Aspects of geometric phases in QFT

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based on work with Marco Baggio and Kyriakos Papadodimas

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Parameter-spaces and geometric phases in QM

(review I)

Frequently in QM the Hamiltonian/spectrum depends on continuous parameters —external couplings $(\lambda_1, \lambda_2, \ldots)$

can be components of a fixed or slowly varying background

The space of λ's is endowed with rich geometric properties that encode important physical effects

Berry phase is one of the most characteristic examples

under adiabatic **cycles** on space of λ's quantum states can pick up a phase

$$|\Psi(\vec{\lambda})\rangle_T = e^{i\gamma}\, e^{-\frac{i}{\hbar}\int_0^T dt\, E_\Psi(t)}\; |\Psi(\vec{\lambda})\rangle_0$$

Berry phase

much like a frame dragged around a curved manifold

- Intrinsic property of the quantum system

  Phases depend only on path *C* in λ-space

$$\gamma = i \oint_C d\vec{\lambda} \, \langle \Psi(\vec{\lambda}) | \partial_{\vec{\lambda}} | \Psi(\vec{\lambda}) \rangle$$

or

$$\gamma = i \oint_C d\vec{\lambda} \, \mathcal{A}(\vec{\lambda})$$

Pancharatnam-Berry connection

connection on a vector bundle of Hilbert spaces

- For *D* degenerate states the *U(1)* phase upgrades to a *U(D)* transform

  $\Rightarrow$ Berry connection is **non-abelian**

- there is a corresponding curvature

$$F_{\mu\nu} = \frac{\partial \mathcal{A}_\nu}{\partial \lambda^\mu} - \frac{\partial \mathcal{A}_\mu}{\partial \lambda^\nu} + [\mathcal{A}_\mu, \mathcal{A}_\nu]$$

- for which one can write a general spectral formula

$$\boxed{\left(F^{(n)}_{\mu\nu}\right)_{ab} = \sum_{m\neq n} \sum_{c,d} \frac{1}{(E_n - E_m)^2} \langle n, b|\partial_\mu H|m, c\rangle g^{cd}_{(m)} \langle m, d|\partial_\nu H|n, a\rangle - (\mu \leftrightarrow \nu)}$$

in sector with energy E_n and degenerate states $|n, a\rangle$

$$g_{ab} = \langle a|b\rangle$$

Berry phase is a physically detectable effect with many **applications…**

The curvature $\left(F_{\mu\nu}^{(n)}\right)_{ab}$ is a very interesting observable

- typically hard to evaluate analytically, but
- if known it helps constrain the parametric dependence of state overlaps and potentially other observables

(examples soon)

Geometric phases in (continuum) QFT?

*tt** equations: a beautiful example from the 90s (Cecotti-Vafa '91)

- 2d QFTs with 4 real supersymmetries
- parameter space: complex (superpotential) couplings
- topologically twisted, spectrum truncated to 1/2-BPS states (chiral - antichiral)

  corresponding operators close among themselves under the OPE **(chiral ring)**

$$\phi_i(x)\phi_j(0) \sim C_{ij}^k\, \phi_k(0) + \text{regular}$$

Cecotti & Vafa computed the Berry curvature of these ground states and found an expression that closes on the chiral ring

$$\left(F_{i\bar{j}}\right)_K^{\ L} = [C_i, \bar{C}_j]_K^L$$
$$\equiv C_{iK}^P g_{P\bar{Q}} C^{*\bar{Q}}_{\bar{j}\bar{R}} g^{\bar{R}L} - g_{K\bar{N}} C^{*\bar{N}}_{\bar{j}\bar{U}} g^{\bar{U}V} C_{iV}^L$$

- RHS is algebraic functional of OPE coefficients and 2-point functions

$$g_{K\bar{L}} = \langle \bar{L}|K\rangle$$

chiral-antichiral overlap → all SUSIES broken

(topological-antitopological → tt*)

- LHS is curvature, involves derivatives wrt complex couplings λ^i

- In a set of conventions LHS is expressed only in terms of derivatives of the 2-point overlaps

Leads to PDEs for coupling dependence of non-SUSY quantities

which Cecotti & Vafa solved in several examples

Unfortunately these ideas were not pushed much further in the context of higher-dimensional QFTs

- *tt** equations in 4d QFTs were unknown until recently and results of analogous power were beyond reach
- Berry phase as an intrinsic quantum property was not explored systematically in QFT

Parallel ideas in QFT

(review II)

The idea to think about the properties of the space of parameters **(theory space)** seriously has been discussed over the years in (continuum) QFT in various contexts

- Wilsonian RG

- Zamolodchikov c-theorem in 2d QFTs

  Zamolodchikov metric

$$g_{ij} = x^4 \langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle \big|_{x^2 = x_0^2}$$

Early ‘90s: much interest in 2d CFTs / worldsheet description of strings

In that context: 2d QFT couplings = spacetime fields

Interest to understand the theory space of exactly marginal couplings (deformations that preserve conformal invariance)

conformal manifolds

It was understood (Kutasov, Sonoda, Zwiebach...) that besides the Zamolodchikov metric the conformal manifolds also possess natural notions of **parallel transport, connection**

Conformal perturbation theory: a connection in theory-space tells us how to compare operators in near-by CFTs

A covariant derivative incorporates regularization prescriptions

$$\nabla_\mu \langle \varphi_1(z_1) \cdots \varphi_n(z_n) \rangle \sim \left[\int d^{d+1}x \, \langle \mathcal{O}_\mu(x) \varphi_1(z_1) \ldots \varphi_n(z_n) \rangle \right]_{regularised}$$

the curvature of this connection involves integrated correlation functions

Roughly:

$$[\nabla_\mu, \nabla_\nu]_{12} \sim \int d^{d+1}x \int d^{d+1}y \, \langle \mathcal{O}_{[\mu}\underbrace{(x)\mathcal{O}}_{\text{antisymmetry}}{}_{\nu]}(y)\varphi_1(z_1)\varphi_2(z_2)\rangle$$

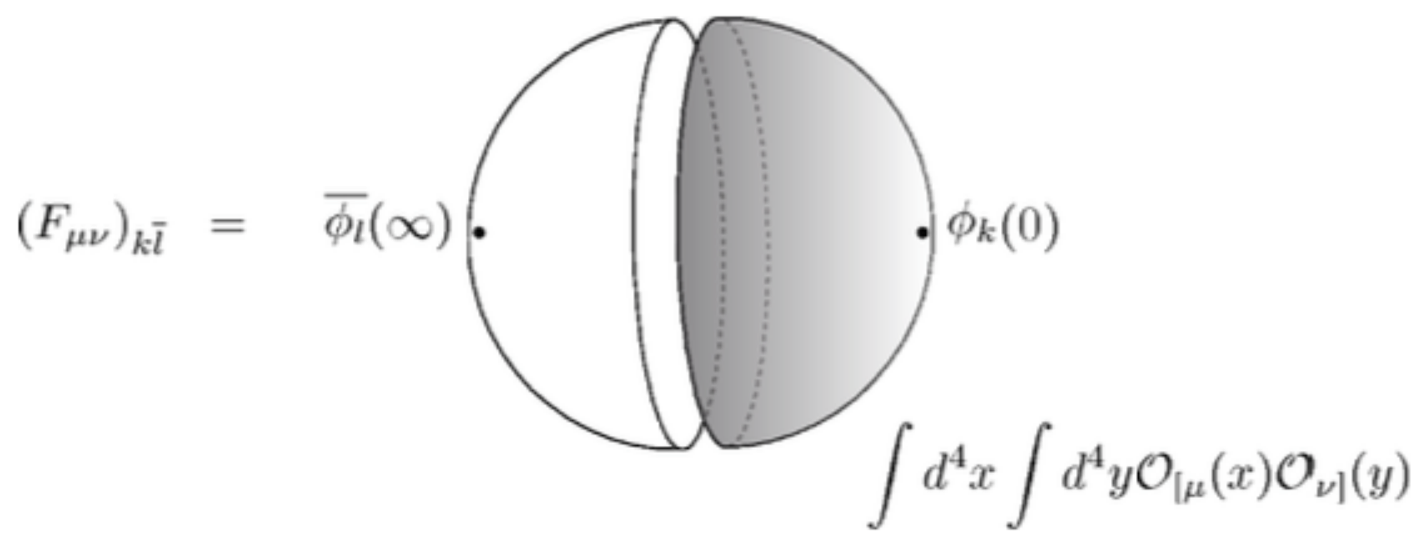
integrations lead to divergences

regularisation leads to non-vanishing commutators ⇒ curvature

Ranganathan-Sonoda-Zwiebach ('93) prescription

(cut out small balls around operators, do not allow collisions, remove divergent pieces)

4-point operator formula for the curvature



$on\ \mathbb{R}^4$

$$(F_{\mu\nu})_{kl} = \frac{1}{(2\pi)^4} \int_{|x|\leq 1} d^4x \int_{|y|\leq 1} d^4y \, \langle \varphi_l(\infty) \mathcal{O}_{[\mu}(x) \mathcal{O}_{\nu]}(y) \varphi_k(0) \rangle$$

Connection in conformal perturbation theory vs Berry connection in QM

They are obviously related concepts…

Q: Can one go from one to the other?

Q: Can one import the geometric phases of QM systematically & more generally in QFT? Computable?

Q: Will this lead to new lessons?

Berry phases in (non-perturbative) QFT

(old meets modern)

Technical approach

Baggio-VN-Papadodimas '17

Berry phase by quantizing QFT in the Hamiltonian framework

Obvious issues:

- UV divergences:

  ☞ renormalize

- IR issues: e.g. continuous spectra

  ☞ put theory on a hypercylinder : $\mathbb{R} \times \mathcal{Q}$, $\mathcal{Q}$ compact

Example 1: Berry phase of photons

- Consider electromagnetism with a θ interaction

e.g. Wilczek, PRL '89

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{64\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

- θ has non-trivial implications if it varies in spacetime, or if there are boundaries/walls

- interested in adiabatic changes of θ in time

- this theory is free: as we change *e* and θ adiabatically the spectrum is unchanged, but the energy eigenstates can rotate

- Hamiltonian deformations

$$\partial_{e^2} H = \frac{1}{e^4} \int d^3x \left(\vec{E}^2 - \vec{B}^2\right) \;\;, \quad \partial_\theta H = \frac{1}{8\pi^2} \int d^3x \, \vec{E} \cdot \vec{B}$$

- Place on $\mathbb{R} \times \mathbb{T}^3$ and quantize in Coulomb gauge

$$\vec{A}(t,\vec{x}) = \sum_{\vec{k}} \sum_{\epsilon=\pm} \sqrt{\frac{\hbar e^2}{2\omega_k V}} \left(\vec{e}_\epsilon(\vec{k}) a_{\vec{k},\epsilon} e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} + \bar{\vec{e}}_\epsilon(\vec{k}) a^\dagger_{\vec{k},\epsilon} e^{i\omega_k t - i\vec{k}\cdot\vec{x}} \right)$$

$$\omega_k = c|\vec{k}| \ , \quad k_i = \frac{2\pi n_i}{R} \ , \quad n_i \in \mathbb{Z} \ , \quad V = R^3$$

- Evaluate the spectral sum in the formula for Berry curvature

  The curvature has non-vanishing components only for identical external states

- for a general multi-photon state $\left|\mathfrak{n}_+, \mathfrak{n}_-\right\rangle$ with $\mathfrak{n}_+$ positive helicity photons, $\mathfrak{n}_-$ negative helicity photons

$$(F_{\tau\bar{\tau}})_{(\mathfrak{n}_+,\mathfrak{n}_-)} = \frac{\mathfrak{n}_+ - \mathfrak{n}_-}{8}\frac{1}{(\mathrm{Im}\tau)^2}$$

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$$

☞ a non-trivial Berry phase for photons

☞ independent of momentum

☞ Implications:

linearly polarized light changes polarization under (e,θ) variation

$$|p_z, \hat{x}\rangle = \frac{1}{\sqrt{2}}\left[|p_z, +\rangle + |p_z, -\rangle\right]$$

$$\Big\downarrow$$

$$|p_z, \hat{\phi}\rangle = \frac{1}{\sqrt{2}}\left[e^{i\phi}|p_z, +\rangle + e^{-i\phi}|p_z, -\rangle\right]$$

☞ experimentally visible? (e.g. topological insulators…)

e.g. Essin-Moore-Vanderbilt, PRL '09

Example 2: conformal manifolds, conformal pert. theory

- In CFT there is a natural correspondence between states and operators

  OPERATOR-STATE correspondence

  in radial quantization or on $\mathbb{R} \times S^d$

$$|I\rangle \sim \mathcal{O}_I(0)|0\rangle$$

- The **Berry connection for states** should map to a natural **connection for operators in Conformal Perturbation Theory**

- Two seemingly different expressions for curvature

Berry

double d-dim integral

$$(F_{\mu\nu})_{IJ} = \sum_{n \notin \mathcal{H}_I} \sum_{a,b \in \mathcal{H}_n} \frac{1}{(\Delta_I - \Delta_n)^2} \langle J|\partial_\mu H|n,a\rangle g^{ab}_{(n)} \langle n,b|\partial_\nu H|I\rangle - (\mu \leftrightarrow \nu)$$

CFT

$$(F_{\mu\nu})_{IJ} = \int_{|x|\leq 1} d^{d+1}x \int_{|y|\leq 1} d^{d+1}y \langle \mathcal{O}_J(\infty)\mathcal{O}_\mu(x)\mathcal{O}_\nu(y)\mathcal{O}_I(0)\rangle - (\mu \leftrightarrow \nu)$$

double (d+1)-dim integral

These expressions compute the same object !

Baggio-VN-Papadodimas '17

Example 3: SUPERconformal manifolds, *tt** equations

Conformal manifolds are rare in non-SUSY QFTs

In SUSY QFTs they are abundant: **superconformal manifolds**

Depending on amount of SUSY, their existence can be argued (non-perturbatively) in QFT, their dimension is known etc...

Also examples in large-N limit in AdS/CFT from (super)gravity solutions

Here focus on **4d N=2 SCFTs** --8 Poincare supersymmetries

(very similar statements for 2d N=(2,2) --4 Poincare SUSIES)

Details of 4d N=2 SUSY

- 4d N=2 has 8 supercharges $Q^i_\alpha, \bar{Q}_{i,\dot{\alpha}}$ $(i = 1, 2 \quad \alpha = \pm)$

  & superconformal partners $S^\alpha_i, \bar{S}^{i,\dot{\alpha}}$

- N=2 chiral primaries: $[\bar{Q}^i_{\dot{\alpha}}, \phi_I] = 0$ (+ complex conjugate)

$$\Delta = \frac{R}{2}$$

chiral ring under the Operator Product Expansion (OPE)

$$\phi_I(x)\phi_J(0) = C^K_{IJ}\,\phi_K(0) + \dots$$

Exactly marginal interactions are descendants of N=2 chiral primaries

$$\mathcal{O}_i = \int d^4\theta\,\phi_i \ , \quad \bar{\mathcal{O}}_j = \int d^4\bar{\theta}\,\bar{\phi}_j$$

Similar to the chiral ring in the 2d Cecotti-Vafa case above

Compute (non-perturbatively) the Berry curvature of chiral primary states

If the computation closes on the chiral ring we get a 4d version of the *tt** equations

Important note: unlike Cecotti-Vafa we consider the physical theory without topological twist!

Berry curvature in sector of N=2 chiral primary states

$$(F_{\mu\nu})_{IJ} = \sum_{n \notin \mathcal{H}_I} \sum_{a,b \in \mathcal{H}_n} \frac{1}{(\Delta_I - \Delta_n)^2} \langle J|\delta_\mu H|n,a\rangle g^{ab}_{(n)} \langle n,b|\delta_\nu H|I\rangle - (\mu \leftrightarrow \nu)$$

N=2 superconformal deformations

conveniently recast as

$$(F_{\mu\nu})_{I\bar{J}} = \lim_{x\to 0} \left(\tilde{F}_{\mu\nu}\right)_{I\bar{J}}$$

$$\left(\tilde{F}_{\mu\nu}\right)_{IJ} = \langle J|\delta_\mu H (H + \hat{R} - x)^{-2} \delta_\nu H|I\rangle - (\mu \leftrightarrow \nu)$$

Insert expressions for δH and compute…

Dolan-Osborn conventions

$$H + \hat{R} = H - \frac{R}{2}$$

After a few elementary steps using SCA relations

$$\left(\tilde{F}_{\mu\nu}\right)_{I\bar{J}} = i\left[\langle\bar{J}|\oint\phi_k\frac{(H+\hat{R})^4}{(H+\hat{R}-x)^2}\oint\bar{\phi}_{\bar{l}}|I\rangle - \langle\bar{J}|\oint\bar{\phi}_{\bar{l}}\frac{(H+\hat{R})^4}{(H+\hat{R}-x)^2}\oint\phi_k|I\rangle\right]$$
$$-4i\left[\langle\bar{J}|\oint\phi_k\frac{(H+\hat{R})^2}{(H+\hat{R}-x)^2}\oint\bar{\phi}_{\bar{l}}|I\rangle - \langle\bar{J}|\oint\bar{\phi}_{\bar{l}}\frac{(H+\hat{R})^2}{(H+\hat{R}-x)^2}\oint\phi_k|I\rangle\right]$$

- regulate by separating integrated ϕ_k insertions in time
- take $x \to 0$ limit
- use $\phi\ \bar{\phi} \sim \ldots$ OPEs

and the result is…

*tt** equations for 4d *N*=2 SCFTs

$$g_{K\bar{L}} = \langle \bar{\phi}_L(\infty)\phi_K(0)\rangle$$

$$\left(F_{i\bar{j}}\right)_{K\bar{L}} = -[C_i, \bar{C}_j]_{K\bar{L}} + g_{i\bar{j}}g_{K\bar{L}}\left(1 + \frac{R}{c}\right)$$

Derived earlier by Papadodimas '09 in conformal perturbation theory with the use of superconformal Ward identities

Rederived here as a **non-perturbative Berry phase**

Are these equations useful?

Example: SU(2) N=2 SCQCD

Baggio-VN-Papadodimas '14

chiral primary operators $\phi_{2n} \propto \left(\mathrm{Tr}\left[\varphi^2\right]\right)^n$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$$

in normalization $\phi_2(x)\phi_{2n}(0) = \phi_{2n+2}(0) + \ldots$

non-trivial info in 2-point functions $\langle \phi_{2n}(x)\bar{\phi}_{2n}(0)\rangle = \frac{g_{2n}(\tau,\bar{\tau})}{|x|^{4n}}$

***tt** equations**

$$\partial_\tau \partial_{\bar{\tau}} \log g_{2n} = \frac{g_{2n+2}}{g_{2n}} - \frac{g_{2n}}{g_{2n-2}} - g_2$$

$$g_0 = 1\ , \quad n = 1, 2, \ldots$$

semi-infinite Toda

the unique physically relevant solution requires further input

for example, knowledge of the Zamolodchikov metric

$$g_2 \sim \langle \mathrm{Tr}(\varphi^2)\mathrm{Tr}(\bar{\varphi}^2)\rangle$$

Recent developments in SUSY QFT have allowed access to this quantity

A beautiful result by Gerschkovitz-Gomis-Komargodski '14 relates g_2 to the 4-sphere partition function

$$g_2 = \partial_\tau \partial_{\bar{\tau}} Z_{S^4}$$

4-sphere PF reduced to a matrix integral using **SUSY localisation**

Pestun '07

Baggio-VN-Papadodimas '14

SU(2) N=2 SCQCD $\langle \left(\mathrm{Tr}\varphi^2\right)^{n_1} \left(\mathrm{Tr}\varphi^2\right)^{n_2} \left(\mathrm{Tr}\bar{\varphi}^2\right)^{n_1+n_2} \rangle$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$$

$$\hat{C}^{(0)}_{2\,2\,4} = \sqrt{\frac{10}{3}}\left(1 - \frac{9\,\zeta(3)}{2\pi^2}\frac{1}{(\mathrm{Im}\tau)^2} + \frac{525\,\zeta(5)}{8\pi^3}\frac{1}{(\mathrm{Im}\tau)^3} + \ldots\right),$$

$$\hat{C}^{(0)}_{2\,4\,6} = \sqrt{7}\left(1 - \frac{9\,\zeta(3)}{\pi^2}\frac{1}{(\mathrm{Im}\tau)^2} + \frac{675\,\zeta(5)}{4\pi^3}\frac{1}{(\mathrm{Im}\tau)^3} + \ldots\right),$$

$$\hat{C}^{(0)}_{2\,6\,8} = 2\sqrt{3}\left(1 - \frac{27\,\zeta(3)}{2\pi^2}\frac{1}{(\mathrm{Im}\tau)^2} + \frac{2475\,\zeta(5)}{8\pi^3}\frac{1}{(\mathrm{Im}\tau)^3} + \ldots\right),$$

$$\hat{C}^{(0)}_{2\,8\,10} = \sqrt{\frac{55}{3}}\left(1 - \frac{18\,\zeta(3)}{\pi^2}\frac{1}{(\mathrm{Im}\tau)^2} + \frac{975\,\zeta(5)}{2\pi^3}\frac{1}{(\mathrm{Im}\tau)^3} + \ldots\right),$$

$$\hat{C}^{(0)}_{4\,4\,8} = 3\sqrt{\frac{14}{5}}\left(1 - \frac{18\,\zeta(3)}{\pi^2}\frac{1}{(\mathrm{Im}\tau)^2} + \frac{825\,\zeta(5)}{2\pi^3}\frac{1}{(\mathrm{Im}\tau)^3} + \ldots\right),$$

$$\hat{C}^{(0)}_{4\,6\,10} = \sqrt{66}\left(1 - \frac{27\,\zeta(3)}{\pi^2}\frac{1}{(\mathrm{Im}\tau)^2} \quad \frac{2925\,\zeta(5)}{4\pi^3}\frac{1}{(\mathrm{Im}\tau)^3} + \ldots\right).$$

+ ALL instanton corrections

This study led to the first **exact non-perturbative** computation of non-trivial 3-point functions in 4d QFTs

Baggio-VN-Papadodimas ’14

In fact, **supersymmetric localization** now allows the complete solution of extremal N-point functions in the N=2 chiral ring

> an in principle complete solution of the 4d *tt** equations

Gerschkovitz-Gomis-Ishtiaque-Karasik-Komargodski-Pufu ’16

Berry phase, *tt** equations

4d N=2 SCFTs

1/2-BPS sectors

Supersymmetric localization on spheres

Non-perturbative correlation functions in flat space

- **useful even at tree level**: compact expressions for complicated Wick contractions

- vast continuous families of interacting SCFTs with **access to highly non-trivial (non-SUSY) data** that probe global structure in theory-space, dualities etc…

- new detailed information beyond large-N limit that could be useful in further probes of **gauge-gravity dualities**

Large-N ’t Hooft limit
SU(N) N=2 SCQCD

$$\langle \mathrm{Tr}\varphi^{n_1}\mathrm{Tr}\varphi^{n_2}\mathrm{Tr}\bar{\varphi}^{n_1+n_2}\rangle$$

$$\langle 2\,,2\,,\overline{4}\rangle_n = \frac{4}{N}\left(1-\frac{3\zeta(3)}{64\pi^4}\lambda^2+\frac{45\zeta(5)}{512\pi^6}\lambda^3+\frac{3(72\zeta(3)^2-1085\zeta(7))}{32768\pi^8}\lambda^4+\dots\right)\,,$$

$$\langle 2\,,4\,,\overline{6}\rangle_n = \frac{4\sqrt{3}}{N}\left(1-\frac{3\zeta(3)}{128\pi^4}\lambda^2+\frac{15\zeta(5)}{256\pi^6}\lambda^3+\frac{99\zeta(3)^2-2275\zeta(7)}{32768\pi^8}\lambda^4+\dots\right)\,,$$

$$\langle 4\,,4\,,\overline{8}\rangle_n = \frac{8\sqrt{2}}{N}\left(1+\frac{15\zeta(5)}{512\pi^6}\lambda^3-\frac{665\zeta(7)}{16384\pi^8}\lambda^4+\dots\right)\,,$$

$$\langle 4\,,6\,,\overline{10}\rangle_n = \frac{4\sqrt{15}}{N}\left(1+\frac{15\zeta(5)}{512\pi^6}\lambda^3-\frac{35(263520\zeta(3)^2-501551\zeta(7))}{32768\pi^8}\lambda^4+\dots\right)\,,$$

$$\dots$$

Baggio-VN-Papadodimas ’16
(also Gomez-Russo ’16, ‘17)

Main messages - outlook

- QFTs can exhibit non-trivial Berry phases
  (examples in this talk: E&M, CFTs)

  Possibly new physically interesting effects?

- Evaluating the Berry phase of QFTs by putting them on different curved manifolds appears to be a useful strategy

  revisit Hamiltonian methods in QFT?

- The Berry phase of 1/2-BPS states in **4d N=2** & **2d N=(2,2)** SCFTs are non-trivial examples where non-perturbative computations are possible

- Very interesting to extend these results to:

  - lower SUSY,

  - non-conformal theories…

  where methods like SUSY localization are not yet available…