Emergent Geometry:

Towards a proof of the AdS/CFT correspondence

David Berenstein

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hep-th/0509015 with D. Correa, S. Vazquez
hep-th/0511104 with D. Correa, hep-th/0605220 with R. Cotta
Works in progress with R. Cotta, S. Vazquez

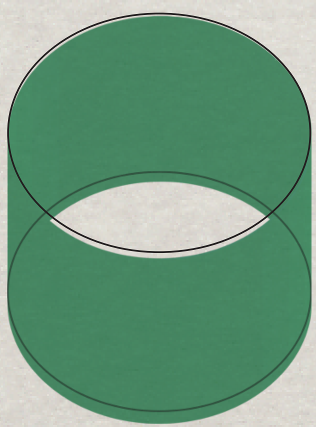
Outline

- AdS/CFT: the basics.
- The operator state correspondence and integrability.
- BPS states and semiclassical CFT dynamics.
- Eigenvalue Distributions.
- All loop string energies.

AdS/CFT: the basics

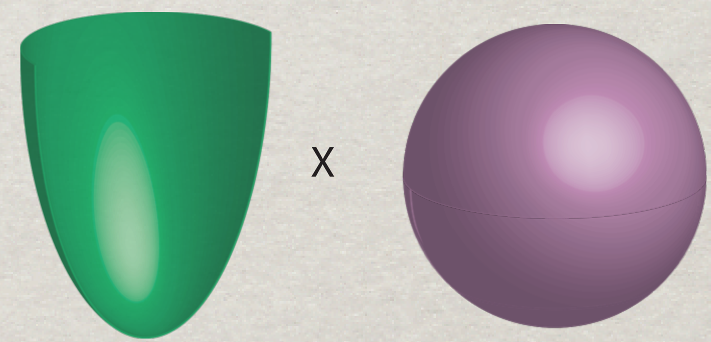
Conjectured exact quantum duality between:

N=4 SYM
on a sphere



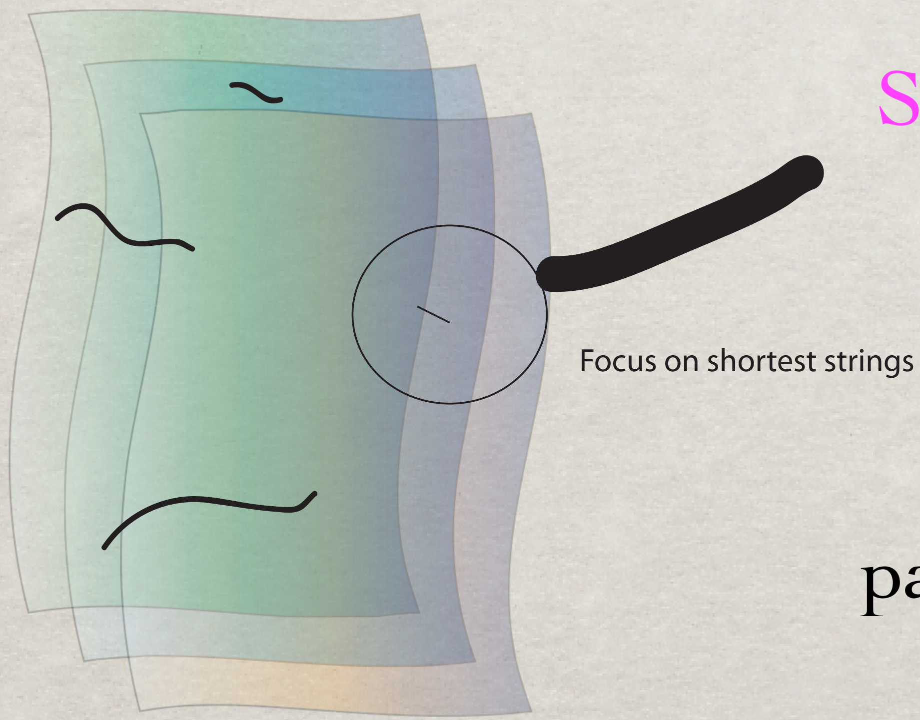
Type IIB Superstring on
asymptotically

$AdS_5 \times S^5$



AdS/CFT conjecture was motivated first by Maldacena

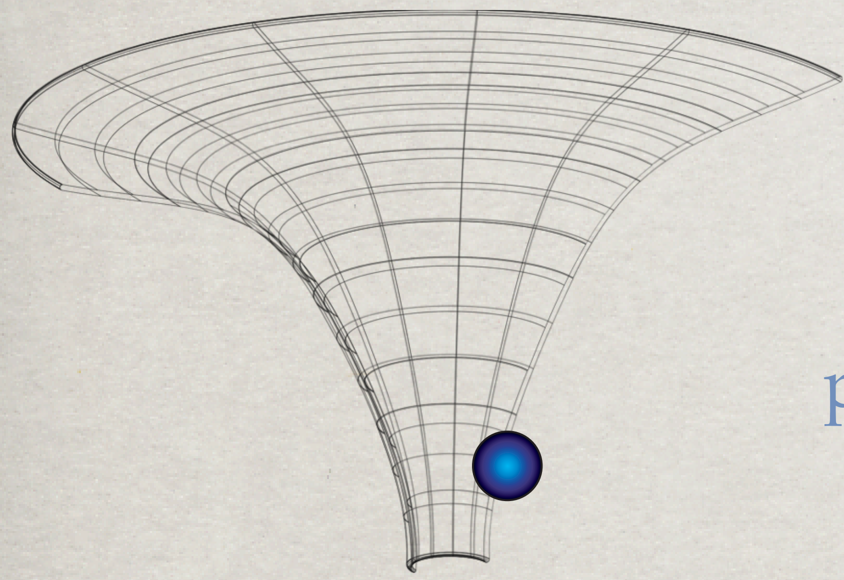
The basic idea is to take a stack of D3-branes and analyze the low energy limit of it's excitations in two different ways:



Shortest strings are massless

Lead to field theory of particles with spin one or less

Branes are massive: they bend spacetime



Lead to black-hole type geometry

Because of gravitational redshift, particles in the near horizon region cost almost no energy to produce.

We can take also a low energy limit by focusing on the near horizon region.

The AdS/CFT correspondence states that these two ways of thinking of the low energy limit are completely equivalent: each is perturbatively reliable in different regimes.

Basic Dictionary

AdS	CFT
Super-isometries	Global symmetries
R^4	$g_{YM}^2 N$
Flux = N	Gauge group U(N)
State	State

The main problem of the AdS/CFT is to show the equivalence of states on both sides: same irreducible representations of the symmetry group.

Technicalities:

Want to work in Lorentzian signature

Gravity works best in global coordinates for AdS

$$ds^2 = -\cosh^2 \rho dt^2 + \sinh^2 \rho d\Omega_3^2 + d\rho^2$$

SYM lives in conformal boundary: $\rho \longrightarrow \infty$

$$S^3 \times \mathbb{R}$$

We understand the states of SYM in free field limit (gauge invariant subset of Fock space).

Only know how to create some states in gravity by adding probes to some classical solution of gravity with required asymptotics. These probes can be strings, D-branes. Don't have systematic theory of quantum gravity, just semiclassical expansion.

We have a lot of reasons why the correspondence should work. (Many tests)

The big question is: how does it work?

More precisely: where does geometry come from?

This talk will present a proposal for the origin of extra dimensional geometry in CFT that passes many consistency tests and seems to give a good expansion of the N=4 SYM at strong 't Hooft coupling.

What we're after

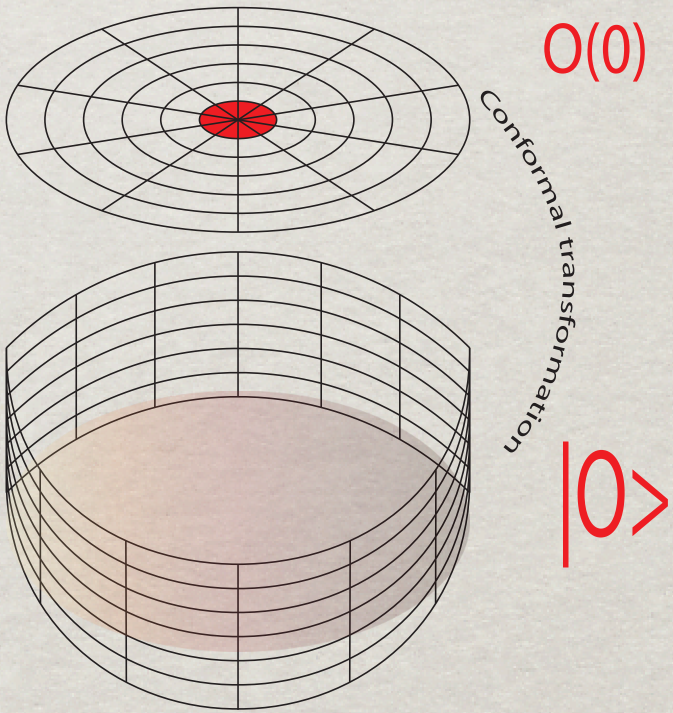
- Some background independence: we need to encode many geometries /topologies. Not just the ground state of the system.
- Focus on BPS dynamics to help tame quantum corrections.
- Universal description of the origin of strings for all classical gravity states.

Operator state correspondence.

The N=4 SYM is a superconformal field theory

In any CFT there is an operator state correspondence

Radial quantization



$$ds^2 = r^2 \left(\frac{dr^2}{r^2} + d\Omega_3^2 \right)$$

$$\mathcal{O}(0) \sim |\mathcal{O}\rangle$$

Hamiltonian is scaling dimension: Need to measure anomalous dimensions.

Complete description of states in SYM:

Local gauge invariant operators inserted at the origin in Euclidean theory.

These are products of traces of fields and their derivatives inserted at origin.

Can make a 1-1 correspondence with states in the gauge invariant Fock space of the field theory on the sphere.

Basic correspondence:

$$\partial^{[n]}\phi^i \sim a^\dagger_{i,[n]}$$

Derivatives of fundamental fields (as operators) get turned into raising operators of partial waves of field on sphere. Each of these counts as a Letter.

It's easy to match quantum numbers: dimensions vs energy, SO(4) quantum numbers.

Each trace is interpreted as a string (Witten, Gubser,Klebanov, Polyakov)

Works only so long as the number of Letters (length of an operator) is less than

$$N^{1/2}$$

In this regime, planar diagrams dominate

Operator description is very convenient for computing Hamiltonian.

First done to one loop in BMN limit. (B., Maldacena, Nastase)

Extended to SO(6) subsector by Minahan and Zarembo. Found an integrable SO(6) spin chain.

Extended to full SU(2,2|4) chain by Beisert and Staudacher, who found a full 1-loop integrable spin chain.

Classical string motion in dual geometry also integrable (Bena, Roiban, Polchinski)

Suggests integrability as a way to match string states around AdS, to quantum spin chain model from resummation of planar diagrams. This is a formidable task, but if integrable, one expects some solution in the form of a Bethe Ansatz. There has been a lot of recent progress in this direction.

Beisert, Dorey, Frolov, Hernandez, Hoffman, Janick, Lopez, Maldacena, Staudacher, Tseytlin ...

This is advocated as a proof of AdS/CFT.

THE BASICS

Need a ground state

$$|0\rangle \sim tr(Z^J)(0) = ZZZZZZZZZZZZZZZZZZZZZ$$

$$Z = X_5 + iX_6$$

This state preserves half of supersymmetries.

Need defects propagating about ground state background

$$|n_1, n_2\rangle \sim tr(Z^{n_1} Y Z^{n_2} Y Z^{n_3} \dots) \sim ZZZZZZZZYZZZZZZZZZYZZZZ$$

We have some type of lattice with various defects on it.

Asymptotic Bethe Ansatz

Eigenfunctions of the Hamiltonian look asymptotically as follows

$$\psi(n_1, n_2) \sim exp(iP_1 n_1 + iP_2 n_2) + S(P_1, P_2)\exp(iP_2 n_1 + iP_1 n_2)$$

This is valid asymptotically for both n large.
S is some type of S-matrix of the defects.

The energy associated to such a state is

$$E - J = \sum_i \sqrt{1 + \frac{g^2 N}{\pi^2}\sin^2(P_i/2)}$$

Dispersion relation

Santambrogio, Zanon. Nice argument for exactness by Beisert

The "proof" is limited:

One background. (Expansion around strict AdS x S)

Captures only strings (with all α' corrections)

Opinion: too algebraic, not enough geometric intuition,

D-branes missing.

We expect correspondence to include "all geometries with asymptotic AdS x S boundary conditions" in some sense.

I will now take a different route to try to address all of these at the same time, but giving away a solvable description of strings (integrability).

There is a lot of SUSY.

Systematic exploration of supersymmetric states usually give good results.

Half BPS states: in free field limit can be described by a gauged matrix quantum mechanics of a harmonic oscillator (equivalent to quantum hall droplet)
(Corley, Jevicki, Ramgoolam, B.)

In gravity there is a classification of regular solutions in terms of a two-coloring of the plane
(Lin, Lunin, Maldacena)

Gravity is like the hydrodynamic description of the quantum hall droplet. Suggestive of the origin of geometry.

BPS matrix model dynamics

- 1/8 BPS states: Chiral ring dynamics.
- Leads to Improved effective low energy dynamics.

Want to study states that respect 1/8 SUSY (Chiral ring)

BPS bound: Energy = Angular momentum

Via operator state correspondence:

$$\mathcal{O}(0) \sim |\mathcal{O}\rangle$$

Any local gauge invariant operator is constructed from traces of fields and derivatives.

$$\partial^{[n]}\phi^i \sim a^\dagger_{i,[n]}$$

Spherical harmonic expansion on sphere

In free field limit all states that correspond to chiral ring can be written in terms of the 3 s-wave complex scalar components: X, Y, Z

This is like dimensional reduction on sphere.

BPS argument: dimensional reduction is an accurate description of the BPS dynamics (even at strong coupling). Semiclassical is often exact.

We want to explore this reduction at strong coupling

A typical result of matrix models is that the typical eigenvalue is of order $N^{1/2}$

$$Z^j = X^{2j-1} + iX^{2j}$$

$$S_{sc} = \int dt\, \mathrm{tr}\left(\sum_{a=1}^{6} \frac{1}{2}(D_t X^a)^2 - \frac{1}{2}(X^a)^2 - \sum_{a,b=1}^{6} \frac{1}{8\pi^2} g_{YM}^2 [X^a, X^b][X^b, X^a]\right)$$

This term dominates at strong coupling. The minimum of potential happens for commuting matrices.

There is still a family of such configurations, and we want to quantize them. Further analysis reveals that this constraint is requiredfor BPS states.

Effective low energy dynamics is a gauged Matrix quantum mechanics of commuting matrices

$$[X^i, X^j] = 0$$

(Minisuperspace approximation)

Can diagonalize all matrices simultaneously, by gauge transformations.

Associate a 6-vector per eigenvalue $\vec{x}_j \simeq (X^i_{jj})$

Eigenvalues are coordinates of particles (a la BFSS)

Classically the eigenvalue dynamics is free.

Going to eigenvalue variables is like going to spherical coordinates.

Measure term from going to eigenvalue basis that affects the effective laplacian (angular variables are dropped)

$$\mu^2 = \prod_{i<j} |\vec{x}_i - \vec{x}_j|^2$$

$$H = \sum_i -\frac{1}{2\mu^2}\nabla_i \mu^2 \nabla_i + \frac{1}{2}|\vec{x}_i|^2$$

Quantum truncation of classical H in mini-superspace

- Terms we set to zero (commutators) are D,F terms of potential. This means we are reducing to dynamics on moduli space of vacua.

- This is well know to be given by N particles in 6 flat dimensions. (Lessons from M(atrix) theory)

Eigenvalue Distributions:

One can find ground state, and absorb square root of the measure in wave functions.

~ free fermion description of hermitian matrix models

$$\psi_0 \sim \exp(-\sum \vec{x}_i^2/2)$$

$$\hat{\psi} = \mu\psi$$

$$|\hat{\psi}_0^2| \sim \mu^2 \exp(-\sum x_i^2) = \exp\left(-\sum \vec{x}_i^2 + 2\sum_{i<j} \log|\vec{x}_i - \vec{x}_j|\right)$$

Square of wave function tells us which configurations are dominant.

Interpret collection of eigenvalues as positions of particles in 6d.

Similar to a Boltzman gas of N Bosons in 6d with a confining potential and logarithmic repulsive interactions.

$$|\hat{\psi}_0^2| \sim \mu^2 \exp(-\sum x_i^2) = \exp\left(-\sum \vec{x}_i^2 + 2\sum_{i<j} \log|\vec{x}_i - \vec{x}_j|\right)$$

$$\exp(-\beta \tilde{H})$$

We want to study the thermodynamics of this ensemble in the saddle point approximation.
(Large N limit, replace sums by integrals)
Introduce a density of bosons (eigenvalues)

Density of bosons is a singular configuration. Symmetries of ensemble suggest the following density of “eigenvalues”

$$\rho = N \frac{\delta(|\vec{x}| - r_0)}{r_0^{2d-1} Vol(S^{2d-1})}$$

We get a round five sphere

$$r_0 = \sqrt{\frac{N}{2}}$$

- BPS wave functions are holomorphic, multiplying the ground state wave function. If multiplying by homogeneous polynomial of degree n, the energy is n.
- This is the same as holomorphic quantization of the moduli space of vacua (complex Kahler manifold)

For example, take this holomorphic function

$$P_n = \sum_i (z_i^1)^n = \mathrm{tr}(Z^n)$$

And multiply the ground state by it.

This is a new wave function, with energy n.

$$a_n^\dagger |\alpha\rangle \sim P_n(Z)\hat{\psi}_\alpha$$

Interpreted as creation of one graviton with momentum n on top of a given state (Witten).

To consider classical solutions, we need coherent states

Naively, we would write

$$\exp(\sum_n t_n P_n(Z))\hat{\psi}_0$$

But these are usually non-normalizable.

Instead, we use

$$\sum_n t_n P_n(Z) = \mathrm{tr} f(Z)$$

Where f is analytic at zero, but only grows logarithmically.

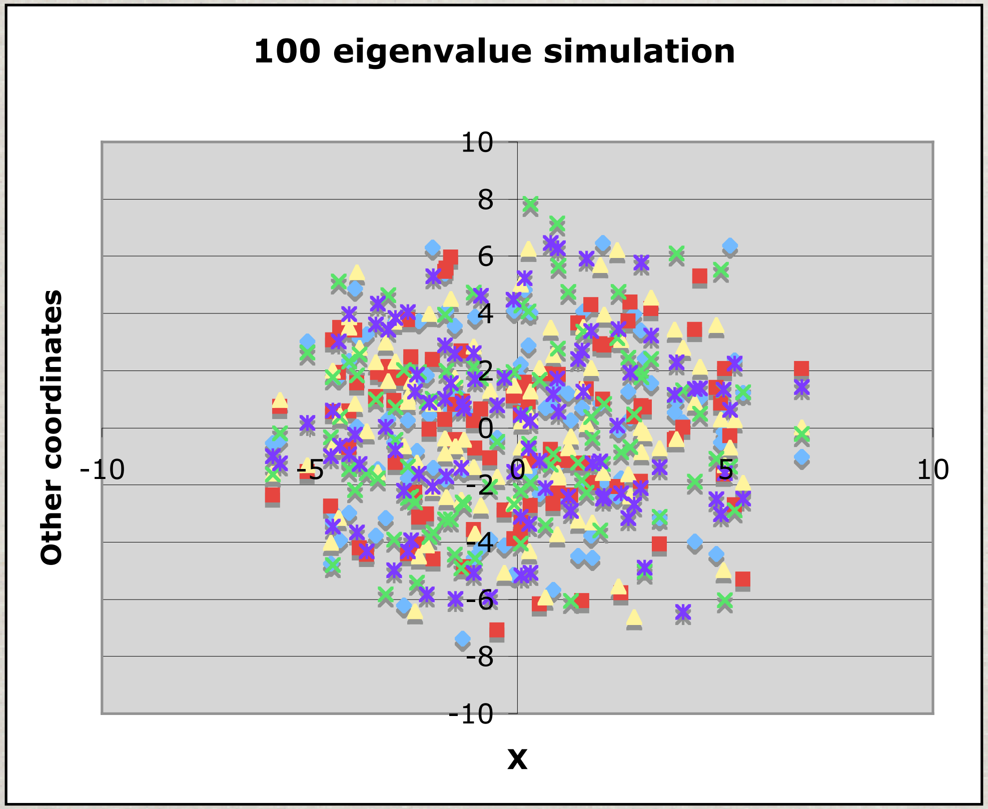
Net result is to change the confining potential, but not the repulsive interactions. We get a deformed distribution.

- Deformation of geometry of eigenvalues parallels deformation of gravity by a classical “coherent state”.

- We identify the eigenvalue distribution with (some aspect of) gravity: Geometry is emergent.

- Different topologies of eigenvalue distributions lead to different spacetime topologies (explicit in Lin, Lunin, Maldacena case.)
- Very few exact results are known. We have the wave functions, but we need the distributions of eigenvalues.

These wave functions can be simulated using Monte-Carlo methods



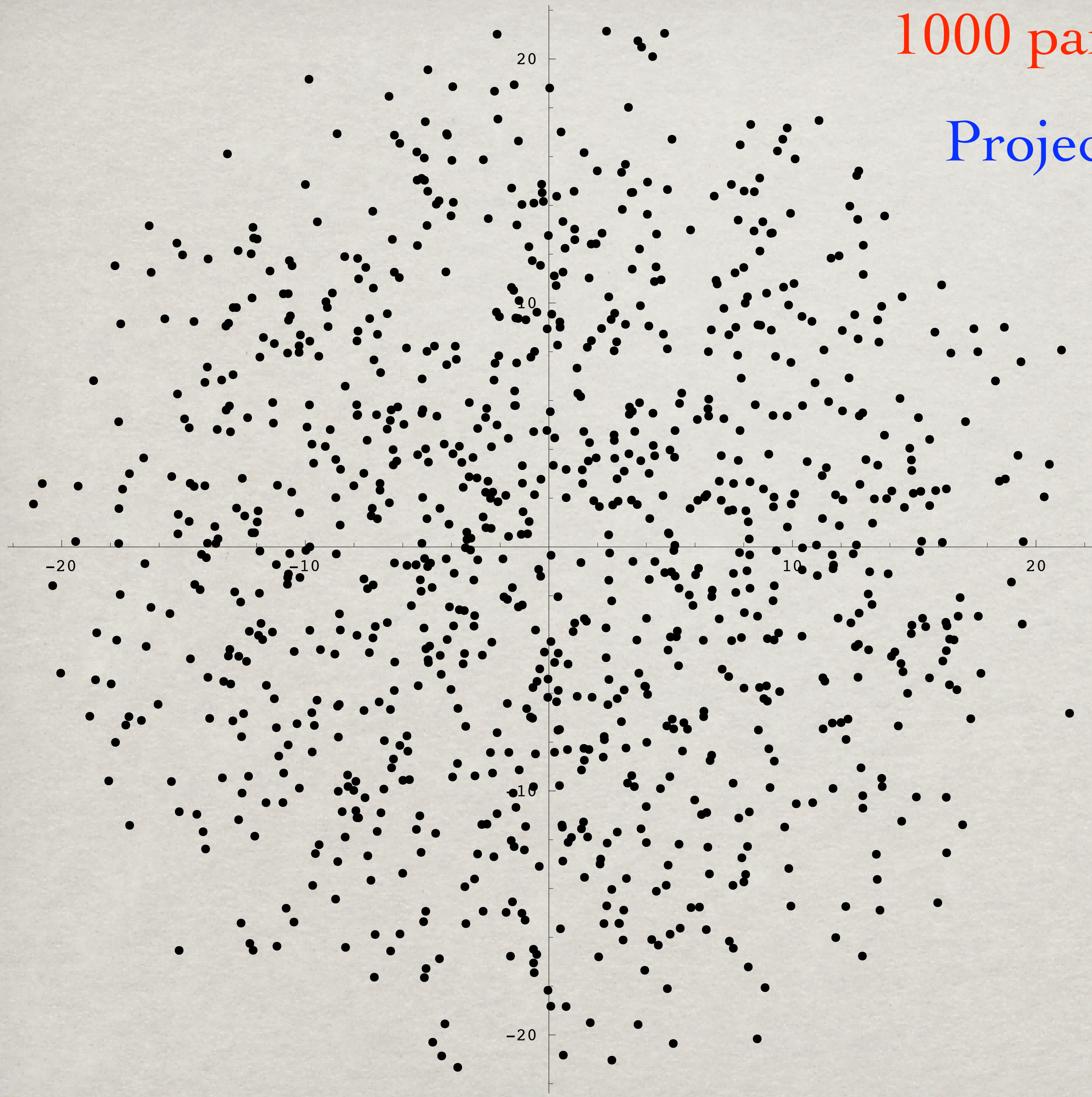
Expected classical radius: 7.07

Numerical radius: 6.63+-0.05

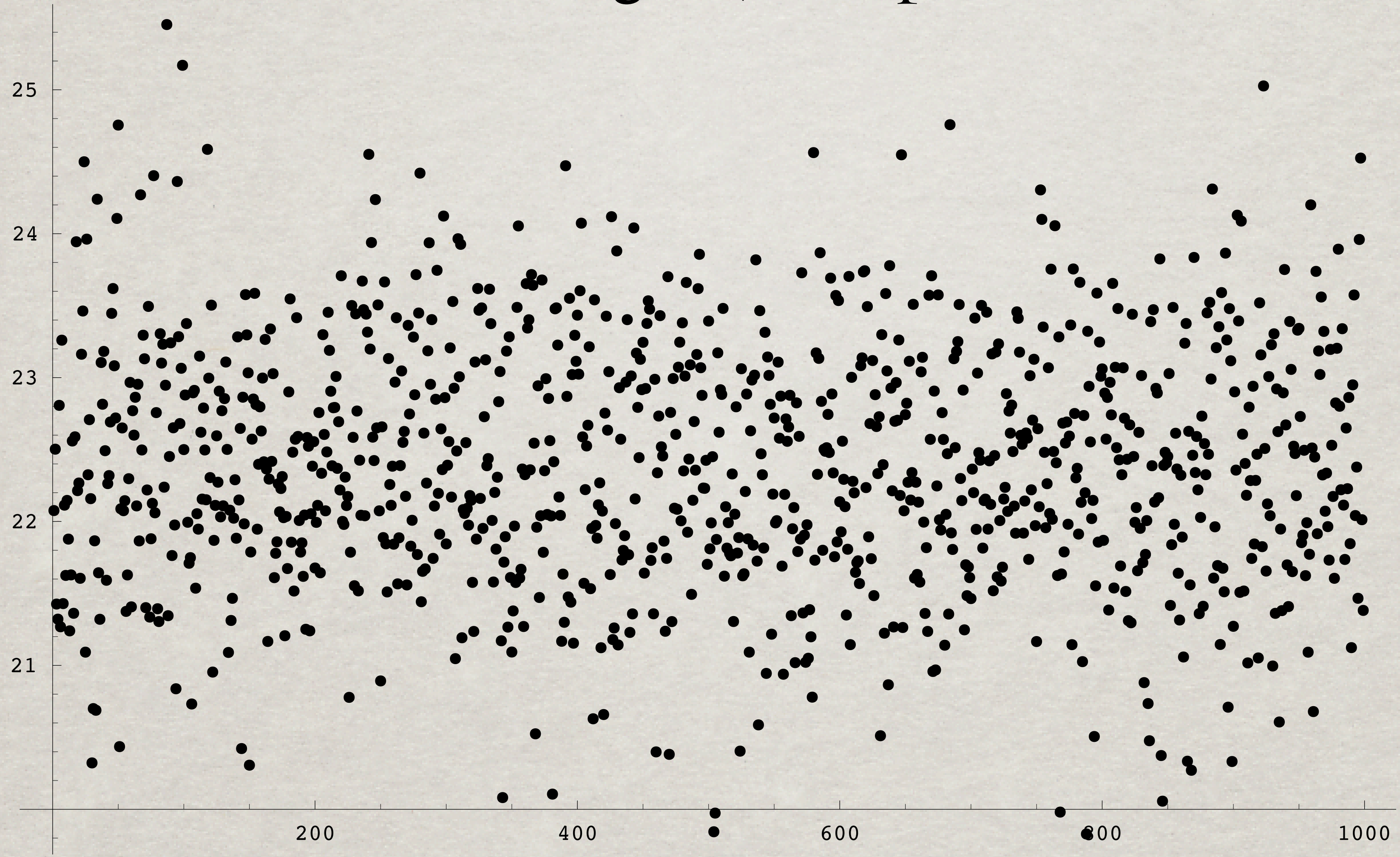
Can check low quadrupole moment.

1000 particle simulation

Projection on 12 plane



Distance from origin (1000 particle simulation).



Empty in center

The full quantum simulation differs from the classical expected result in an interesting way.

Thermal fluctuations of Boltzman system are quantum fluctuations in wave function.

These should be important quantum gravity effects.

Many wave functions can be simulated, they give rise to different geometries.

Matching to gravity needs a better dictionary:

Eigenvalues are D-branes (giant gravitons)

Experience with this type of setup in the half BPS case shows that the D-branes are a geometric locus where the sphere of the boundary shrinks to zero size. (LLM analysis)

The simulations capture this degeneration locus, not the full geometry.

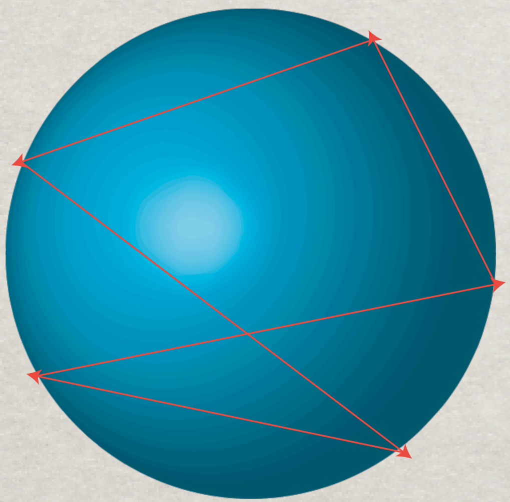
Approximations that lead to commuting matrices improve at strong coupling! Off-diagonal modes become heavy.

$$S_{sc} = \int dt\, \mathrm{tr} \left(\sum_{a=1}^{6} \frac{1}{2}(D_t X^a)^2 - \frac{1}{2}(X^a)^2 - \sum_{a,b=1}^{6} \frac{1}{8\pi^2} g_{YM}^2 [X^a, X^b][X^b, X^a] \right)$$

$$E_{osc} = \sqrt{1 + \frac{1}{2\pi^2} g_{YM}^2 |\vec{x}_j - \vec{x}_{j'}|^2}$$

With the typical radius, the energies scale like square root of the ‘t Hooft coupling. Diagonal modes are slow degrees of freedom in a Born-Oppenheimer approx.

Can also suggest origin of string scale: off-diagonal modes are massive and can be represented by lines joining eigenvalues (points on the sphere): STRING BITS. Need to dress them with gravity (eigenvalues).



One can also verify string tension

(D.B, D. Correa, S. Vazquez)

One can calculate string energies for BMN states:

$$O_k \sim \sum_{l=0}^{J} \exp(ikl/J)\mathrm{tr}(Z^{l-1}[Y,Z]Z^{J-l-1}[X,Z])$$

Diagonal modes

Off-diagonal modes

One treats the off-diagonal modes as free fields.

One calculates the energy of the BMN state assuming the off-diagonal modes don't affect the diagonal ones to first order.

The calculation of energies can be done in a saddle point and stationary phase approximation.

$$E \sim \frac{\langle \psi_k | H^{total} | \psi_k \rangle}{\langle \psi_k | \psi_k \rangle}$$

Details of state description

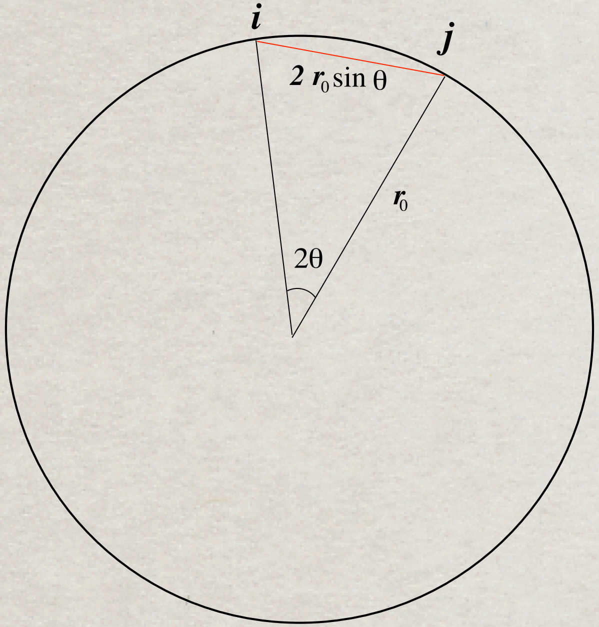
$$|\psi_k\rangle \sim \sum_{l=0}^{J} \exp(ikl/J) \sum_{j,j'} z_j^l Y_{j'}^{\dagger j} z_{j'}^{J-l} X_j^{\dagger j'} \hat{\psi}_0 |0\rangle_{od} \ ,$$

Manipulations of formulae lead to the following expression

$$\langle E^{osc} \rangle = \frac{\int \prod dx^i |\hat{\psi}_0|^2 \sum_{j,j'} |\sum_l \exp(ikl/J) z_j^l z_{j'}^{J-l}|^2 2\sqrt{1 + \frac{g_{YM}^2}{2\pi^2} |\vec{x}_j - \vec{x}_{j'}|^2}}{\int \prod dx^i |\psi_0|^2 \sum_{j,j'} |\sum_l \exp(ikl/J) z_j^l z_{j'}^{J-l}|^2} \ .$$

Localizes on sphere

Localizes on |z| maximal (diameter), and particular phase between z, z'



Result localizes to a string bit of particular fixed length

$$\theta = k/2J$$

$$\langle E^{osc} \rangle = 2\sqrt{1 + \frac{g_{YM}^2 N}{\pi^2} \sin^2(k/2J)}$$

Matches BMN string calculation to all orders in the ‘t Hooft coupling exactly, by taking J large k fixed. (Small angles)

- Matches the formula of Santambrogio and Zanon for all loop anomalous dimensions.

- It also matches the conjectures from the ``all loop Bethe ansatz" dispersion relation (Arutyunov, Beisert, Dippel, Staudacher)

- Surprisingly, this also matches the recent classical Nambu-Goto calculation (Hofman, Maldacena), including the full geometrical interpretation. The eigenvalue geometry above preceded the Nambu-Goto calculation.

- The origin of strings by string-bits is robust for other BPS geometries (different eigenvalue distributions)

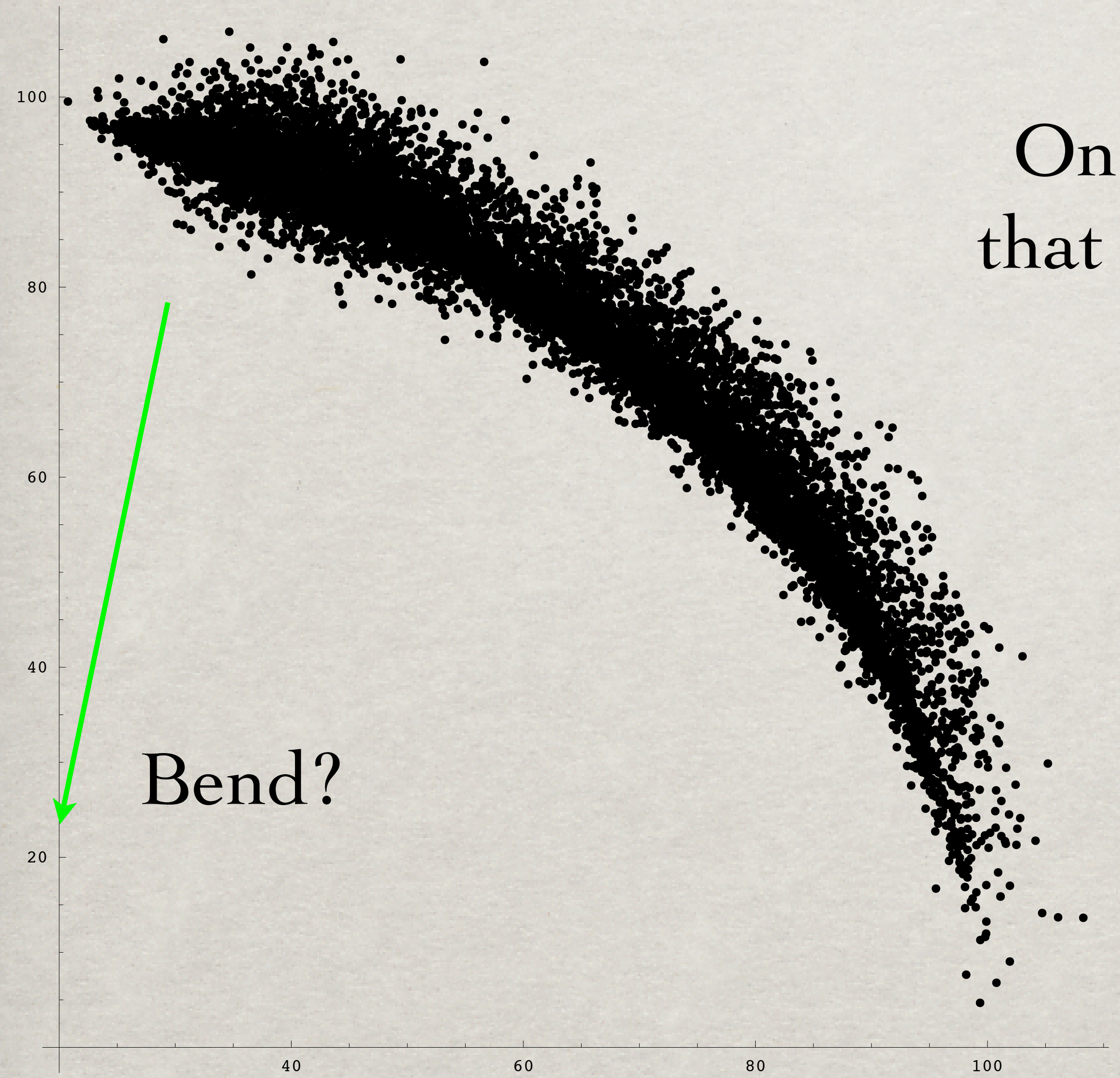
- Incomplete picture because off-diagonal modes are treated as free fields.

- Suggests breakdown of all-loop Bethe ansatz for small J, but it's not clear if it happens and where it happens. Number of magnons ~ number of off-diagonal modes.

- One should be able to change the number of string bits to make smooth slow strings moving on the sphere (or other geometries)

Puzzles

One can try a wavefunction that corresponds to a “donut”.



Does not match topological LLM intuition

There is no analytic result on the “surfaces” that the simulation produces, other than they are submanifolds, possibly with a boundary. Only the most symmetric case is understood.

Simulations can be a source of intuition to get to understand gravity better, including all quantum corrections!

A lot left to do!

Conclusion

- A lot of recent progress on Integrability of string.
- Reduction in low energy degrees of freedom to eigenvalues.
- Non-trivial repulsion of eigenvalues gives geometry as thermodynamic saddle point of wave function.
- Off-diagonal modes give string bits for all geometries.
- The approach gives a strong coupling expansion that is quantitative (can reproduce giant magnons, etc).
- The problems can be simulated on a computer!