\alpha},p_{j}) = \tfrac{1}{2} \{(p_{1}p_{4})(p_{i}p_{j}) - (p_{1}p_{i})(p_{j}p_{4}) - (p_{1}p_{j})(p_{i}p_{4}) + i\varepsilon_{\mu\nu\rho\lambda}p_{1}^{\mu}p_{1}^{\nu}p_{j}^{\rho}p_{4}^{\lambda}\}/(p_{1}p_{4}) \; ,$ $G(p_n, p_i) = E(p_n, p_5)E(p_n, p_6),$

Tanis	- Press Production
Matrices K(I, J)[I =	-11, J = 1-11].
Matrix K ⁽⁴⁾	Matrix K ⁽²⁾
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{smallmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0$
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$
Matrix K ⁽⁴⁾	
0 0 0 0 1 1 0 1 1 0 -1	3 3 0 3 0 0 0 3 0 0
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	3 3 0
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{smallmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3$
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{smallmatrix} 0 & 0 & 3 & 0 & 3 & 3 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0$
Matrix $K_{\sigma}^{(4)}$	Matrix $K_{\sigma}^{(2)}$
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{smallmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0$
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$
Matrix K ⁽⁴⁾	Matrix K(2)
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$

S.J. Parke, T.R. Taylor / Four gluon production									
		00404000-00							
		0000-00000-							
		0000000000000							
	000-000-000								
		• • - - - • • • • • • •							
	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0								
	0070-000000 it								
	~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~								
		0000-000000							
		0000000000-							
	~~~~~~	•••• <del>~</del> •• <del>,</del> •••							
9		00000-70000							
ž									
Mat		00							
	00000000								
		00000-70000							
	.								
1	00-0-000000	00							
	نْ * * * * * * * * * * * * * * * * *	00-0-00-00							
	0000-000 fi								

	0000-000								
	00	0000-00-000							
	0000-000	0000000-00							
		0000-000000							

S.J. Parke, T.R. Taylor / Four gluon proc where ε is the totally antisymmetric tensor, $\varepsilon_{xyzt} = 1$. For the future use, we define

 $F(p_n, p_i) = \{(p_1, p_4)(p_i, p_i) + (p_1, p_i)(p_i, p_4) - (p_1, p_i)(p_i, p_4)\}/(p_1, p_4). \quad (10)$

Note that when evaluating A₀ and A₂ at crossed configurations of the momenta care must be taken with the implicit dependence of the functions E, F and G on the momenta p_1, p_4, p_5, p_6 . The diagrams D_2^G are listed below:

 $D_{2}^{G}(1) = \frac{\delta_{2}}{\sum_{s,s,s,s,s,t}} \{ [(p_{2} - p_{5})(p_{3} - p_{6})][(p_{1} - p_{4})(p_{5} + p_{6})] - [(p_{2} - p_{5})(p_{3} + p_{6})] \} \}$ $\times [(p_1 - p_4)(p_3 - p_6)] + [(p_2 + p_5)(p_3 - p_6)][(p_1 - p_4)(p_2 - p_5)]],$

 $D_2^G(2) = \frac{1}{\delta_{22}\delta_{22}} \{ 2E(p_2 - p_5, p_3 - p_6) - 2E(p_3 - p_6, p_2 - p_5) + \delta_2[(p_2 - p_5)(p_3 - p_6)] \}$

 $D_2^G(3) = \frac{4}{s_{25}s_{36}t_{125}} \{ [(p_1 + p_2 - p_5)(p_4 + p_3 - p_6)] E(p_2, p_3) \}$ $-[(p_1 + p_2 - p_5)(p_4 - p_1 + p_6)]E(p_2, p_6)$ $-[(p_1-p_2+p_3)(p_4+p_2-p_6)]E(p_5, p_2)$ $+[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5, p_6)$ $-[p_1(p_2-p_5)]E(p_3-p_6, p_3+p_6)-[p_4(p_3-p_6)]E(p_2+p_5, p_2-p_5)$ $+ \delta_2 [p_1(p_2 - p_5)] [p_4(p_3 - p_6)] \}$

 $D_2^{\mathcal{O}}(4) = \frac{-2}{\delta_{22} f_{22}} \left\{ E(p_3 - p_6, p_3 + p_6) - \delta_2 [p_4(p_3 - p_6)] \right\},$

 $D_2^G(5) = \frac{-2}{\sum_{s=1,s=1}^{s}} \{ E(p_2 + p_5, p_2 - p_5) - \delta_2[p_1(p_2 - p_5)] \},\$

 $D_2^G(6) = \frac{\delta_2}{t_{exc}},$

 $D_2^G(7) = \frac{4}{s_{12}s_{14}s_{12}} \{ [(p_1 + p_2 - p_5)(p_4 + p_3 - p_6)] E(p_2, p_3) \}$

 $-[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2,p_6)-[p_4(p_3-p_6)]E(p_2,p_2-p_5)\}\,,$ $D_2^{G}(8) = \frac{4}{S_{14}S_{14}S_{14}S_{14}} \{ [(p_1 + p_2 - p_5)(p_4 + p_3 - p_6)] E(p_2, p_3) \}$

 $-[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5,p_3)-[p_1(p_2-p_5)]E(p_3-p_6,p_3)\},$

S.J. Parke, T.R. Taylor / Four gluon production $D_2^{O}(9) = \frac{4}{\sum (x_5 + f_{12})} \{ [(p_1 - p_2 + p_5)(p_4 + p_3 - p_6)] E(p_5, p_3) \}$ $-[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5,p_6)+[p_4(p_3-p_6)]E(p_5,p_2-p_5)],$ $D_2^G(10) = \frac{4}{p_1 p_2} \left[(p_1 + p_2 - p_5)(p_4 - p_3 + p_6) \right] E(p_2, p_6)$ $-[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5,p_6)+[p_1(p_2-p_5)]E(p_3-p_6,p_6)\}$ $D_2^{\rm G}(11) = \frac{\delta_2}{s_{10}t_{10}} [s_{35} - s_{56} + s_{36}],$ $D_2^G(12) = \frac{-\delta_2}{s_1 t_2} [s_{23} - s_{26} - s_{36}],$ $D_2^{\rm G}(13) = \frac{\delta_2}{s_{14}s_{16}s_{14}} [s_{12} - s_{24}] [s_{35} - s_{56} + s_{36}],$ $D_2^G(14) = \frac{\delta_2}{s_{14}s_{36}t_{145}} [s_{15} - s_{45}][s_{23} - s_{26} - s_{36}],$ $D_2^G(15) = \frac{\delta_2}{s_1, s_2} (p_1 - p_4)(p_3 - p_6),$ $D_2^G(16) = \frac{-4}{s_{12}s_{24}f_{124}} [s_{35} - s_{56} + s_{36}] E(p_2, p_2),$ $D_2^{\rm G}(17) = \frac{4}{s_{23} - s_{26} - s_{36}} [E(p_5, p_5)],$ $D_2^G(18) = \frac{-4}{\sum_{s=1}^{s} \sum_{s=1}^{s}} [2(p_1 + p_2)(p_3 - p_6) - s_{36}] E(p_2, p_5),$ $D_2^G(19) = \frac{-2}{s_1 - s_2} E(p_2, p_3 - p_6),$ $D_2^G(20) = \frac{2}{e_1 e_2} E(p_3 - p_6, p_5),$ $D_2^G(21) = \frac{-4}{s_{26}s_{26}-s_{56}+s_{25}} [s_{26}-s_{56}+s_{25}] E(p_3, p_3),$ $D_2^{\rm G}(22) = \frac{4}{s_{16}s_{23}t_{146}} [s_{23} - s_{35} - s_{25}]E(p_6, p_6),$

0 1 1 0 -1	1 2 1 -2	1 0 1 1	0 2 0 1 0	2 2 0 0 2	2 0 0 -1	4 0 0 -2	0 4 0 0 0	0 0 2 -1	0 0 2 4 0	-2 0 -1 0 4			0 3 0 0 0	0 0 0 0	0 0 3 0	000000000000000000000000000000000000000	0 3 3 0	0 0 3 0	0 0 3 0	0 0 3 0	0 3 0 0	
Matrix $K_{\sigma}^{(4)}$						Matrix $K_{\phi}^{(2)}$																
4 2 0 2 0 1 0 1 0 0 0 0	2 4 0 1 0 1 1 0 1 0	0 4 2 1 1 1 2 1 0	2 1 2 0 1 2 1 0 1 0 0	0 2 1 0 0 0 4 2 2	1 0 1 2 0 0 0 0 0 1 2 0	0 1 1 0 0 0 2 4 0	1 1 0 0 0 0 0 2 0 1	0 2 1 4 1 2 2 0 0 4	0 1 0 2 2 4 0 0 0 -2	0 0 0 2 0 0 1 -4 -2 4			0 0 0 0 0 0 0 0 0 0 3 0 3 0 3	0 0 0 3 0 0 3 0 0 0 0 0	0 0 0 0 0 0 3 0 0 0 0 0	000000000000000000000000000000000000000	0 3 0 0 0 0 3 0 0 0 0 0 0 0	0 0 0 3 3 0 0 0 3 3 0 0 0 0 3	0 3 0 3 3 3 0 0 0 0 0 0	0 3 0 0 0 0 0 3 0 0 0 0 0 0 0 0 0 0 0 0	300000000000000000000000000000000000000	
_			1	Matr	ix A	(4)									_	1	Mate	ix K	(2)			
0 1 -1 1 1 2 0 0	1 -2 -1 2 0 1 4 2 0	-1 0 0 1 1 1 -1 0	-1 0 1 0 2 1 0 1 -1 0	1 0 0 1 -1 -1 0 -2 1	1 0 1 2 -1 0 1 -2 2 4 -1	0 1 1 1 - 1 0 - 1 4 8 - 1	1 1 0 -2 -1 0 2 -2 0 2 -2 0	2 4 -1 -2 2 4 2 1 0 -2	0 2 1 -1 2 4 8 -2 0 0 0	0 0 1 -1 -2 0 2			3 0 0 0 3 3 0 0 0 0 0 0	3 0 0 3 3 0 0 0 0 0	0 3 3 0 0 3 0 0 0 0 0	0033003003	0 0 3 3 3 0 0 3 0 0 0 0	3 0 0 3 3 0 0 0 -3	3 0 0 3 3 0 0 0 0 0 0 0	0 0 3 3 3 0 0 3 0 0 0 0	0 0 0 0 0 0 0 0 0 3 3 0	
1 1 1	D ^G 2() D ^G 2() D ^G 2()	24) 25) 26)	$=\frac{1}{s_2}$ $=\frac{1}{s_1}$ $=\frac{1}{s_1}$	-2 5 ⁵ 34 2 16 ⁵ 25 -2	E(s p2- P65 (p25	.J. P - Ps P2 -	^p arke , p ₃) - p ₅)	; T.),),	R. 7.	aylor ,	/ For	er glu	108	prodi	uctio						
	D20(27)	=	2	E	(p3	- p.	₽6	э,													

 $D_2^G(28) = \frac{2}{c_1 c_2} E(p_5, p_2 - p_5),$

 $D_2^G(29) = \frac{-2}{r_1} E(p_3 - p_{6*} p_3),$

where $\delta_2 = 1$.

 $D_2^{\rm G}(30) = \frac{4}{s_{12}s_{44}t_{125}} \left[(p_1 + p_2 - p_5)(p_4 + p_3 - p_6) - t_{125} \right] E(p_2, p_3) ,$

 $D_2^{\rm Q}(31) = \frac{4}{\sum_{1 \le 1} \sum_{j < 1} [(p_1 + p_2 - p_5)(p_4 - p_3 + p_6) + t_{125}]E(p_2, p_6),$

 $D_2^G(32) = \frac{4}{p_2 p_3 p_4 p_5} [(p_1 - p_2 + p_5)(p_4 + p_3 - p_6) + t_{125}] E(p_5, p_3),$

 $D_2^{\rm G}(33) = \frac{4}{s_{15}s_{46}t_{155}} [(p_1 - p_2 + p_5)(p_4 - p_3 + p_6) - t_{125}]E(p_5, p_6),$

 $D_0^F(1) = \frac{4}{s_{15}s_{34}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5)$

 $D_0^{\rm F}(3) = \frac{4}{\sum_{s,s} \sum_{s=1}^{s} (p_{s}, p_{s}) E(p_{3}, p_{5}) - F(p_{5}, p_{3}) E(p_{6}, p_{5})}$

 $-[F(p_1, p_6) - \frac{1}{2}s_{16} - \frac{1}{2}s_{14} + \frac{1}{2}s_{46}]E(p_5, p_5)],$

+[$F(p_2, p_3) + \frac{1}{2}s_{34}$] $E(p_6, p_5) - F(p_6, p_3)E(p_2, p_5)$ }

+ [$F(p_6, p_3) + s_{34}$] $E(p_5, p_5)$ }, $D_0^F(2) = \frac{-4}{s_{14}s_{24}s_{14}} \{ [F(p_6, p_2) + \frac{1}{2}s_{16}] E(p_3, p_5) \}$

where $\delta_2 = 1$. The diagrams D_0^G are obtained from D_2^G by replacing δ_2 by $\delta_0 = 0$ and the functions $E(p, p_1)$ by $G(p, p_2)$. The diagrams D_0^F are listed below:

418	S.J. Parke, T.R. Taylor / Four gluon production
	$D_0^P(4) = \frac{4}{s_{25}s_{34}t_{125}} \{F(p_2, p_3)E(p_5, p_5) - F(p_5, p_3)E(p_2, p_5)\}$
	+ $[F(p_5, p_2) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_3, p_5)$

 $D_0^{\rm F}(5) = \frac{2}{s_{12}s_{23} - s_{23} + s_{25}} [E(p_6, p_5)],$

 $D_0^{\rm F}(6) = \frac{2}{s_{26}s_{26} f_{136}} [s_{56} - s_{26} - s_{25}] E(p_3, p_5),$

 $D_0^{\mathbb{P}}(7) = \frac{4}{s_{25}s_{36}t_{125}} \{ [F(p_5, p_2) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}] E(p_3, p_5) \}$

+ [$F(p_2, p_3) + \frac{1}{4}t_{125}$] $E(p_5, p_5) - [F(p_5, p_3) + \frac{1}{4}t_{125}]E(p_2, p_5)$ }, $D_0^{\rm F}(8) = \frac{1}{c_1 c_2} E(p_3 - p_6, p_5),$

 $D_0^{\rm F}(9) = \frac{2}{s_{15}s_{15}s_{15}} [s_{35} - s_{56} + s_{36}] E(p_2, p_5),$

 $D_0^F(10) = \frac{2}{s_{12}s_{23} - s_{26} - s_{36}} [E(p_5, p_5)],$

 $D_0^{\rm F}(11) = \frac{1}{2s_{14}s_{25}s_{35}} \{ [s_{23} + s_{35} - s_{26} - s_{56}] E(p_2 - p_5, p_5)$

 $-[s_{23}+s_{26}-s_{35}-s_{56}]E(p_3-p_6,p_5)-[s_{23}+s_{56}-s_{35}-s_{26}]E(p_2+p_5,p_5)\}.$ The diagrams D_0^s are listed below:

 $D_0^{\rm S}(1) = \frac{1}{s_{24}s_{24}(s_{34}-s_{46}+s_{36})[s_{12}-s_{15}-s_{25}]},$

 $D_0^{\rm S}(2) = \frac{1}{s_{12} - s_{24} - s_{14}} [s_{12} - s_{24} - s_{14}] [s_{35} - s_{56} + s_{36}],$

 $D_0^{\rm S}(3) = \frac{1}{s_{11}s_{22} + s_{45}} [s_{15} - s_{45} + s_{14}] [s_{23} - s_{26} - s_{36}],$

 $D_0^{\rm S}(4) = \frac{1}{s_{14}s_{34}s_{145}} [s_{15} + s_{25} - s_{12}] [s_{34} - s_{46} + s_{36}],$

 $D_0^{\rm S}(5) = \frac{1}{s_{15}s_{24}f_{156}} [s_{56} - s_{15} - s_{16}] [s_{23} - s_{24} - s_{34}],$

 $D_0^{\rm S}(6) = \frac{1}{s_{15}s_{34}t_{125}} [s_{46} - s_{34} - s_{36}] [s_{12} - s_{25} - s_{15}],$

S.J. Parke, T.R. Taylor / Four gluan production 419 $D_0^{S}(7) = \frac{1}{s_{24}s_{24}t_{125}} [s_{36} - s_{46} + s_{34}] [s_{12} - s_{15} - s_{25}],$ $D_0^{S}(8) = \frac{1}{s_{16}s_{25}t_{146}} [s_{25} + s_{35} - s_{23}][s_{14} - s_{46} + s_{16}],$ $D_0^{S}(9) = \frac{1}{s_{25}s_{44}t_{144}} [s_{14} + s_{34} - s_{13}] [s_{26} - s_{56} + s_{25}],$ $D_0^{\rm S}(10) = \frac{1}{s_{24}s_{26}} (p_2 - p_5)(p_3 - p_6) ,$ $D_0^{\rm S}(11) = \frac{1}{s_{14}s_{24}}(p_1 - p_4)(p_3 - p_6),$ $D_0^{\rm S}(12) = \frac{1}{s_{14}s_{24}} (p_6 - p_1)(p_2 - p_5) ,$ $D_0^{\rm S}(13) = \frac{1}{s_{14}s_{14}} (p_5 - p_1)(p_3 - p_4) ,$ $D_0^{\rm S}(14) = \frac{1}{s_{24}s_{14}} (p_2 - p_5)(p_3 - p_4),$ $D_0^{S}(15) = \frac{1}{s_{14}s_{25}s_{36}} \left[((p_2 + p_5)(p_3 - p_6)) \right] ((p_1 - p_4)(p_2 - p_5)]$ $+[(p_2-p_5)(p_3-p_6)][(p_1-p_4)(p_3+p_6)]$

 $+ [(p_1 + p_4)(p_2 - p_5)][(p_1 - p_4)(p_3 - p_6)]],$ $D_0^{S}(16) = \frac{2}{s_{16}s_{14}s_{25}} \{ [(p_2 - p_5)(p_3 + p_4)] [(p_1 - p_6)(p_3 - p_4)] \}$

> + $[(p_1 + p_4)(p_3 - p_4)][(p_1 - p_4)(p_2 - p_5)]$ + $[(p_1 - p_6)(p_2 + p_5)][(p_1 - p_4)(p_2 - p_5)]]$.

(13)

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagram D are calculated by using eqs. (1)-(13). The result is substituted to eq. (8) to obtain the vectors $\mathcal{B}_{2,2}$ and $\mathcal{B}_{2,3}$ After generating the vectors $\mathcal{B}_{2,3}$ $\mathcal{B}_{2,3}$, $\mathcal{B}_{2,3}$, $\mathcal{B}_{2,3}$, $\mathcal{B}_{3,3}$, $\mathcal{B}_{3,4}$, $\mathcal{B}_{4,5}$ have proportial permutations of nomenta, eq. (6) is used to obtain the functions $\mathcal{A}_{4,4}$ and $\mathcal{A}_{5,5}$ Finally, the total cross section is calculated by using eq. (5). The FORTRAN 5 program based on such a scheme generates the Monte Carlo points in less than a second on the heterotic CDC CYBER 175/875.

Given the complexity of the final result, it is very important to have some reliable testing procedures available for numerical calculations. Usually in QCD, the multi gluon amplitudes are tested by checking the gauge invariance. Due to the specific

 $D_2^{\rm Q}(23) = \frac{4}{s_{12}s_{22}s_{23}} [2(p_1 + p_6)(p_2 - p_5) + s_{25}]E(p_6, p_5),$

Background amplitudes crucial for e.g. colliders





[3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

We thank Keith Ellis, Chris Quigg and especially, Estia Eichten for many useful discussions and encouragement during the course of this work. We acknowledge he hospitality of Aspen Center for Physics, where this work was being completed

 $D_2^{\rm G}(16) = \frac{-4}{s_{12}s_{36}t_{124}} [s_{35} - s_{56} + s_{36}] E(p_2, p_2),$ $D_2^{\rm G}(17) = \frac{4}{s_{23} - s_{26} - s_{36}} [E(p_5, p_5)],$ $D_2^{\rm G}(18) = \frac{-4}{\sum_{s=1}^{s} \sum_{s=1}^{s} [2(p_1 + p_2)(p_3 - p_6) - s_{36}]E(p_2, p_5),$ $D_2^G(19) = \frac{-2}{r_1 r_2} E(p_2, p_3 - p_6),$ $D_2^G(20) = \frac{2}{e_1 e_2} E(p_3 - p_6, p_5),$ $D_2^G(21) = \frac{-4}{s_{26}s_{26}-s_{56}+s_{25}} [s_{26}-s_{56}+s_{25}] E(p_3, p_3),$ $D_2^{\rm G}(22) = \frac{4}{s_{16}s_{23}t_{146}} [s_{23} - s_{35} - s_{25}]E(p_6, p_6),$ $D_2^{\rm Q}(23) = \frac{4}{s_{12}s_{22}s_{23}} [2(p_1 + p_6)(p_2 - p_5) + s_{25}]E(p_6, p_5),$

 $A_{\binom{0}{2}}(p_1, p_2, p_3, p_4, p_5, p_6)$

vhere D. D., D., and D. are 11-cor

		P
the second second second second second		
$D_2^{\rm G}(32) = \frac{4}{s_{51}s_{54}t_{125}} \left[(p_1 - p_2 + p_5)(p_4 + p_3 - p_6) + t_{125} \right] E(p_5, p_5) ,$		
$D_2^G(33) = \frac{4}{s_{12}s_{46}t_{125}} \left[(p_1 - p_2 + p_5)(p_4 - p_3 + p_6) - t_{125} \right] E(p_5, p_6) ,$	(11)	
where $\delta_2 = 1$.		101000
The diagrams D_0^G are obtained from D_2^G by replacing δ_2 by $\delta_0 = 0$ and the fur-	nctions	011402
$E(p_n p_j)$ by $G(p_n p_j)$.		
The diagrams D_0^F are listed below:		100 C
		0.000
$D_0^{\mathbf{F}}(1) = \frac{4}{s_{15}s_{34}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5)\}$		121212
+ [$F(p_6, p_3) + s_{34}$] $E(p_5, p_5)$ },		1202
$D_0^F(2) = \frac{-4}{s_{16}s_{25}s_{34}} \{ [F(p_6, p_2) + \frac{1}{2}s_{16}] E(p_3, p_5)$		E VE
+[$F(p_2, p_3) + \frac{1}{2}s_{34}$] $E(p_6, p_3) - F(p_6, p_3)E(p_2, p_3)$ }		0.0150
$D_0^{\rm F}(3) = \frac{4}{s_{15}s_{36}t_{125}} \{F(p_5, p_6)E(p_5, p_5) - F(p_5, p_5)E(p_6, p_5)$		0.00
$- [F(p_{3}, p_{6}) - \frac{1}{2}s_{36} - \frac{1}{2}s_{34} + \frac{1}{2}s_{46}]E(p_{5}, p_{5})\},$		10000
		1.00
		100000
		100 C 200 C

 $D_0^{\rm F}(11) = \frac{1}{2s_{14}s_{25}s_{36}} \{ [s_{23} + s_{35} - s_{26} - s_{56}] E(p_2 - p_5, p_5) \}$ $-[s_{23}+s_{26}-s_{35}-s_{56}]E(p_3-p_6,p_5)-[s_{23}+s_{56}-s_{35}-s_{26}]E(p_2+p_5,p_5)\}.$ The diagrams D_0^s are listed below: $D_0^{\rm S}(1) = \frac{1}{s_{25}s_{36}t_{125}} [s_{34} - s_{46} + s_{36}] [s_{12} - s_{15} - s_{25}],$ $D_0^{\rm S}(2) = \frac{1}{s_{12} - s_{24} - s_{14}} [s_{35} - s_{56} + s_{36}],$ $D_0^{S}(3) = \frac{1}{s_{15} - s_{45} + s_{14}} [s_{23} - s_{26} - s_{36}],$ $D_0^{\rm S}(4) = \frac{1}{s_{15}s_{26}t_{125}} [s_{15} + s_{25} - s_{12}] [s_{34} - s_{46} + s_{36}],$ $D_0^{\rm S}(5) = \frac{1}{s_{15}s_{24}t_{156}} [s_{56} - s_{15} - s_{16}] [s_{23} - s_{24} - s_{34}],$ $D_0^{\rm S}(6) = \frac{1}{s_{15}s_{34}t_{125}} [s_{46} - s_{34} - s_{36}] [s_{12} - s_{25} - s_{15}],$

 $\frac{1}{s_{14}s_{25}s_{36}}\left\{\left[(p_2+p_5)(p_3-p_6)\right]\left[(p_1-p_4)(p_2-p_5)\right]\right]$ $+[(p_2-p_5)(p_3-p_6)][(p_1-p_4)(p_3+p_6)]$ $+ [(p_1 + p_4)(p_2 - p_5)][(p_1 - p_4)(p_3 - p_6)]],$ $D_0^{S}(16) = \frac{2}{s_{16}s_{14}s_{25}} \{ [(p_2 - p_5)(p_3 + p_4)] [(p_1 - p_6)(p_3 - p_4)] \}$ $+[(p_1+p_6)(p_3-p_4)][(p_1-p_6)(p_2-p_5)]$ + $[(p_1 - p_6)(p_2 + p_5)][(p_1 - p_4)(p_2 - p_5)]]$. The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams D are calculated by using eqs. (1)-(13). The result is substituted to eq. (8) to obtain the vectors $\mathcal{B}_{2,2}$ and $\mathcal{B}_{2,2}$ After generating the vectors $\mathcal{B}_{2,2}$ $\mathcal{B}_{2,2}$, $\mathcal{B}_{2,2}$, $\mathcal{B}_{2,3}$, $\mathcal{B}_{2,3}$, $\mathcal{B}_{3,4}$, $\mathcal{B}_{4,5}$, but consist let $\mathcal{A}_{4,5}$ Finally, the total cross section is calculated by using eq. (5). The FORTRAN 5 program based on such a scheme generates ten Monte Carlo points in less than a second on the heterotic CDC CYBER 175/875. Given the complexity of the final result, it is very important to have some reliable testing procedures available for numerical calculations. Usually in QCD, the multi gluon amplitudes are tested by checking the gauge invariance. Due to the specific

 Within six months, Parke-Taylor stumbled on a simple guess—unquestionably a *theorist's delight*:

Within six months, Parke-Taylor stumbled on a simple guess—unquestionably a *theorist's delight*:

Amplitude for *n*-Gluon Scattering [*PRL* 56 (1986)]

Stephen J. Parke and T. R. Taylor

Fermi National Accelerator Laboratory, Batavia, Illinois 60510 (Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

 Within six months, Parke-Taylor stumbled on a simple guess—unquestionably a *theorist's delight*:



 $\langle 1 2 \rangle^4$ $\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \langle 5 6 \rangle \langle 6 1 \rangle$

Amplitude for *n*-Gluon Scattering [*PRL* 56 (1986)]

Stephen J. Parke and T. R. Taylor

Fermi National Accelerator Laboratory, Batavia, Illinois 60510 (Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

 $p_a^{\mu} \equiv \sigma^{\mu}_{\alpha\dot{\alpha}}\lambda^{\alpha}_a\lambda^{\dot{\alpha}}_a$

 $\langle a b \rangle \equiv \det(\lambda_a, \lambda_b)$

$$[a b] \equiv \det(\widetilde{\lambda}_a, \widetilde{\lambda}_b)$$

[van der Waerden (1929)]

 Within six months, Parke-Taylor stumbled on a simple guess—unquestionably a *theorist's delight*:



$$\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\,\cdots\,\langle n\,1\rangle}$$

Amplitude for *n*-Gluon Scattering [*PRL* 56 (1986)]

Stephen J. Parke and T. R. Taylor

Fermi National Accelerator Laboratory, Batavia, Illinois 60510 (Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

$$p_a^{\mu} \equiv \sigma^{\mu}_{\alpha\dot{\alpha}}\lambda^{\alpha}_a\widetilde{\lambda}^{\dot{\alpha}}_a$$

$$\langle a b \rangle \equiv \det(\lambda_a, \lambda_b)$$

$$[a b] \equiv \det(\widetilde{\lambda}_a, \widetilde{\lambda}_b)$$

[van der Waerden (1929)]

 Within six months, Parke-Taylor stumbled on a simple guess—unquestionably a *theorist's delight*:



$$\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\,\cdots\,\langle n\,1\rangle}$$

Goal: make the simplicity of amplitudes **manifest** in the way we compute them, *dramatically* extending the reach of the predictions we can make for experiment

Simplex Sigillum Veri

To what extent can QFT be reformulated without reference to (simplicity-spoiling) redundancies?

To what extent can QFT be reformulated without reference to (simplicity-spoiling) redundancies?

To what extent can QFT be reformulated without reference to (simplicity-spoiling) redundancies?

Answer: most (all?) of perturbation theory

To what extent can QFT be reformulated without reference to (simplicity-spoiling) redundancies?

Answer: most (all?) of perturbation theory

Building Blocks:

 tree amplitudes and *on-shell functions*—recursion relations, scattering equations, (twistor) string theory, cluster varieties, the amplituhedron, ...

To what extent can QFT be reformulated without reference to (simplicity-spoiling) redundancies?

Answer: most (all?) of perturbation theory

Building Blocks:

 tree amplitudes and *on-shell functions*—recursion relations, scattering equations, (twistor) string theory, cluster varieties, the amplituhedron, ...

Loop Integrands: unitarity, Q-cuts, bootstraps, ...

To what extent can QFT be reformulated without reference to (simplicity-spoiling) redundancies?

Answer: most (all?) of perturbation theory

Building Blocks:

 tree amplitudes and *on-shell functions*—recursion relations, scattering equations, (twistor) string theory, cluster varieties, the amplituhedron, ...

Loop Integrands: unitarity, Q-cuts, bootstraps, ...
 Loop Integration: symbology, motives, bootstraps, symmetry-preserving regularization/evaluation, ...

Basic Building Blocks: On-Shell Functions

Η απλότητα του σχεδίου κάνει εύκολη την κατασκευή

(the simplicity of the design makes it easy to build)

The Vernacular of the S-Matrix

 On-Shell Functions: scattering amplitudes, and functions built thereof—as *networks* of amplitudes

The Vernacular of the S-Matrix

 On-Shell Functions: scattering amplitudes, and functions built thereof—as *networks* of amplitudes


On-Shell Functions: scattering amplitudes, and functions built thereof—as *networks* of amplitudes



 On-Shell Functions: scattering amplitudes, and functions built thereof—as *networks* of amplitudes



Locality: amplitudes independent, so multiplied

 On-Shell Functions: scattering amplitudes, and functions built thereof—as *networks* of amplitudes



Locality: amplitudes independent, so multiplied
Unitarity: internal particles unseen, so summed

 On-Shell Functions: scattering amplitudes, and functions built thereof—as *networks* of amplitudes



Locality: amplitudes independent, so multiplied
Unitarity: internal particles unseen, so summed

$$f_{\Gamma} \equiv \prod_{i \in I} \left(\sum_{h_i, c_i} \int d^{d-1} \text{LIPS}_i \right) \prod_{v \in V} \mathcal{A}_v$$

 On-Shell Functions: scattering amplitudes, and functions built thereof—as *networks* of amplitudes







Locality: amplitudes independent, so multiplied
Unitarity: internal particles unseen, so summed

$$f_{\Gamma} \equiv \prod_{i \in I} \left(\sum_{h_i, c_i} \int d^{d-1} \text{LIPS}_i \right) \prod_{v \in V} \mathcal{A}_v$$

 On-Shell Functions: scattering amplitudes, and functions built thereof—as *networks* of amplitudes







defined for all *all* quantum field theories *exclusively* in terms of physical (observable) states
can be used to reconstruct *all* loop amplitudes

$$f_{\Gamma} \equiv \prod_{i \in I} \left(\sum_{h_i, c_i} \int d^{d-1} \text{LIPS}_i \right) \prod_{v \in V} \mathcal{A}_v$$

What's Special about Massless, 4d? • Massless, 3-particle amplitudes in 4 dimensions: $h_{1} - \begin{pmatrix} h_{2} \\ = \langle 12 \rangle^{h_{3}-h_{1}-h_{2}} \langle 23 \rangle^{h_{1}-h_{2}-h_{3}} \langle 31 \rangle^{h_{2}-h_{3}-h_{1}} \\ h_{1} + h_{2} + h_{3} \leq 0 \end{pmatrix}$





Primitive On-Shell Functions

 On-shell functions built from 3-point vertices edges label states (which dictate the vertices)



The Grassmannian Correspondence $f_{\Gamma} \equiv \prod_{i} \left(\sum_{h_{i}, q_{i}} \int d^{3} \text{LIPS}_{i} \right) \prod_{v} \mathcal{A}_{v}$ The Grassmannian Correspondence $f_{\Gamma} \equiv \prod_{i} \left(\sum_{h_{i}, q_{i}} \int d^{3} \text{LIPS}_{i} \right) \prod_{v} \mathcal{A}_{v} \equiv \int \Omega_{C} \ \delta(C, p, h)$

The Grassmannian Correspondence $f_{\Gamma} \equiv \prod_{i} \left(\sum_{h_{i}, q_{i}} \int d^{3} \text{LIPS}_{i} \right) \prod_{v} \mathcal{A}_{v} \equiv \int \Omega_{C} \ \delta(C, p, h)$

On-Shell Physics

- on-shell diagrams and functions
- physical symmetries
 - trivial symmetries (identities)
- functional relations of observables

Grassmannian Geometry

- {strata $C \in G(k, n)$, volume-form Ω_C }
- volume-preserving diffeomorphisms

 cluster coordinate mutations
- homological identities among strata

The Grassmannian Correspondence $f_{\Gamma} \equiv \prod_{i} \left(\sum_{h_{i}, q_{i}} \int d^{3} LIPS_{i} \right) \prod_{v} \mathcal{A}_{v} \equiv \int \Omega_{C} \ \delta(C, p, h)$ On-Shell Physics• on-shell diagrams and functions• physical symmetries- trivial symmetries (identities)• Cluster coordinate mutations

functional relations of observables

homological identities among strata



$\frac{\text{The Grassmannian Correspondence}}{f_{\Gamma} \equiv \prod_{i} \left(\sum_{h_{i}, q_{i}} \int d^{3} \text{LIPS}_{i} \right) \prod_{v} \mathcal{A}_{v} \equiv \int \Omega_{C} \ \delta(C, p, h)}$

On-Shell Physics

- on-shell diagrams and functions
- physical symmetries
 - trivial symmetries (identities)
- functional relations of observables

Grassmannian Geometry

- {strata $C \in G(k, n)$, volume-form Ω_C }
- volume-preserving diffeomorphisms

 cluster coordinate mutations
 - homological identities among strata

◆ general characteristics: $k \equiv 2n_B + n_W - n_I$ $d \equiv n + n_I - n_V$

$\frac{\text{The Grassmannian Correspondence}}{f_{\Gamma} \equiv \prod_{i} \left(\sum_{h_{i}, q_{i}} \int d^{3} \text{LIPS}_{i} \right) \prod_{v} \mathcal{A}_{v} \equiv \int \Omega_{C} \ \delta(C, p, h)}$

On-Shell Physics

- on-shell diagrams and functions
- physical symmetries
 - trivial symmetries (identities)
- functional relations of observables

Grassmannian Geometry

- {strata $C \in G(k, n)$, volume-form Ω_C }
- volume-preserving diffeomorphisms

 cluster coordinate mutations
 - homological identities among strata

★ general characteristics: k ≡ 2n_B + n_W - n_I
 ★ reducibility into functions with non-degenerate Ω_C

 \Rightarrow

The Grassmannian Correspondence $f_{\Gamma} \equiv \prod_{i} \left(\sum_{h_{i}, q_{i}} \int d^{3} \text{LIPS}_{i} \right) \prod_{v} \mathcal{A}_{v} \equiv \int \Omega_{C} \ \delta(C, p, h)$

 \Leftrightarrow

On-Shell Physics

- on-shell diagrams and functions
- physical symmetries
 - trivial symmetries (identities)
- functional relations of observables

Grassmannian Geometry

- {strata $C \in G(k, n)$, volume-form Ω_C }
- volume-preserving diffeomorphisms

 cluster coordinate mutations

homological identities among strata

 general characteristics: k ≡ 2n_B + n_W - n_I d ≡ n + n_I - n_V

 reducibility into functions with non-degenerate Ω_C
 volume-preserving diffeomorphisms correspond to active symmetry transformations:
 ∫Ω_C δ(C, p, h) → ∫Ω_{C'}δ(C', p, h) = ∫Ω_C δ(C', p, h) = ∫Ω_C δ(C, p', h') = ∫Ω_C δ(C, p', h') = ∫Ω_C δ(C, p', h) = ∫Ω_C ∂(C, p', h) = ∫Ω_C ∂(C, p', h) = ∫Ω_C ∂(C,

The Grassmannian Correspondence $f_{\Gamma} \equiv \prod_{i} \left(\sum_{h_{i}, q_{i}} \int d^{3} \text{LIPS}_{i} \right) \prod_{v} \mathcal{A}_{v} \equiv \int \Omega_{C} \ \delta(C, p, h)$

On-Shell Physics

- on-shell diagrams and functions
- physical symmetries
 - trivial symmetries (identities)
- functional relations of observables

Implications:

finite in number
each enjoys infinitedimensional symmetries
span rational functions, distributions, & integrals

Grassmannian Geometry

- {strata $C \in G(k, n)$, volume-form Ω_C }
- volume-preserving diffeomorphisms

 cluster coordinate mutations
- homological identities among strata

$\frac{\text{The Grassmannian Correspondence}}{f_{\Gamma} \equiv \prod_{i} \left(\sum_{h_{i}, q_{i}} \int d^{3} \text{LIPS}_{i} \right) \prod_{v} \mathcal{A}_{v} \equiv \int \Omega_{C} \ \delta(C, p, h)}$

On-Shell Physics

- on-shell diagrams and functions
- physical symmetries
 - trivial symmetries (identities)
- functional relations of observables

Implications:

finite in number
each enjoys infinitedimensional symmetries
span rational functions, distributions, & integrals

Grassmannian Geometry

- {strata $C \in G(k, n)$, volume-form Ω_C }
- volume-preserving diffeomorphisms

 cluster coordinate mutations
- homological identities among strata
- *Questions (math/physics)*: •how many?
- do these extend to full scattering amplitudes?
 a *functional* basis for
 - amplitude integrands?14



Some provocative recent claims:

- the S-matrices of (asymptotically-)flat (4d) theories of massless particles enjoy *infinite dimensional* symmetries
- soft theorems are Ward identities for these symmetries

[Strominger; Strominger *et al*] [**JB**, Haco, Hawking, Perry (2017)]



Some provocative recent claims:

- the S-matrices of (asymptotically-)flat (4d) theories of massless particles enjoy *infinite dimensional* symmetries
- soft theorems are Ward identities for these symmetries

[Strominger; Strominger *et al*] [**JB**, Haco, Hawking, Perry (2017)]

◆ Proving the finiteness of N=8?



Some provocative recent claims:

- the S-matrices of (asymptotically-)flat (4d) theories of massless particles enjoy *infinite dimensional* symmetries
- soft theorems are Ward identities for these symmetries

[Strominger; Strominger *et al*] [**JB**, Haco, Hawking, Perry (2017)]

- ◆ Proving the finiteness of N=8?
- Connections to the Yangian?





Two identities among on-shell diagrams:





Two identities among on-shell diagrams:





Web of Dualities for Planar SYM



 Recently published, authoritative description of the foundations of this story, with a full treatment for planar, maximally supersymmetric Yang-Mills

 Recently published, authoritative description of the foundations of this story, with a full treatment for planar, maximally supersymmetric Yang-Mills





 Recently published, authoritative description of the foundations of this story, with a full treatment for planar, maximally supersymmetric Yang-Mills







Grassmannian Positroids, Plabic Graphs, & Scattering Amplitudes in $\mathcal{N}=4$ SYM Jacob L. Bourjaily, 2012

JB (2012)]

 Recently published, authoritative description of the foundations of this story, with a full treatment for planar, maximally supersymmetric Yang-Mills







Grassmannian Positroids, Plabic Graphs, & Scattering Amplitudes in $\mathcal{N}=4$ SYM Jacob L. Bourjaily, 2012

JB (2012)

CAMBRIDGE UNIVERSITY PRESS Memorandum of Agreement BETWEEN the Syndicate of the Press of the University of Cambridge (hereinafter referred to as 'the Syndicate') on behalf of the Chancellor, Masters and Scholars of the University of Cambridge, The Edinburgh Building, Shaftesbury Road, Cambridge, CB2 8RU UK of the one part and Professor Nima Arkani-Hamed Dr. Jacob Bouriaily Institute for Advanced Study Harvard University Department of Physics School of Natural Sciences Einstein Drive 17 Oxford Street Princeton, NJ 08540 Cambridge, MA 02138 TIS arkani@ias.edu bourjaily@fas.harvard.edu Professor Freddy Cachazo Professor Alexander Goncharov Perimeter Institute for Theoretical Yale University Physics Department of Mathematics 31 Caroline Street N. 10 Hillhouse Avenue Waterloo, N2L 2Y5 New Haven, CT 06520 Canada US fcachazo@perimeterinstitute.ca alexander.goncharov@yale.edu Professor Alexander Postnikov Dr. Jaroslav Trnka Massachusetts Institute of Technolog California Institute of Technology Department of Mathematics Department of Physics 77 Massachusetts Avenue 1200 East California Blvd. Cambridge, MA 02139-4307 Pasadena, CA 91125 US trnka@caltech.edu apost@math.mit.edu

LS.ACA.MMXIII

19

 Recently published, authoritative description of the foundations of this story, with a full treatment for planar, maximally supersymmetric Yang-Mills







Grassmannian Positroids, Plabic Graphs, & Scattering Amplitudes in $\mathcal{N}=4$ SYM Jacob L. Bourjaily, 2012

JB (2012)

(a) (i) First Serial (the right to publish one or more extracts from the Work in successive issues of a periodical or newspaper beginning before publication of the Syndicate's first edition of the Work) 50%

(ii) Second (and subsequent) Serial (the right to publish one or more extracts from the Work in successive issues of a periodical or newspaper beginning at or following publication of the Syndicate's first edition of the Work) 50%

(iii) Dramatisation/Documentary 50%

(iv) Film 50%

(v) Single-Voice Readings 50%(vi) Translation 50%

(vi) Translation 50

(vii) Anthology 50%

(viii) Digest (the right to publish an abridgement or condensation of the Work) 50% (ix) Single Issue (the right to publish the complete Work in a single issue of a journal, periodical or newspaper) 50%

(x) Mechanical Reproduction (the right to reproduce the Work by mechanical means in audio or video or a combination of both except in so far as such reproduction is covered by (iv) hereof) 50%

(b) (i) Quotation and Extract 50%

(ii) Reprographic Reproduction 50%

The Author understands that works published by the Syndicate are included under the terms of this Agreement in the non-exclusive licensing schemes operated by such Reproduction Rights Organisations as the Copyright Licensing Agency (UK), the Copyright Clearance Center (USA) and the Copyright Agency Limited (Australia) and that any payments due for the copying of the Work under the said schemes shall be made in accordance with the licence terms then prevailing.

(c) (i) Any book club, reprint or adapted edition published in the English language by another publisher under licence from the Syndicate for a royalty payment 50% of the Syndicate's receipts from such royalties

(ii) Any book club, reprint or adapted edition manufactured by the Syndicate and sold to the licensee at a royalty-inclusive price 10% of the Syndicate's receipts from such sales
(d) The Author understands that the Syndicate may grant permission without charge to reproduce the Work in Braille, large type or other format provided such use is solely for the visually impaired and on a non-profit basis.

(e) The Author accepts that the Syndicate may enter the text of the Work into the corpus of written English which informs the Syndicate's publications in the field of English Language Teaching for the purpose of linguistic analysis and research (but not for publication), on the understanding that all entries to the Corpus are anonymised, that access to the Corpus is restricted, and that the Corpus and any citing of text from the Corpus are directly controlled by the Syndicate.

Author's copies

LS.ACA.MMXI

12. (a) Each Author shall receive five free copies of the hardback edition of the Work and five free copies of any paperback edition. Further copies for personal use (but not for resale) may be bought by the Author as follows: from the Syndicate's Americas Branch at a discount of 40% off the American list price where the Author lives in America; from the Syndicate's Australia Branch at a discount of 40% off the Australian list price where the Author lives in Australia or New Zealand; from the Syndicate's Asia Branch at a discount of 40% off the Asian list price where the

5

 Recently published, authoritative description of the foundations of this story, with a full treatment for planar, maximally supersymmetric Yang-Mills

NIMA ARKANI-HAMED JACOB BOURJAILY FREDDY CACHAZO ALEXANDER GONCHAROV ALEXANDER POSTNIKOV JAROSLAV TRNKA GRASSMANNIAN GEOMETRY OF SCATTERING AMPLITUDES





(a) (i) First Serial (the right to publish one or more extracts from the Work in successive issues of a periodical or newspaper beginning before publication of the Syndicate's first edition of the Work) 50%

(ii) Second (and subsequent) Serial (the right to publish one or more extracts from the Work in successive issues of a periodical or newspaper beginning at or following publication of the Syndicate's first edition of the Work) 50%

(iii) Dramatisation/Documentary 50%

(iv) Film 50%



Grassmannian Positroids, Plabic Graphs, & Scattering Amplitudes in $\mathcal{N}=4$ SYM Jacob L. Bourjaily, 2012

JB (2012)

The Author understands that works published by the Syndicate are included under the terms of this Agreement in the non-exclusive licensing schemes operated by such Reproduction Rights Organisations as the Copyright Licensing Agency (UK), the Copyright Clearance Center (USA) and the Copyright Agency Limited (Australia) and that any payments due for the copying of the Work under the said schemes shall be made in accordance with the licence terms then prevailing.

(c) (i) Any book club, reprint or adapted edition published in the English language by another publisher under licence from the Syndicate for a royalty payment 50% of the Syndicate's receipts from such royalties

(ii) Any book club, reprint or adapted edition manufactured by the Syndicate and sold to the licensee at a royalty-inclusive price 10% of the Syndicate's receipts from such sales
(d) The Author understands that the Syndicate may grant permission without charge to reproduce the Work in Braille, large type or other format provided such use is solely for the visually impaired and on a non-profit basis.

(e) The Author accepts that the Syndicate may enter the text of the Work into the corpus of written English which informs the Syndicate's publications in the field of English Language Teaching for the purpose of linguistic analysis and research (but not for publication), on the understanding that all entries to the Corpus are anonymised, that access to the Corpus is restricted, and that the Corpus and any citing of text from the Corpus are directly controlled by the Syndicate.

Author's copies

12. (a) Each Author shall receive five free copies of the hardback edition of the Work and five free copies of any paperback edition. Further copies for personal use (but not for resale) may be bought by the Author as follows: from the Syndicate's Americas Branch at a discount of 40% off the American list price where the Author lives in America, from the Syndicate's Australia Branch at a discount of 40% off the Australia or New Zealand; from the Syndicate's Asia Branch at a discount of 40% off the Australia or New Zealand; from the Syndicate's Asia Branch at a discount of 40% off the Asian list price where the Author lives in Australia or New Zealand; from the Syndicate's Asia Branch at a discount of 40% off the Asian list price where the Author lives in Australian Syndicate's Asia Branch at a discount of 40% off the Asian list price where the Author lives in Australian Syndicate's Asia Branch at a discount of 40% off the Asian list price where the Author lives in Australian Syndicate's Asia Branch at a discount of 40% off the Asian list price where the Author lives in Australian Syndicate's Asia Branch at a discount of 40% off the Asian list price where the Author lives in Australian Syndicate's Asia Branch at a discount of 40% off the Asian list price where the Author lives in Australian Syndicate's Asia Branch at a discount of 40% off the Asian list price where the Author lives in Australian Syndicate's Asia Branch at a discount of 40% off the Asian list price where the Author lives in Australian Syndicate's Asia Branch at a discount of 40% off the Asian list price where the Author lives in Australian Syndicate's Asia Branch at a discount of 40% off the Asian list price where the Author lives in Australian Syndicate's Asia Branch at a discount of 40% off the Asian list price where the Author lives in Australian Syndicate's Asia Branch at a discount of 40% off the Asian list price where the Author lives in Australian Syndicate's Asia Branch at a discount of 40% off the Asian list price where the Author liv

LS.ACA.MMXIII

- Two natural roads toward generalization:
 - non-planar theories with maximal supersymmetry
 - planar theories without (or less) supersymmetry

Two natural roads toward generalization:

- non-planar theories with maximal supersymmetry
- planar theories without (or less) supersymmetry

An experimental / phenomenological approach:

- construct all (reduced) diagrams
- enumerate all inequivalent varieties that result
- directly classify their functional relations and symmetries

Two natural roads toward generalization:

- non-planar theories with maximal supersymmetry
- planar theories without (or less) supersymmetry

An experimental / phenomenological approach:
construct all (reduced) diagrams

- enumerate all inequivalent varieties that result
- directly classify their functional relations and symmetries

Implications for physics far beyond amplitudes
Explorations Beyond Planar SYM

Two natural roads toward generalization:

- non-planar theories with maximal supersymmetry
- planar theories without (or less) supersymmetry

An experimental / phenomenological approach:
construct all (reduced) diagrams

- enumerate all inequivalent varieties that result
- directly classify their functional relations and symmetries

Implications for physics far beyond amplitudes

 Implications for diverse areas of mathematics graph theory, combinatorics, algebraic geometry, …

 For k=2 (MHV) on-shell functions, planar tree amplitudes (Parke-Taylor) form a complete basis:

 $\widetilde{f}_{\Gamma} = \sum PT(\sigma_1, \dots, \sigma_n) \text{ [Arkani-Hamed, JB, et al. (2014)]} \\ \{\sigma \in (\mathfrak{S}_n/\mathbb{Z}_n) | \forall \tau \in T : \sigma_{\tau_1} < \sigma_{\tau_2} < \sigma_{\tau_3} \}$



For k=2 (MHV) on-shell functions, planar tree amplitudes (Parke-Taylor) form a complete basis:

 $\widetilde{f}_{\Gamma} = \sum PT(\sigma_1, \dots, \sigma_n) \text{ [Arkani-Hamed, JB, et al. (2014)]} \\ \{\sigma \in (\mathfrak{S}_n/\mathbb{Z}_n) | \forall \tau \in T : \sigma_{\tau_1} < \sigma_{\tau_2} < \sigma_{\tau_3} \}$

For k=3 (NMHV) and n=6, a complete classification of non-planar on-shell functions and their relations was recently completed [JB, Franco, Galloni, Wen (2016)]

 For k=2 (MHV) on-shell functions, planar tree amplitudes (Parke-Taylor) form a complete basis:

 $\widetilde{f}_{\Gamma} = \sum_{\{\sigma \in (\mathfrak{S}_n/\mathbb{Z}_n) | \forall \tau \in T: \sigma_{\tau_1} < \sigma_{\tau_2} < \sigma_{\tau_3}\}} [Arkani-Hamed, JB, et al. (2014)]$

For k=3 (NMHV) and n=6, a complete classification of non-planar on-shell functions and their relations was recently completed [JB, Franco, Galloni, Wen (2016)]



 For k=2 (MHV) on-shell functions, planar tree amplitudes (Parke-Taylor) form a complete basis:

 $\widetilde{f}_{\Gamma} = \sum PT(\sigma_1, \dots, \sigma_n) \text{ [Arkani-Hamed, JB, et al. (2014)]} \\ \{\sigma \in (\mathfrak{S}_n/\mathbb{Z}_n) | \forall \tau \in T : \sigma_{\tau_1} < \sigma_{\tau_2} < \sigma_{\tau_3} \}$

For k=3 (NMHV) and n=6, a complete classification of non-planar on-shell functions and their relations was recently completed [JB, Franco, Galloni, Wen (2016)]



Prescriptive Approaches to Perturbation Theory (prior to loop integration)

 Loop *integrands*, being rational functions, are determinable by their cuts: *on-shell functions*

- Loop *integrands*, being rational functions, are determinable by their cuts: *on-shell functions*
- For many quantum field theories, on-shell all-loop recursion relations do (or almost certainly) exist:



[Arkani-Hamed, **JB**, Cachazo, Caron-Huot, Trnka (2010)] [Benincasa (2015-6); **JB**, Caron-Huot, Benincasa (*in prep*)]

- Loop *integrands*, being rational functions, are determinable by their cuts: *on-shell functions*
- For many quantum field theories, on-shell all-loop recursion relations do (or almost certainly) exist:



complications:

[Arkani-Hamed, **JB**, Cachazo, Caron-Huot, Trnka (2010)] [Benincasa (2015-6); **JB**, Caron-Huot, Benincasa (*in prep*)]

spurious propagators, mixture of components, ...

- Loop *integrands*, being rational functions, are determinable by their cuts: *on-shell functions*
- For many quantum field theories, on-shell all-loop recursion relations do (or almost certainly) exist:



♦ complications:
 [Arkani-Hamed, JB, Cachazo, Caron-Huot, Trnka (2010)]
 [Benincasa (2015-6); JB, Caron-Huot, Benincasa (*in prep*)]

spurious propagators, mixture of components, ...
someday may prove *ideal*; usefulness is moot today

The Unitarity-Based Approach

 Pick an arbitrary (but complete) basis of Feynman integrals, and use cuts to determine coefficients (given as on-shell functions)



[Bern, Dixon, Dunbar, Kosower]

Extremely general and powerful, with important applications at the LHC—e.g., BlackHat @ NLO

000 9



[Berger, Bern, et al.]



 $\langle 1 2 \rangle^4$ $\overline{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\cdots\langle n1\rangle}$





Fx7

(000)

[vergu (2009)]	.3				
<text><text><section-header><text><text><equation-block><equation-block><text><text><text></text></text></text></equation-block></equation-block></text></text></section-header></text></text>	<section-header><section-header><section-header><section-header><section-header><section-header><text><equation-block><text></text></equation-block></text></section-header></section-header></section-header></section-header></section-header></section-header>	$\frac{1}{4} \left(z_{n-1}^{2} z_{n-1}^{2} - 2 z_{n-1}^{2} z_{n-1}^{2} + z_{n-$	$0 \qquad (2)$ 1. One matrixe lay and one massive lay attached $-\frac{1}{4}r_{n-2n}^{2}x_{n-1n+2}^{2}x_{n+1n+1}^{2}x_{n+1}^{2}x_{n+1n+1}^{2} \qquad (2)$ 1. One matrixe lay and one massive lay attached $-\frac{1}{4}r_{n-2n}^{2}x_{n-1n+2}^{2}x_{n+1n+1}^{2}x_{n+1n+1}^{2}x_{n+1n+1}^{2} \qquad (2)$ $\frac{1}{4}(-x_{n}^{2}x_{n+1n+1}^{2}+x_{n+1n+1}^{2}+x_{n+1n+1}^{2}x_{n+1n+1}^{2}+x_{n+1n+1}^{2}x_{n+1n+1}^{$	$\begin{array}{c} & 0 & (12) \\ & 0 & (3) \\ \hline & 0 & (3) \\ \hline & 0 & (4) \\ \hline & 0 & (4) \\ \hline & 0 & (5) $	$-\frac{1}{4} \begin{bmatrix} a - a + 1 a + 2 \\ a + 3 a + 4 a - 2 \end{bmatrix} (2)$ $\frac{1}{4} (z_{-\lambda a + 1}^{2} z_{-\lambda + 1}^{2} z$
$-\frac{1}{2} \begin{bmatrix} a & a+1 & b-1 \\ b+1 & c-1 & c \end{bmatrix} $ (3) 1. Kining double-box topologies $-\frac{4}{3} \begin{bmatrix} a +1 & b-1 \\ b+1 & a-1 & a \end{bmatrix} + \frac{4}{3} \begin{bmatrix} a & a+1 \\ a-1 & b \end{bmatrix} \begin{bmatrix} b & b+1 \\ a-1 & a \end{bmatrix} = \frac{1}{4} \begin{bmatrix} a'_{1-1} & a'_{1-1} & b'_{1-1} & a \end{bmatrix} = \frac{1}{4} \begin{bmatrix} a'_{1-1} & a'_{1-1} & b'_{1-1} & a \end{bmatrix} = \frac{1}{4} \begin{bmatrix} a'_{1-1} & a'_{1-$	<section-header> C. Increment entroped e</section-header>	<text><equation-block><equation-block><equation-block><equation-block><equation-block><equation-block><equation-block><equation-block><equation-block><equation-block><equation-block><equation-block></equation-block></equation-block></equation-block></equation-block></equation-block></equation-block></equation-block></equation-block></equation-block></equation-block></equation-block></equation-block></text>	<text><text><text><section-header><text><text><equation-block><equation-block></equation-block></equation-block></text></text></section-header></text></text></text>	<text><equation-block><text><text><text><equation-block><text><text><text><equation-block><equation-block><equation-block><equation-block><equation-block><equation-block><equation-block></equation-block></equation-block></equation-block></equation-block></equation-block></equation-block></equation-block></text></text></text></equation-block></text></text></text></equation-block></text>	<text><text><equation-block><text><text><text><text><text><text><text><equation-block><equation-block><equation-block></equation-block></equation-block></equation-block></text></text></text></text></text></text></text></equation-block></text></text>





 Rather than starting from an arbitrary basis of loop integrands, tailor each to *manifestly* match one cut

 Rather than starting from an arbitrary basis of loop integrands, tailor each to *manifestly* match one cut

one loop:
 [JB, Caron-Huot, Trnka (2013)



 Rather than starting from an arbitrary basis of loop integrands, tailor each to *manifestly* match one cut

one loop:
 [JB, Caron-Huot, Trnka (2013)



 Rather than starting from an arbitrary basis of loop integrands, tailor each to *manifestly* match one cut

one loop:
 [JB, Caron-Huot, Trnka (2013)]



 Rather than starting from an arbitrary basis of loop integrands, tailor each to *manifestly* match one cut

one loop:
[JB, Caron-Huot, Trnka (2013)]

two loop: [JB, Trnka (2015)]

 Rather than starting from an arbitrary basis of loop integrands, tailor each to *manifestly* match one cut



 Rather than starting from an arbitrary basis of loop integrands, tailor each to *manifestly* match one cut

one loop:
 [JB, Caron-Huot, Trnka (2013)]

two loop: [JB, Trnka (2015)]



 Rather than starting from an arbitrary basis of loop integrands, tailor each to manifestly match one cut

one loop:
 [JB, Caron-Huot, Trnka (2013)



 This procedure continues to work at three loops: [JB, Herrmann, Trnka]

This procedure continues to work at three loops:
 [JB, Herrmann, Trnka]



This procedure continues to work at three loops:
 [JB, Herrmann, Trnka]

k

 $\left(k \right)$

i j

 Application to non-planar amplitudes is also now known—to all orders(!)
 [JB, Herrmann, McLeod, Stankowicz, Trnka (*in prep*)]

i j

This procedure continues to work at three loops:
 [JB, Herrmann, Trnka]

i j

. k /

k

 Application to non-planar amplitudes is also now known—to all orders(!)
 [JB, Herrmann, McLeod, Stankowicz, Trnka (*in prep*)]

Construction of N=8 integrands?

i

↓ J

 As an illustration of the current state of the art, consider the case of 2-to-2 scattering in planar SYM

 As an illustration of the current state of the art, consider the case of 2-to-2 scattering in planar SYM



[The Parking Spot Escalation (29 Nov. 2012)]

 As an illustration of the current state of the art, consider the case of 2-to-2 scattering in planar SYM



[*The Parking Spot Escalation* (29 Nov. 2012)]

 As an illustration of the current state of the art, consider the case of 2-to-2 scattering in planar SYM

1997





[*The Parking Spot Escalation* (29 Nov. 2012)]



[Bern, Rozowsky, Yan]








 As an illustration of the current state of the art, consider the case of 2-to-2 scattering in planar SYM



Bern, Rozowsky, Yan 199 Bern, Dixon, Smirnov 2005 Bern, Czakon, et al 2007 Bern, Carrasco, et al] Bern, Carrasco, et al 2009 2011 **JB**, DiRe, et al

 As an illustration of the current state of the art, consider the case of 2-to-2 scattering in planar SYM

6 7







 As an illustration of the current state of the art, consider the case of 2-to-2 scattering in planar SYM

5678

loop	#Feynman	#DCI ints	# <i>f</i> -graphs	1997	- [Bern, Rozowsky, Yan]
1	940	1	1		
2	47,380	1	1		
3	4,448,500	2	1		
4	6.723x10 ⁸	8	3	2005	
5	1.483×10^{11}	34	7	2007	[Bern, Czakon, <i>et al</i>]
6	4.484×10^{13}	229	26	2009	[Bern, Carrasco, <i>et al</i>]
7	1.780×10^{16}	1,873	127	2011	$\mathbf{JB}, \text{DiRe}, et al]$
8	8.969x10 ¹⁸	19,949	1,060		
9	5.592x10 ²¹	247,856	10,525	2015	JB , Heslop, Tran]
10	4.226x10 ²⁴	3,586,145	136,433	2010	generalized unitarity
					soft-collinear bootstrap 28
				(amplitude / correlator bootstrap

The soft-collinear bootstrap: the four-point amplitude is *uniquely* fixed by the criterion that its logarithm is at most log-squared divergent: [JB, DiRe, et al]

 $\operatorname{Res}(\log \mathcal{A}_4) = 0$ $\stackrel{\ell^2 \to 0}{\langle \ell a \rangle, [\ell a] \to 0}$

The soft-collinear bootstrap: the four-point amplitude is *uniquely* fixed by the criterion that its logarithm is at most log-squared divergent: [JB, DiRe, et al]

 $\underset{\substack{\ell^2 \to 0 \\ \langle \ell a \rangle, [\ell a] \to 0}}{\operatorname{Res}(\log \mathcal{A}_4)} = 0 \quad \left(\log \mathcal{A}_4 \right)^{(\ell)} = \mathcal{A}_4^{(\ell)} - \mathcal{A}_4^{(\ell-1)} \mathcal{A}_4^{(1)} - \dots - \frac{1}{\ell} \left(\mathcal{A}_4^{(1)} \right)^{\ell}$

◆ The soft-collinear bootstrap: the four-point amplitude is *uniquely* fixed by the criterion that its logarithm is at most log-squared divergent: [JB, DiRe, et al] $\underset{\ell^2 \to 0}{\operatorname{Res}(\log A_4) = 0} \quad (\log A_4)^{(\ell)} = \mathcal{A}_4^{(\ell)} - \mathcal{A}_4^{(\ell-1)} \mathcal{A}_4^{(1)} - \dots - \frac{1}{\ell} \left(\mathcal{A}_4^{(1)} \right)^{\ell}$

[**JB**, Heslop, Tran]

Some surprising tensions discovered at 8 loops:

 $\langle \ell a \rangle, [\ell a] \rightarrow 0$

The soft-collinear bootstrap: the four-point amplitude is *uniquely* fixed by the criterion that its logarithm is at most log-squared divergent: [JB, DiRe, *et al*]
 Res(logA₄)=0 (logA₄)^(ℓ)=A₄^(ℓ)-A₄^(ℓ-1)A₄⁽¹⁾-...- ¹/_ℓ (A₄⁽¹⁾)^ℓ
 ^{ℓ²→0}

[JB, Heslop, Tran]

Some surprising tensions discovered at 8 loops:
finite terms even on-shell—with elliptic cuts(!)



The soft-collinear bootstrap: the four-point amplitude is *uniquely* fixed by the criterion that its logarithm is at most log-squared divergent: [JB, DiRe, *et al*]
 Res(logA₄)=0 (logA₄)^(ℓ)=A₄^(ℓ)-A₄^(ℓ-1)A₄⁽¹⁾-...- ¹/_ℓ (A₄⁽¹⁾)^ℓ
 ^(ℓ)

[JB, Heslop, Tran]

Some surprising tensions discovered at 8 loops:
finite terms even on-shell—with elliptic cuts(!)
individually divergent integrals even off-shell(!)



There is a lossless translation between the *off-shell* correlator and the *on-shell* scattering amplitude!

$$\mathcal{F}^{(\ell)} \equiv \frac{1}{2} \frac{1}{\xi^{(4)}} \frac{\mathcal{G}_4^{(\ell)}(x_i)}{\mathcal{G}_4^{(0)}(x_i)} \qquad \xi^{(n)} \equiv \prod_{a=1}^n p_a^2 (p_a + p_{a+1})^2$$
$$\lim_{\substack{4\text{-point} \\ \text{light-like}}} \left(\xi^{(4)} \mathcal{F}^{(\ell)}\right) = \frac{1}{2} \left(\mathcal{A}_4(x_i)^2\right)^{(\ell)} = \mathcal{A}_4^{(\ell)} + \mathcal{A}_4^{(\ell-1)} \mathcal{A}_4^{(1)} + \dots$$

There is a lossless translation between the *off-shell* correlator and the *on-shell* scattering amplitude!





There is a lossless translation between the off-shell correlator and the on-shell scattering amplitude!



There is a lossless translation between the off-shell correlator and the on-shell scattering amplitude!



 Importantly, the *four*-point correlator contains (complete?) information of *all n*-point amplitudes!

$$\lim_{\substack{n \text{-point} \\ \text{light-like}}} \left(\xi^{(n)} \mathcal{F}^{(\ell)} \right) = \frac{1}{2} \sum_{k=0}^{n-4} \mathcal{A}_n^k \mathcal{A}_n^{n-k-4} / (\mathcal{A}_n^{n-4,(0)})$$

Simplicity Surviving Loop Integration and Future Directions

Can Simplicity Survive Integration?

- Loop integration remains a serious challenge for preserving simplicity of observable quantities
 - finite observables given in terms of divergent quantities requiring regularization (*is this necessary*?)
 - most regularization schemes *severely* break symmetries known to exist for finite observables
 - most versitile integration *techniques* spoil symmetries along the way
- The *traditional* toolbox for loop integration can be theoretically opaque/computationally intractable

Consider again the Parke-Taylor 2-to-4 amplitude;
divergences captured by BDS, leaving a finite *remainder*

Consider again the Parke-Taylor 2-to-4 amplitude;

- divergences captured by BDS, leaving a finite remainder
- Heroically computed by Del Duca, Duhr, Smirnov in 2010, in terms of 'Goncharov' polylogarithms

Consider again the Parke-Taylor 2-to-4 amplitude;

divergences captured by BDS, leaving a finite remainder

 Heroically computed by Del Duca, Duhr, Smirnov in 2010, in terms of 'Goncharov' polylogarithms

The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

Vittorio Del Duca

PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy E-mail: vittorio.del.duca@cern.ch

Claude Duhr

Institute for Particle Physics Phenomenology, University of Durham Durham, DH1 3LE, U.K. E-mail: claude.duhr@durham.ac.uk

Vladimir A. Smirnov

Nuclear Physics Institute of Moscow State University Moscow 119992, Russia E-mail: smirnov@theory.sinp.msu.ru

[Del Duca, Duhr, Smirnov (2010)]

Consider again the Parke-Taylor 2-to-4 amplitude; divergences captured by BDS, leaving a finite *remainder*

Here in 2	$\begin{split} & \frac{d_{n}^{(0)}}{2\pi} \left(n \cdot n \cdot n \cdot n \right) = \\ & \frac{1}{2\pi^{2}} \left(c \left(\frac{1}{2\pi^{2}} \cdot \frac{n \cdot n - 1}{2\pi^{2}} \right) + \frac{1}{2\pi^{2}} c \left(\frac{1}{2\pi^{2}} \cdot \frac{1}{2\pi^{2}} + \frac{n \cdot n - 1}{2\pi^{2}} \right) + \frac{1}{2\pi^{2}} c \left(\frac{1}{2\pi^{2}} \cdot \frac{n - 1}{2\pi^{2}} + \frac{1}{2\pi^{2}} + \frac{1}{2\pi$	$ \begin{array}{l} \frac{1}{4} G\left(\frac{1}{1+m},\frac{m-1}{m+m-1},0,1\right)-\frac{1}{4} G\left(\frac{1}{1+m},\frac{m-1}{m+m-1},0,\frac{1}{m},\frac{1}{m+m-1}\right)+\\ \frac{1}{4} G\left(\frac{1}{1+m},\frac{m-1}{m+m-1},1,0,1\right)-\frac{1}{4} G\left(\frac{1}{1+m},\frac{m-1}{m+m-1},1,\frac{1}{m},0,1\right)+\\ \frac{1}{4} G\left(\frac{1}{1+m},\frac{m-1}{m+m-1},1,0,1\right)-\frac{1}{4} G\left(\frac{1}{1+m},\frac{m-1}{m+m-1},1,\frac{1}{m},0,1\right)+\\ \frac{1}{4} G\left(\frac{1}{1+m},\frac{m-1}{m+m-1},1,0,1\right)-\frac{1}{4} G\left(\frac{1}{1+m},\frac{m-1}{m+m-1},1,1,1\right)-\\ \frac{1}{4} G\left(\frac{1}{1+m},\frac{m-1}{m+m-1},1,0,1\right)-G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1\right)+\\ \frac{1}{4} G\left(\frac{1}{1+m},\frac{m-1}{m+m-1},1,1,1\right)-G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1\right)-\frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m+m-1},1,1\right)-\\ \frac{1}{4} G\left(\frac{1}{1+m},\frac{1}{m},\frac{1}{m+m-1},1,1,1\right)-G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1\right)-\frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m+m-1},1,1\right)-\\ \frac{1}{4} G\left(\frac{1}{1+m},\frac{1}{m+m-1},1,1,1,1\right)-G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1,1\right)-\\ \frac{1}{4} G\left(\frac{1}{1+m},\frac{1}{m+m-1},1,\frac{1}{m+m-1},1,1,1\right)-G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m+m-1},1,1,1\right)-\\ \frac{1}{4} G\left(\frac{1}{1+m},\frac{1}{m+m-1},1,\frac{1}{m},1,1,1\right)-\frac{1}{4} G\left(\frac{1}{1+m},\frac{1}{m+m-1},1,1,1\right)-\\ \frac{1}{4} G\left(\frac{1}{1+m},\frac{1}{m+m-1},1,\frac{1}{m},1,1,1\right)+\frac{1}{4} G\left(\frac{1}{1+m},\frac{1}{m+m-1},1,1,1\right)-\\ \frac{1}{4} G\left(\frac{1}{1+m},\frac{1}{m+m-1},1,\frac{1}{m},1,1,1\right)+\frac{1}{4} G\left(\frac{1}{1+m},\frac{1}{m+m-1},1,1,1\right)-\\ \frac{1}{4} G\left(\frac{1}{1+m},\frac{1}{m+m-1},1,\frac{1}{m},1,1,1\right)+\\ \frac{1}{4} G\left(\frac{1}{1+m},\frac{1}{m+m-1},1,1,1\right)-G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1,1\right)-\\ \frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m+m-1},1,1,1\right)-G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1,1\right)-\\ \frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1,1\right)-\frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1\right)-\\ \frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1,1\right)-\frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1\right)-\frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1\right)-\\ \frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1\right)+\frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1\right)-\\ \frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1\right)+\frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1\right)-\\ \frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1,1\right)+\frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1\right)-\frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1\right)-\\ \frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1\right)+\frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1\right)-\frac{1}{4} G\left(\frac{1}{m},\frac{1}{m},\frac{1}{m},1\right)-\frac{1}{4} G\left(\frac{1}{m},\frac{1}{m}$	$ \begin{split} & \mathcal{G}\left(\frac{1}{n}(8,8,\frac{1}{n_{1}}(1)+\frac{1}{2}\mathcal{G}\left(\frac{1}{n_{1}}(8,8,\frac{1}{n_{1}}+n_{1}}\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{n_{1}}(8,8,\frac{1}{n_{1}}+n_{1}}\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{n_{1}}(8,\frac{1}{n_{1}}+n_{1}}\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{n_{1}}(8,\frac{1}{n_{1}}+n_{1}}\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{n_{1}}(8,\frac{1}{n_{1}}+n_{1}}\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{n_{1}}+n_{$	$\begin{split} &\frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) + \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1-\infty} \frac{1}{n} \right) - \frac{1}{4^2} \left(6 \max - \frac{1}{1$	$ \begin{split} &\frac{1}{4^2} \left(\frac{1}{1-w} \sin \frac{1}{w} \frac{1}{w} + 1 \right) + \frac{1}{4^2} \left(\frac{1}{1-w} \sin \frac{w-1}{w+w-1} + 1 \right) - \\ &\frac{1}{4^2} \left(\frac{1}{1-w} \cos \theta_1 \theta_1 \right) + \frac{1}{4^2} \left(\frac{1}{1-w} \cos \theta_1 \theta_1 \right) + \frac$	$ \begin{split} &\frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} & \sin \mu \right) - \frac{1}{2^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} & \sin \mu \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) + \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) - \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) + \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) + \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) + \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) + \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) + \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) + \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) + \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) + \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu \frac{1}{1-u} \right) + \frac{1}{4^2} \left(\frac{1}{1-u} & \sin \mu 1$	10V ns
	$ \begin{split} & \frac{1}{2^{2}} \left(\frac{1}{1-u} \max \frac{1}{1-u} \right) + \frac{1}{2^{2}} \left((u-1) + \frac{1}{1-u} + \frac{1}{1$	$ \begin{array}{l} \frac{1}{4} G\left(\left(\frac{1}{1-w}, \frac{w_{n-1}}{w_{n-1}}, \frac{1}{1+v} \right) B(w_{n}) - \\ \frac{1}{4} C\left(\left(\frac{1}{1-w}, \frac{w_{n-1}}{w_{n-1}}, \frac{1}{1+v} \right) B(w_{n}) + \frac{1}{4} C\left(\frac{1}{w_{n}}, \frac{w_{n-1}}{w_{n-1}}, \frac{1}{1+v} \right) B(w_{n}) - \\ \frac{1}{4} C\left(\frac{1}{1-w}, \frac{w_{n-1}}{w_{n-1}}, \frac{1}{1+v} \right) B(w_{n}) + \frac{1}{4} C\left(\frac{1}{w_{n}}, \frac{w_{n-1}}{w_{n-1}}, \frac{1}{1+w} \right) B(w_{n}) - \\ \frac{1}{4} C\left(\frac{1}{1-w}, \frac{w_{n-1}}{w_{n-1}}, \frac{1}{1+w} \right) B(w_{n}) + \frac{1}{4} C\left(\frac{1}{w_{n}}, \frac{w_{n-1}}{w_{n-1}}, \frac{1}{1+w}, \frac{1}{1+w} \right) B(w_{n}) - \\ \frac{1}{4} C\left(\frac{1}{1-w}, \frac{w_{n-1}}{w_{n-1}}, \frac{1}{1+w}, \frac{1}{1+w} \right) B(w_{n}) + \frac{1}{4} C\left(\frac{1}{w_{n}}, \frac{w_{n-1}}{w_{n-1}}, \frac{1}{1+w}, \frac{1}{1+w} \right) B(w_{n}) - \\ \frac{1}{4} C\left(\frac{1}{w_{n}}, \frac{w_{n-1}}{w_{n-1}}, \frac{1}{1+w}, \frac{1}{1+w}, \frac{1}{1+w} \right) B(w_{n}) + \\ \frac{1}{4} C\left(\frac{1}{w_{n}}, \frac{w_{n-1}}{w_{n-1}}, \frac{1}{1+w}, \frac{1}{1+w}, \frac{1}{1+w} \right) B(w_{n}) + \\ \frac{1}{4} C\left(\frac{1}{w_{n}}, \frac{w_{n-1}}{w_{n-1}}, \frac{1}{1+w}, $	$ \begin{split} & \frac{1}{2} \left(\left(\frac{1}{1-n} \left(\frac{1}{n} + \frac{1}{n} + \frac{1}{n} \right) H(0, n) + \frac{1}{2} \left(\left(\frac{1}{1-n} + \frac{1}{n} + \frac{1}{$	$ \begin{array}{l} \frac{1}{2} \left(\left(\frac{1}{1-w} + m_{w} \right) \right) H(0,w) + \frac{1}{2} \left(\left(\frac{1}{1-w} + m_{w} \right) \right) H(0,w) + \\ \frac{1}{2} \left(\left(\frac{1}{1-w} + m_{w} \right) \right) H(0,w) + \frac{1}{2} \left(\left(\frac{1}{1-w} + m_{w} \right) \right) H(0,w) + \\ \frac{1}{2} \left(\left(\frac{1}{1-w} + m_{w} \right) \right) H(0,w) + \frac{1}{2} \left(\left(\frac{1}{1-w} + m_{w} \right) \right) H(0,w) + \\ \frac{1}{2} \left(\left(n,w + \frac{1}{1-w} + m_{w} \right) \right) H(0,w) + \frac{1}{2} \left(\left(n,w + \frac{1}{1-w} + m_{w} \right) \right) H(0,w) + \\ \frac{1}{2} \left(\left(n,w + \frac{1}{1-w} + m_{w} \right) \right) H(0,w) + \frac{1}{2} \left(\left(n,w + \frac{1}{1-w} + m_{w} + m_{w} \right) \right) H(0,w) + \\ \frac{1}{2} \left(\left(n,w + \frac{1}{1-w} + m_{w} $	$ \begin{array}{l} \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min_{i} 1, \frac{1}{1}, \frac{1}{1} \right) H(0, \alpha) + \frac{1}{2} \left(\left(\min$	$ \begin{array}{l} \frac{1}{2^2} \left(\left(\frac{1}{1-w}, \gamma_{111}, \right) H\left((u_1 u_1 + \frac{1}{4^2} \left(\left(\frac{1}{1-w}, \gamma_{111}, \right) \right) H\left((u_1 u_1 - \frac{1}{4^2} \right) \right) \\ \frac{1}{4^2} \left(\left(u_{111}, \frac{1}{1-w}, 1 \right) H\left((u_1 u_1 + \frac{1}{4^2} \left(\left(u_{111}, \frac{1}{1-w}, \gamma_{111}, 1 \right) \right) H\left((u_1 u_1 - \frac{1}{4^2} \right) \right) \\ \frac{1}{4^2} \left(\left(u_{111}, \frac{1}{1-w} \right) \right) H\left((u_1 u_1 + \frac{1}{4^2} \left(\left(u_{111}, \frac{1}{1-w}, \gamma_{111}, 1 \right) \right) H\left((u_1 u_1 - \frac{1}{4^2} \right) \right) \\ \frac{1}{4^2} \left(u_{111}, \frac{1}{1-w}, \gamma_{111} \right) H\left((u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) \right) H\left((u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left((u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) \right) H\left((u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) \right) H\left((u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w}, \gamma_{111}, 1 \right) H\left(u_1 + \frac{1}{4^2} \left(\frac{1}{1-w$	
	$ \begin{split} & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} 1 \right) H(0, m) + \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} 1 \right) H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} 1 \right) H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} 1 \right) H(0, m) H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} 1 \right) H(0, m) H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} 1 \right) H(0, m) H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} 1 \right) H(0, m) H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} 1 \right) H(0, m) H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} 1 \right) H(0, m) H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} 1 \right) H(0, m) H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} 1 \right) H(0, m) H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}{4^2} \mathcal{G} \left(\max_{i=1}^n \frac{1}{1-\max_{i=1}^n} H(0, m) + \\ & \frac{1}$	$\begin{split} \frac{1}{2^{4}} \mathcal{H}\left(0,1,(u+u)\right) + \frac{1}{2^{4}} \mathcal{H}\left(0,1,u\right) + \frac{1}{4^{4}} \mathcal{H}\left(0,1,0,0\right) \mathcal{H}\left(0,\frac{u+u+u}{u}\right) + \frac{1}{2^{4}} \mathcal{H}\left(0,1,(u+u)\right) + \frac{1}{4^{4}} \mathcal{H}\left(0,1,(u+u)\right) + \frac{1}{2^{4}} \mathcal{H}\left(0,1,(u+u)\right) + \frac{1}{4^{4}} \mathcal{H}\left(0,1,(u+u)\right) + \frac{1}{2^{4}} \mathcal{H}\left(1,(u+u)\right) + \frac{1}{2^{4}} \mathcal{H}\left($	$ \begin{array}{l} & \left[\begin{array}{c} \frac{1}{2} H(y_{n,y}) H\left(0,0,1,\frac{y_{n}+y_{n-1}}{y_{n-1}}\right) - H\left(0,y_{n}\right) H\left(0,0,1,\frac{y_{n}+y_{n-1}}{y_{n-1}}\right) - \\ & H\left(0,y_{n}\right) H\left(0,0,1,\left(y_{n}+y_{n}\right)\right) - \frac{1}{2} H\left(0,y_{n}\right) H\left(0,0,1,\frac{y_{n}+y_{n-1}}{y_{n-1}}\right) - \\ & \frac{1}{2} H\left(y_{n,y}\right) H\left(0,0,1,\frac{y_{n}+y_{n-1}}{y_{n-1}}\right) - \frac{1}{2} H\left(0,y_{n}\right) H\left(0,0,1,\frac{y_{n}+y_{n-1}}{y_{n-1}}\right) - \\ & \frac{1}{2} H\left(y_{n,y}\right) H\left(0,1,1,\frac{y_{n}+y_{n-1}}{y_{n-1}}\right) - \frac{1}{2} H\left(0,y_{n}\right) H\left(0,1,0,y_{n}\right) - \\ & \frac{1}{2} H\left(y_{n,y}\right) H\left(0,1,1,\frac{y_{n}+y_{n-1}}{y_{n-1}}\right) - \frac{1}{2} H\left(0,y_{n}\right) H\left(0,1,0,y_{n}\right) - \\ & \frac{1}{2} H\left(y_{n,y}\right) H\left(0,1,1,\frac{y_{n}+y_{n-1}}{y_{n-1}}\right) - \frac{1}{2} H\left(0,y_{n}\right) H\left(1,0,y_{n}\right) - \\ & \frac{1}{2} H\left(y_{n,y}\right) H\left(1,0,y_{n}\right) + \frac{1}{2} H\left(y_{n}\right) H\left(1,0,y_{n}\right) + \frac{1}{2} H\left(y_{n}\right) H\left(1,0,y_{n}\right) - \\ & \frac{1}{2} H\left(y_{n,y}\right) H\left(1,0,y_{n}\right) + \frac{1}{2} H\left(y_{n}\right) H\left(1,0,y_{n}\right) + \frac{1}{2} H\left(y_{n}\right) H\left(1,0,y_{n}\right) - \\ & \frac{1}{2} H\left(y_{n,y}\right) H\left(1,0,y_{n}\right) + \frac{1}{2} H\left(y_{n}\right) H\left(1,0,y_{n}\right) + \frac{1}{2} H\left(y_{n}\right) H\left(1,0,y_{n}\right) - \\ & \frac{1}{2} H\left(y_{n,y}\right) H\left(1,0,y_{n}\right) + \frac{1}{2} H\left(y_{n}\right) H\left(1,0,y_{n}\right) + \\ & \frac{1}{2} H\left(y_{n}\right) H\left(1,0,y_{n}\right) - \frac{1}{2} H\left(y_{n}\right) H\left(1,0,y_{n}\right) - \\ & \frac{1}{2} H\left(y_{n}\right) H\left(1,0,y_{n}\right) + \frac{1}{2} H\left(y_{n}\right) H\left(1,0,y_{n}\right) - \\ & \frac{1}{2} H\left(y_{n}\right) H\left(y_{n}\right) + \\ & \frac{1}{2} H\left(y_{n}\right) + \\ &$	$ \begin{split} \mathcal{H} & = \frac{1}{2n} e^{2} d T \left(u_{n,n} \right) \mathcal{H} \left(+ \frac{1}{u_{n,n}} \right) + \frac{1}{3n} e^{2} d T \left(u_{n,n} \right) \mathcal{H} \left(+ \frac{1}{u_{n,n}} \right) + \frac{1}{3n} e^{2} d T \left(u_{n,n} \right) \mathcal{H} \left(+ \frac{1}{u_{n,n}} \right) + \frac{1}{3n} e^{2} d T \left(u_{n,n} \right) \mathcal{H} \left(+ \frac{1}{u_{n,n}} \right) + \frac{1}{3n} e^{2} d T \left(u_{n,n} \right) \mathcal{H} \left(+ \frac{1}{u_{n,n}} \right) + \frac{1}{n} e^{2} d T \left(u_{n,n} \right) \mathcal{H} \left(+ \frac{1}{u_{n,n}} \right) + \frac{1}{n} e^{2} d T \left(u_{n,n} \right) \mathcal{H} \left(u_{n,n$	$ \begin{split} &\frac{1}{2} f'(u_{0,0}) W \left(0,1,1,\frac{1}{u_{0,1}} \right) + \frac{1}{2} f'(u_{0,0}) W \left(0,1,1,\frac{1}{u_{0,1}} \right) - \frac{1}{2} f'(u_{0,0,1}) W \left(0,1,1,\frac{1}{u_{0,1}} \right) - \frac{1}{2} f'(u_{0,0,1},1,\frac{1}{u_{0,1}} - \frac{1}{2} f'(u_{0,0,1},1,\frac{1}{u_{0,1}} - \frac{1}{u_{0,1}} - \frac{1}{2} f'(u_{0,0,1},1,\frac{1}{u_{0,1}} - \frac{1}{u_{0,1}} \right) - \frac{1}{2} f'(u_{0,0,1,1,1},\frac{1}{u_{0,1}} - \frac{1}{2} f'(u_{0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1$	<text><list-item><list-item><list-item><list-item><list-item><section-header><list-item><list-item><list-item><list-item><list-item><list-item><list-item><list-item><list-item><list-item><list-item><list-item><list-item><list-item><list-item><list-item><list-item><list-item><list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></section-header></list-item></list-item></list-item></list-item></list-item></text>	

33

Consider again the Parke-Taylor 2-to-4 amplitude;

• divergences captured by BDS, leaving a finite remainder

 Heroically computed by Del Duca, Duhr, Smirnov in 2010, in terms of 'Goncharov' polylogarithms

Classical Polylogarithms for Amplitudes and Wilson Loops

A. B. Goncharov,¹ M. Spradlin,² C. Vergu,² and A. Volovich²

¹Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA ²Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA

We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.

Consider again the Parke-Taylor 2-to-4 amplitude;

• divergences captured by BDS, leaving a finite remainder

 Heroically computed by Del Duca, Duhr, Smirnov in 2010, in terms of 'Goncharov' polylogarithms

Classical Polylogarithms for Amplitudes and Wilson Loops

A. B. Goncharov,¹ M. Spradlin,² C. Vergu,² and A. Volovich²

¹Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA ²Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA

We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.

$$R(u_1, u_2, u_3) = \sum_{i=1}^{3} \left(L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right) \\ - \frac{1}{8} \left(\sum_{i=1}^{3} \text{Li}_2(1 - 1/u_i) \right)^2 + \frac{J^4}{24} + \frac{1}{2} \zeta_2 \left(J^2 + \zeta_2 \right)$$

33

 Iterated integrals: symbology and the symbolic bootstrap method for entire amplitudes

[Dixon *et al*.; ...]

 Iterated integrals: symbology and the symbolic bootstrap method for entire amplitudes

Integration by parts (& algebraic geometry thereof)

- Iterated integrals: symbology and the symbolic bootstrap method for entire amplitudes
- Integration by parts (& algebraic geometry thereof)
- Differential equations for iterated integration
 [Caron-Huot, Henn; Drummond, Trnka; ...]

- Iterated integrals: symbology and the symbolic bootstrap method for entire amplitudes
- Integration by parts (& algebraic geometry thereof)
- Differential equations for iterated integration
 [Caron-Huot, Henn; Drummond, Trnka; ...]
- ◆ Symmetry-preserving regularization schemes $p_a^2 \mapsto p_a^2 + \delta \frac{(p_{a-1} + p_a)^2 (p_a + p_{a+1})^2}{(p_{a-1} + p_a + p_{a+1})^2}$ [JB, Caron-Huot, Trnka (2013)]

- Iterated integrals: symbology and the symbolic bootstrap method for entire amplitudes
- Integration by parts (& algebraic geometry thereof)
- Differential equations for iterated integration
 [Caron-Huot, Henn; Drummond, Trnka; ...]
- ◆ Symmetry-preserving regularization schemes $p_a^2 \mapsto p_a^2 + \delta \frac{(p_{a-1} + p_a)^2 (p_a + p_{a+1})^2}{(p_{a-1} + p_a + p_{a+1})^2}$ [JB, Caron-Huot, Trnka (2013)]
- (Nearly) finite integrals for finite observables
 JB, Dixon, Dulat, Panzer, (in prep)]

Using the 'dual-conformal' regularization scheme,

$$p_a^2 \mapsto p_a^2 + \delta \frac{(p_{a-1} + p_a)^2 (p_a + p_{a+1})^2}{(p_{a-1} + p_a + p_{a+1})^2}$$

all planar loop integrals take the form:

Using the 'dual-conformal' regularization scheme,

$$p_a^2 \mapsto p_a^2 + \delta \frac{(p_{a-1} + p_a)^2 (p_a + p_{a+1})^2}{(p_{a-1} + p_a + p_{a+1})^2}$$

all planar loop integrals take the form: 2L

$$I \mapsto \sum_{k=0} I_k \log^k(\delta)$$

[**JB**, Dixon, Dulat, Panzer, (*in prep*)]

Using the 'dual-conformal' regularization scheme,

$$p_a^2 \mapsto p_a^2 + \delta \frac{(p_{a-1} + p_a)^2 (p_a + p_{a+1})^2}{(p_{a-1} + p_a + p_{a+1})^2}$$

all planar loop integrals take the form: $L \rightarrow \sum_{k=1}^{2L} L \rightarrow k(s)$

$$I \mapsto \sum_{k=0}^{k} I_k \log^k(\delta)$$

[**JB**, Dixon, Dulat, Panzer, (*in prep*)]

 Coefficients can be calculated individually as manifestly finite (easy to integrate) integrals

Using the 'dual-conformal' regularization scheme,

$$p_a^2 \mapsto p_a^2 + \delta \frac{(p_{a-1} + p_a)^2 (p_a + p_{a+1})^2}{(p_{a-1} + p_a + p_{a+1})^2}$$

all planar loop integrals take the form:

$$I \mapsto \sum_{k=0}^{2L} I_k \log^k(\delta)$$

[JB, Dixon, Dulat, Panzer, (*in prep*)]

- Coefficients can be calculated individually as manifestly finite (easy to integrate) integrals
 - which are *manifestly* dual-conformal if the initial integral were; if not, then: as an expansion of DCI ints, with non-DCI coefficients

Objectives for Going Forward

36
Objectives for Going Forward

Develop new techniques for integration

make evaluation as simple and easy as finding integrands

Objectives for Going Forward

Develop new techniques for integration

make evaluation as simple and easy as finding integrands

Exploit simplicity into powerful new technology *better representations of amplitudes* prior to integration

Objectives for Going Forward

Develop new techniques for integration

make evaluation as simple and easy as finding integrands

Exploit simplicity into powerful new technology *better representations of amplitudes* prior to integration

Reformulate foundations using only observables
make no reference to unobservable quantities





Questions?



























CAN WE FIX IT? Why you're not allowed to mend your own stuff









