Infrared divergences for quantum fields in cosmological spacetimes

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QFT in curved spacetimes

- Want a quantum theory of gravity, but gravity is non-renormalisable, so this is tricky
- Possible contenders are
  - Loop quantum gravity
  - Modified gravity
  - String theory
- Quantum field theory in curved spacetimes

IR divergences of the graviton two point function

- It is known (and we see later) that the graviton mode functions are proportional to $p^{-2\nu}$, where $\nu \geqslant \frac{n-1}{2}$ leads to IR div
- Shown in Ref. [1] that large, non-compactly supported, gauge transformations can be used, in de Sitter space, to make the graviton two-point function infrared finite
- The mode functions in the integrand of the p-integration in the linearised Weyl-tensor (a local gauge invariant quantity) correlator behaves better in the infrared by a factor of p^2

FLRW metric

Background metric

$$g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} \tag{1}$$

Einstein's equations in the presence of a source:

$$R_{\mu\nu} = \frac{1}{2}\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}_{\lambda}\right] \tag{2}$$

where

$$T_{\mu\nu} = (\rho + \varrho)u_\mu u_\nu + \varrho g_{\mu\nu} \tag{3}$$

We will consider

$$g_{\mu\nu} \to g_{\mu\nu} + H_{\mu\nu} \tag{4}$$

Gauge choice

TTS gauge

$$\nabla^{\mu} H_{\mu\nu} = 0 \tag{5}$$

$$H = H^{\mu}_{\mu} = 0 \tag{6}$$

$$u^{\mu} H_{\mu\nu} = 0 \tag{7}$$

In the TTS gauge, EE become

$$\frac{1}{a^n}\frac{\partial}{\partial\eta}\left(a^{n-2}\frac{\partial}{\partial\eta}H^j_i\right) - \frac{1}{a^2}\delta^{lm}\tilde{\nabla}_l\tilde{\nabla}_m H^j_i = 0 \tag{8}$$

Mode sum

Rescale the perturbation such that

$$h_{\mu\nu} = a^{-2} H_{\mu\nu} \tag{9}$$

Mode sum:

$$h_{ij}(\eta, \mathbf{p}) = \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}} \sum_s \left[a_s(\mathbf{p}) \gamma_{ij}^{(s,\mathbf{p})}(\eta, \mathbf{p}) + a_s(\mathbf{p})^\dagger \gamma_{ij}^{(s,\mathbf{p})}(\eta, \mathbf{p})^* \right] \tag{10}$$

where

$$\gamma_{ij}^{(s,\mathbf{p})}(\eta, \mathbf{x}) = \epsilon_{ij}^{(s)}(\mathbf{p}) f_{\mathbf{p}}(\eta) e^{i\mathbf{p}\cdot\mathbf{x}} \tag{11}$$

TTS gauge, so constant, symmetric, polarisation tensors $\epsilon^{s}_{ij}(\mathbf{p})$ satisfy

$$\epsilon^{(s)}_{ij}(\mathbf{p})p_i = 0 \tag{12}$$

$$\epsilon^{(s)}_{ii}(\mathbf{p}) = 0 \tag{13}$$

$$\epsilon^{(s)}_{ij}(\mathbf{p})^*\epsilon^{(s')}_{ij}(\mathbf{p}) = 1 \tag{14}$$

The function $f_{\mathbf{p}}(\eta)$ satisfies

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + (n-2)\frac{a'(\eta)}{a(\eta)}\frac{\mathrm{d}}{\mathrm{d}\eta} + p^2\right] f_{\mathbf{p}}(\eta) = 0 \tag{15}$$

We write this solution as

$$f_{\mathbf{p}}(\eta) = A(p)F_{\mathbf{p}}^{(1)}(\eta) - iB(p)F_{\mathbf{p}}^{(2)}(\eta) \tag{16}$$

where $F_{\mathbf{p}}^{\sigma}$, $\sigma = 1, 2$, are real and $F_{\mathbf{p}}^{\sigma} \to F_{\mathbf{0}}^{\sigma}$ as $p \to 0$.

Vacuum choice:

$$A(p) \to \infty \tag{17}$$
$$B(p) \to 0 \tag{18}$$

as $p \to 0$

The two point function

The two point function is

$$\langle 0|h_{ij}(\eta,\mathbf{x})h_{i'j'}(\eta',\mathbf{x}')|0\rangle = \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}} \sum_s \gamma_{ij}^{(s,\mathbf{p})}(\eta,\mathbf{x})\gamma_{i'j'}^{(s,\mathbf{p})}(\eta',\mathbf{x}')^* \tag{19}$$

In the IR limit, this becomes

$$\langle 0|h_{ij}(\eta,\mathbf{x})h_{i'j'}(\eta',\mathbf{x}')|0\rangle \approx \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}} |A(p)|^2 e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} F_{\mathbf{p}}^{(1)}(\eta)F_{\mathbf{p}}^{(1)}(\eta')^* \sum_s \epsilon_{ij}^{(s)}(\mathbf{p})\epsilon_{i'j'}^{(s)}(\mathbf{p}) \tag{20}$$

The gauge transformation

Gauge freedom (coordinate transformation)

$$\delta h_{\mu\nu} = a^{-2}\mathcal{L}_{\tilde{\xi}}\left(a^2\eta_{\mu\nu}\right) = 2\partial_{(\mu}\tilde{\xi}_{\nu)} - 2Ha\eta_{\mu\nu}\tilde{\xi}_0 \tag{21}$$

Rescale as before such that

$$\tilde{\xi}_\mu = a^2\xi_\mu \tag{22}$$

The transformation vector is of the form [2]

$$\xi_0 = 0 \tag{23}$$

$$\xi_i = -\frac{1}{2}A(p)F_0^{(1)}(\eta)\left[\epsilon_{il}^{(s)}(\mathbf{p})x^l(1 + i\mathbf{p}\cdot\mathbf{x}) - \frac{i}{2}\epsilon_{lm}^{(s)}(\mathbf{p})x^l x^m p_i\right] e^{-\rho^4 p^4} \tag{24}$$

The transformed mode functions are

$$\gamma'_{ij} = \epsilon^{(s)}_{ij}(\mathbf{p})A(p)\Big[(1+i\mathbf{p}\cdot\mathbf{x})\Big(F^{(1)}_{\mathbf{p}}(\eta) - F^{(1)}_{0}(\eta)\Big) - \frac{1}{2}(\mathbf{p}\cdot\mathbf{x})^2 F^{(1)}_{\mathbf{p}}(\eta) + \mathcal{O}(p^3)\Big] \quad (25)$$

Therefore, to lowest order in p,

$$\gamma'_{ij} \propto \epsilon^{(s)}_{ij}(\mathbf{p})A(p)p^2 \quad (26)$$

so

$$\langle 0|h_{ij}(\eta,\mathbf{x})h_{i'j'}(\eta',\mathbf{x}')|0\rangle \propto \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}}|A(p)|^2 p^4 \sum_s \epsilon^{(s)}_{ij}(\mathbf{p})\epsilon^{(s)}_{i'j'}(\mathbf{p})^* \quad (27)$$

Inflation Background

- Proposed by Guth, Sato et.al.
- Solution to the flatness, horizon, and monopole problems of the Big Bang model
- Period of exponential expansion in the early universe

Slow roll Background

Inflation is caused by a scalar field slowly rolling down a potential.

Have the following conditions:

- flat potential
- curvature of the potential slope is small

Slow roll parameters

Introduce the slow roll parameters:

$$\text{Hubble Parameter:} \quad H \equiv -\frac{a'}{a^2} \tag{28}$$

$$\text{Measure of slope:} \quad \epsilon \equiv -\frac{H'}{aH^2} \tag{29}$$

$$\text{Measure of curvature:} \quad \delta \equiv \frac{\epsilon'}{2Ha\epsilon} \tag{30}$$

$\epsilon > 0$ leads to IR divergences

Slow roll example

Scale factor:

$$a(\eta) = D(-\eta)^{-\frac{1}{1-\epsilon}} \tag{31}$$

where D is a constant.

The field equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + (n-2)\frac{a'(\eta)}{a(\eta)}\frac{\mathrm{d}}{\mathrm{d}\eta} + p^2\right] f_{\mathbf{p}}(\eta) = 0 \tag{32}$$

becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + (n-2)\frac{\lambda}{(-\eta)}\frac{\mathrm{d}}{\mathrm{d}\eta} + p^2\right] f_{\mathbf{p}}(\eta) = 0 \tag{33}$$

We want this solution in the form

$$f_{\mathbf{p}}(\eta) = A(p)F_{\mathbf{p}}^{(1)}(\eta) - iB(p)F_{\mathbf{p}}^{(2)}(\eta) \quad (34)$$

where $F_{\mathbf{p}}^{\sigma} \to F_{\mathbf{0}}^{\sigma}$ as $p \to 0$

We have

$$F_0^{(1)}(\eta) = 1 \quad (35)$$

and

$$F_0^{(2)}(\eta) = \int \mathrm{d}\eta \, [a(\eta)]^{-(n-2)} = -\frac{D^{-(n-2)}}{2\nu}(-\eta)^{2\nu} + \text{constant} \quad (36)$$

In the infrared limit, we see that

$$f_{\mathbf{p}}(\eta) = \left[\frac{2^{\nu-1}\Gamma(\nu)}{\sqrt{\pi}D^{\frac{1}{2}(n-2)}} p^{-\nu} F_0^{(1)}(\eta) - i \frac{\sqrt{\pi}D^{\frac{1}{2}(n-2)}}{2^{\nu}\Gamma(\nu)} p^{\nu} F_0^{(2)}(\eta) \right] \quad (37)$$

To lowest order in p,

$$A(p) = \frac{2^{\nu-1}\Gamma(\nu)}{\sqrt{\pi}D^{\frac{1}{2}(n-2)}} p^{-\nu}\,, \quad (38)$$

$$B(p) = \frac{\sqrt{\pi}D^{\frac{1}{2}(n-2)}}{2^{\nu}\Gamma(\nu)} p^{\nu}\,, \quad (39)$$

so

$$F_{\mathbf{p}}^{(1)}(\eta) - F_0^{(1)}(\eta) \propto p^2 \quad (40)$$

$$\langle 0|h_{ij}(\eta,\mathbf{x})h_{i'j'}(\eta',\mathbf{x}')|0\rangle \propto \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}}\left(\frac{2^{\nu-1}\Gamma(\nu)}{\sqrt{\pi}D^{\frac{1}{2}(n-2)}}\right)^2 p^{-2\nu}p^4\sum_s \epsilon_{ij}^{(s)}(\mathbf{p})\epsilon_{i'j'}^{(s)}(\mathbf{p}) \tag{41}$$

This integral is convergent for

$$\epsilon > \frac{4}{n+2} \tag{42}$$

This is in agreement with the results of Ref. [4]

Thanks for listening!

[1] *de Sitter invariance of the dS graviton vacuum,* A. Higuchi, D. Marolf, and I. A. Morrison, Class. Quant. Grav, 28:245012,(2011)

[2] *Infrared divergences for free quantum fields in cosmological spacetimes,* A. Higuchi, N. Rendell, arXiv:1711.03964, (2017)

[3] *The Weyl tensor correlator in cosmological spacetimes,* M. B. Frb, JCAP, 2014:010, (2014)

[4] *Some Inconvenient Truths*, R. Woodard, arXiv:1506.04252, (2015)