# Scalar hairy black holes in four dimensions are unstable

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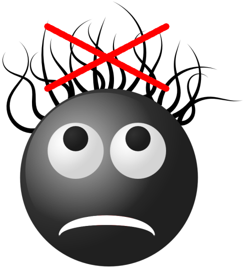
# Background

- 1970s - The *no-hair* theorems[Car71, Rob75] were derived, clearly influencing black hole physics in asymptotically flat, 4D for the decades to follow



# Background

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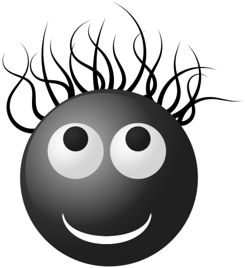
These theorems state that if (M, g) is a stationary, axisymmetric, four-dimensional asymptotically flat vacuum spacetime that is suitably regular on and in the vicinity of a connected event horizon, then it is isometric to a member of the Kerr family.

# Background

- 1970s - The *no-hair* theorems[Car71, Rob75] were derived, clearly influencing black hole physics in asymptotically flat, 4D for the decades to follow
- Not all assumptions have a firm physical motivation. In 2012 hairy Kerr black holes (HBH) were found by Herdeiro and Radu [HR14] by coupling gravity to a complex massive scalar field



# Scalar clouds and existence of HBHs

- Einstein gravity minimally coupled to a complex massive scalar

$$S = \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g} \left( \frac{R}{16\,\pi\, G} - \nabla_a \psi^* \nabla^a \psi - \mu^2 |\psi|^2 \right) .$$

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- Combine with *superradiant scattering* [ZE79, Det80], the wave analog of the Penrose process, at the onset of which HBHs branch off from Kerr.

# Construction of HBHs

- One way of constructing them - using the DeTurck method [HKW10, FLW11, FW17] - solving a modified Einstein equation

$$R_{ab} - \nabla_{(a}\xi_{b)} = 8\pi\left[2\nabla_{(a}\psi^*\nabla_{b)}\psi + g_{ab}\mu^2\psi^*\psi\right],$$

where $\xi^a = g^{bc}\left[\Gamma^a_{bc}(g) - \Gamma^a_{bc}(\mathfrak{g})\right]$ and $\mathfrak{g}$ - a reference metric.

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- Have to verify *a posteriori* that $\xi = 0$, as to get a solution to Einstein equation.
- Generic ansatz, reference metric based on obtaining Kerr asymptotically and boundary conditions $\Rightarrow$ Moduli space of HBH solutions by varying the angular velocity $\Omega_{HBH}$ and $\tilde{m}$.

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- Were they to turn unstable with a very short lifetime, their astrophysical significance might not be that huge.
  - Furthermore, the end-point of the superradiant instability of Kerr will remain undetermined.
- The first step to answering those questions is a linear mode analysis

# Perturbing the HBHs

Consider small changes in both metric and scalar field

$$g_{ab} = g_{ab}^{(0)} + h_{ab} \,,$$
$$\psi = \psi^{(0)} + \eta \,,$$

giving us an equation for the perturbed scalar field

$$\Box^{(0)}\eta - \mu^2\eta - \hat{L}^{(0)}\psi^{(0)} = 0 \,,$$
$$\hat{L}^{(0)} = \bar{h}^{ab}\nabla_a^{(0)}\nabla_b^{(0)} - \nabla_a^{(0)}\bar{h}^{ad}\nabla_d^{(0)} - \frac{1}{2}\mu^2\bar{h} \,,$$

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Ideally, one would like to have $\hat{L}^{(0)}\psi^{(0)} = 0$ so that we have

$$\Box^{(0)}\eta - \mu^2\eta = 0$$

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# QNM spectra

We have computed the QNM spectrum of the HBHs with $\tilde{m} = 1$ for perturbations with $m = 1$ and $m = 2$.

- The $m = 1$ modes correspond to shifts in the phase space of HBHs - altering the mass and angular momentum of the scalar cloud.

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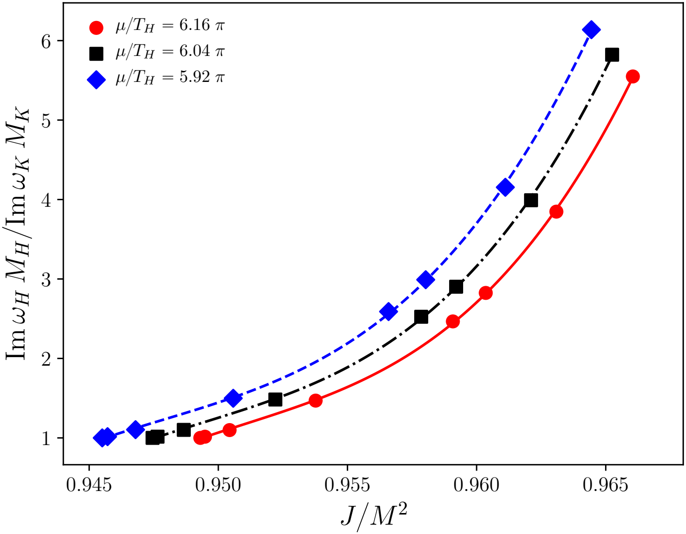
- The $m = 1$ modes correspond to shifts in the phase space of HBHs - altering the mass and angular momentum of the scalar cloud.
- The modes with $m = 2$ are *always* unstable in the regions where the $\tilde{m} = 1$ HBHs exist.

# QNM spectra


$\mu/T_H = 6.16\ \pi$
$\mu/T_H = 6.04\ \pi$
$\mu/T_H = 5.92\ \pi$
$\mathrm{Im}\,\omega_H\,M_H/\mathrm{Im}\,\omega_K\,M_K$
1
2
3
4
5
6
0.945
0.950
0.955
0.960
0.965
$J/M^2$

# Astrophysical significance

- Compare the growth rate of the lowest lying mode with $m = 1$ around Kerr, with the lowest lying mode with $m = 2$ around a HBH at the same $J/M^2$ and fixed scalar field mass $\mu M$.

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Even though the superradiant instability for the HBHs is three orders of magnitude slower than that for Kerr, the timescales involved in both processes are significantly smaller than the age of the universe.

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- Superradiance is an efficient process and the growth rates of the instability are such that the Hairy black holes are rendered astrophysically insignificant.
- We conjecture that black holes with Proca hair will meet a similar fate, the idea being that even though hair develops around the black hole, the black hole itself resembles Kerr and should be unstable to modes with higher $m$.

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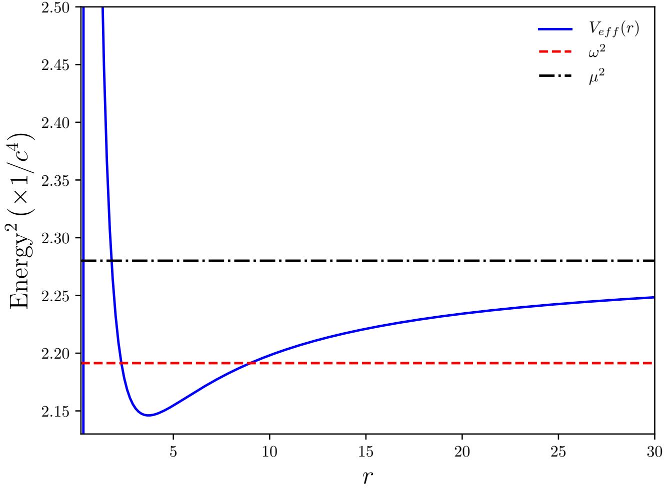
## Trapping region



# Ansatz metric

The most generic ansatz for a stationary, axisymmetric spacetime with a $t-\phi$ reflection symmetry

$$\mathrm{d}s^2 = -F(x,z)x^2\mathrm{d}t^2 + r_0^2\Bigg\{\frac{4C(x,z)}{(1-x^2)^4}\mathrm{d}x^2 + \frac{A(x,z)(1-z^2)^2}{(1-x^2)^2}\left[\mathrm{d}\phi - (1-x^2)^2W(x,z)\frac{\mathrm{d}t}{r_0}\right]^2 + \frac{4D(x,z)}{(1-x^2)^2(2-z^2)}\left[\mathrm{d}z + B(x,z)\mathrm{d}x\right]^2\Bigg\},$$

where $r_0$ - outer event horizon and $A$, $B$, $C$, $D$, $F$, $W$ are unknown function of $(x,z)$.

# Perturbing the HBHs

- We still have to deal with the diffeomorphism invariance of GR

$$h_{ab} \to h_{ab} + \mathcal{L}_\chi g^{(0)} \,,$$
$$\eta \to \eta + \mathcal{L}_\chi \psi^{(0)} \,,$$

where $\chi$ is of the same order as $h_{ab}$ and $\eta$.

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where $\chi$ is of the same order as $h_{ab}$ and $\eta$.

- To this end first set

$$\nabla_a^{(0)} \bar{h}^{ad} = P^d(\bar{h}, \bar{h}_{ab})\,.$$

# Perturbing the HBHs

- We have to show such a choice can be made independetly of the form of $P$, hence we gauge transform the condtion

$$\Box^{(0)}\chi_d + R^{(0)}_{da}\chi^a + \nabla^{(0)}_a \bar{h}^a{}_d - P_d - P^{(\chi)}_d = 0\,,$$

It is essential that $P_d$ can only depend on $h_{ab}$ but not on its derivatives. The same follows for $P^{(\chi)}_d$ - its gauge transform.

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- Moreover, it can be deduced that $P^{(\chi)}_d$ can also only depend on first derivatives of $\chi_d$, implying that the principal symbol of the above equation is governed by $\Box^{(0)}$.

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- Confirming that we can set $\nabla_a^{(0)} \bar{h}^{ad} = P^d$ for any choice of $P^d$ that can depend at most on $h_{ab}$, but not on its derivatives.
- With this gauge choice setting $\hat{L}^{(0)} \psi^{(0)} = 0$ is equivalent to solving

$$\bar{h}^{ab} \nabla_a \nabla_b \psi^{(0)} - P^d \nabla_d^{(0)} \psi^{(0)} - \frac{1}{2} \mu^2 \bar{h} \psi^{(0)} = 0.$$

leaving us with

$$\Box^{(0)} \eta - \mu^2 \eta = 0\,.$$

# Residual gauge freedom

Residual gauge transformations satisfy

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We devise a test to distinguish pure gauge from physical modes based on the fact that the former necessarily produce a metric perturbation that diverges exponentially at large distances, thus becoming incompatible with of asymptotic flatness.

# Residual gauge freedom

- Perform a Frobenius analysis near asymptotic infinity ($x = 1$)

$$\eta = e^{-\frac{\Gamma}{1-x}}(1-x)^{\kappa}\tilde{\eta}(t,x,\theta,\phi)$$

where $\tilde{\eta}(t,x,\theta,\phi)$ - polynomial in $(1-x)$ and $\Gamma\,,\kappa \in \mathbb{C}$.

$$\psi^{(0)} = e^{-\frac{\tilde{\Gamma}}{1-x}}(1-x)^{\tilde{\kappa}}\tilde{\psi}^{(0)}(t,x,\theta,\phi)\,,$$

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- Recall

$$\eta \to \eta + \mathcal{L}_{\chi}\psi^{(0)}\,.$$

# Residual gauge freedom

- Assume $\tilde{\Gamma} > \mathrm{Re}(\Gamma)$. If $\eta + \mathcal{L}_\chi \psi^{(0)}$ is to vanish then the residual gauge perturbation $\hat{\chi}$ has to blow up exponentially as $x \to 1^-$.

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- Therefore, the above argument shows that modes with $\tilde{\Gamma} > \mathrm{Re}(\Gamma)$ are necessarily physical.

# Theorem from Wald

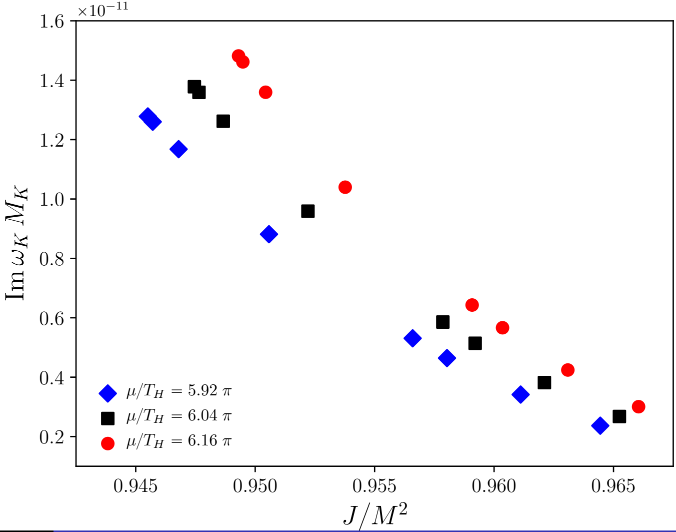
## Theorem

*Let $M, g_{ab}$ be a globally hyperbolic spacetime(or a globally hyperbolic region of an arbitrary spacetime) and let $\nabla_a$ be any derivative operator. Let $\Sigma$ be a smooth Cauchy surface. Consider the system of $n$ linear equations for $n$ unknown functions $\phi_1, \ldots, \phi_n$ of the form*

$$\Box\phi_i + \sum_j (A_{ij})^a \nabla_a\phi_j + \sum_j B_{ij}\phi_j + C_i = 0.$$

*Then the equation has a well posed initial value formulation on $\Sigma$. More precisely, given arbitrary smooth initial data, $(\phi_i, n^a\nabla_a\phi_i)$ for $i = 1, \ldots, n$ on $\Sigma$ there exists a unique solution of the equation throughout $M$. Furthermore, the map from intial data on $\Sigma$ to solutions in any fixed compact region of spacetime is continuous for the norms defined on the solutions and on the initial data. Finally, a variation of the initial data outside of a closed subset, $S$, of $\Sigma$ does not affect the solution in $D(S)$.*

The imaginary parts of the QNM spectra of massive scalar field perturbations, $m = 2$, around Kerr ($\omega_K$) BHs.



The imaginary parts of the QNM spectra of massive scalar field perturbations, $m = 2$, around hairy ($\omega_H$) BHs.