Solvable irrelevant deformations of CFT_2 and AdS_3

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1701.05576, 1707.05800, 1711.02690, 1805.06286, 1806.09667, 1808.02492, 1808.08978

Deformations of CFT_2

In the last two years there has been some work on irrelevant deformations of CFT_2 , that lead to solvable non-local theories. Two types of deformations have been studied.

One is $T\bar{T}$ deformed CFT_2 ,

- F. Smirnov and A. Zamolodchikov, 1608.05499.
- A. Cavaglia, S. Negro, I. Szecsenyi and R. Tateo, 1608.05534.

These authors computed the spectrum of the theory on a circle of radius R.

They found that for a state with energy E_n and momentum P_n in the undeformed theory, the energy at finite $T\bar{T}$ coupling t is given by

$$\mathcal{E}_n(E_n, P_n, \lambda) = \frac{1}{\pi\lambda R}\left(\sqrt{1 + 2\pi\lambda R E_n + \lambda^2\pi^2 R^2 P_n^2} - 1\right)$$

where $\lambda = \frac{t}{R^2}$ is a dimensionless coupling, which can be thought of as the coupling t at the scale R.

The qualitative features of the spectrum are different for the two different signs of the coupling t.

- t>0: the energies are all real (for coupling below some critical value – we will get back to this later). The spectrum interpolates between the Cardy spectrum of the original CFT, and a Hagedorn spectrum for sufficiently high energies. In particular, the high energy physics is not governed by a fixed point. In that sense, the theory is non-local.

- $t<0$: for sufficiently large undeformed energies, the energies in the deformed theory become complex. One of the open questions in this field is what is the status of this theory? Is it inconsistent? If it is consistent, should one omit from the spectrum the states with complex energies? If not, what is the physics associated with the apparent non-unitarity?

The second class of deformations is $J\bar{T}$ deformed CFT_2 ,

- M. Guica, 1710.08415.
- S. Chakraborty, A. Giveon, DK, 1806.09667

Here, we start with a CFT that has a holomorphic U(1) current, J(x), and add to the Lagrangian the term

$$\mathcal{L}_{\rm int} = \mu J(x)\overline{T}(\overline{x})$$

At higher order in μ, one can adjust the Lagrangian such that the theory satisfies the following properties:

- The (xx) component of the stress tensor, $T_{xx} = T$, remains holomorphic, $\bar{\partial}T = 0$.
- The current J remains holomorphic, $\bar{\partial}J = 0$.

Generalizing the analysis of the $T\bar{T}$ case, one finds the following spectrum of energies and charges

$$\begin{aligned}\mathcal{E}_n^{(+)}(\widehat{\mu}) &= -\frac{2}{\pi^2\widehat{\mu}^2 kR}\sqrt{(1+\pi Q_n\widehat{\mu})^2+\pi^2\widehat{\mu}^2 kR(P_n-E_n)} \\ &\quad +\frac{1}{\pi^2\widehat{\mu}^2 kR}\left(2+2\pi Q_n\widehat{\mu}+\pi^2\widehat{\mu}^2 kP_nR\right), \\ \mathcal{Q}_n^{(+)}(\widehat{\mu}) &= \frac{1}{\pi\widehat{\mu}}\sqrt{(1+\pi Q_n\widehat{\mu})^2+\pi^2\widehat{\mu}^2 kR(P_n-E_n)}-\frac{1}{\pi\widehat{\mu}},\end{aligned}$$

Here $\hat{\mu} = \mu/R$ is the coupling μ at the scale R, E_n and P_n are the energy and momentum of the state at $\mu = 0$, and k is the anomaly of the U(1) current J, $\langle J(x)J(0)\rangle = \frac{k}{x^2}$.

Note that this fixes the normalization of the charge in the deformed theory. The charge of states depends on the coupling.

In this case, for both signs of the (real) coupling μ the spectrum has similar properties to those of $T\bar{T}$ deformed CFT with t<0. Highly excited states in the undeformed theory give rise after deformation to states with complex energies and charges.

Many of the questions raised above are relevant for this case as well.

Deformations of AdS_3

If a CFT_2 has an AdS_3 dual, one can study deformations of the type described above in the bulk theory. The direct analogs of $T\bar{T}$ and J$\bar{T}$ in AdS_3 are double trace operators. The corresponding perturbations were studied in a number of papers, starting (for $T\bar{T}$ with $t < 0$) with

L. McGough, M. Mezei, H. Verlinde, 1611.03470.

A second construction involves adding to the Lagrangian of the CFT_2 certain single trace operators, which have many properties in common with $T\bar{T}$ and $J\bar{T}$. In the bulk description they correspond to adding to the worldsheet Lagrangian of the theory certain truly marginal current-current deformations.

In the $T\bar{T}$ case they have been studied in the papers mentioned on the first slide. For $J\bar{T}$ in

- S. Chakraborty, A. Giveon, DK, 1806.09667.
- L. Apolo, W. Song, 1806.10127.

We next provide a few details on the worldsheet construction of these single trace deformations, and their interpretation in bulk gravity and boundary CFT.

String theory on AdS_3 (stabilized by NS B field) has three left-moving worldsheet currents $J^a(z)$, $a = 3, +, -$, that form an SL(2,R) current algebra, and their right- moving analogs $\bar{J}^a(\bar{z})$. These currents play an important role in the construction of the spacetime Virasoro symmetry of the theory. In particular, the zero modes of J^-, J^3, J^+ give the spacetime SL(2,R) generators L_{-1}, L_0, L_1, respectively.

In terms of these currents, one can show that the single trace analog of the $T\bar{T}$ deformation corresponds to adding to the worldsheet Lagrangian the term

$$\delta L = \lambda J^{-}(z)\,\bar{J}^{-}(\bar{z})$$

This is an intriguing result: the deformation is irrelevant in spacetime but is marginal on the worldsheet!

Thus, from the worldsheet string theory point of view, we do not expect to lose control of the theory when we turn on λ. And, since this is a current-current deformation, we expect the deformed theory to be solvable. All this is reminiscent of $T\bar{T}$.

Holography implies that the worldsheet deformation δL must correspond to a deformation of the boundary Lagrangian by the term $\lambda D(x)$, where $D(x)$ is an operator of dimension (2,2). This operator was constructed in KS (1999). It is a quasi-primary of the spacetime Virasoro, and has the same OPE with the spacetime stress tensor as the operator $T\bar{T}$ of the boundary CFT.

I next provide some technical details as to its construction.

The left-moving SL(2,R) worldsheet currents can be combined into the single current

$$J(x;z) = 2xJ_3(z) - J^+(z) - x^2 J^-(z)$$

Where x=position on the boundary, z=position on the worldsheet. The left-moving spacetime stress tensor takes the form (in the bosonic string)

$$T(x) = \frac{1}{2k}\int d^2z(\partial_x J \partial_x \Phi_1 + 2\partial_x^2 J \Phi_1)\bar{J}(\bar{x};\bar{z})$$

The operator $\Phi_1(x;z)$ is a certain dimension zero primary of worldsheet Virasoro.

The operator D(x) takes the form

$$D(x) = \int d^2z (\partial_x J \partial_x + 2\partial_x^2 J)(\partial_{\bar{x}} \bar{J} \partial_{\bar{x}} + 2\partial_{\bar{x}}^2 \bar{J})\Phi_1$$

This operator has spacetime scaling dimension (2,2). Hence it is a supergravity field. It is essentially the massive dilaton on AdS_3. One can show that

$$\int d^2x D(x, \bar{x}) \simeq \int d^2z J^-(z) \bar{J}^-(\bar{z})$$

A simple way to think about the difference between the single and double trace $T\bar{T}$ - type operators is the following. Suppose the boundary CFT had the symmetric product form M^N/S_N , where M is itself a CFT. Denoting by T_i the stress tensor of the i'th copy of M, we can consider the following two deformations:

(*) Single trace: $\sum_i T_i\bar{T}_i$

(*) Double trace: $\sum_{ij} T_i\bar{T}_j$

The single trace deformation corresponds to a $T\bar{T}$ deformation of the building block of the symmetric product, M. The double trace to a $T\bar{T}$ deformation of the full CFT, M^N/S_N.

The above discussion can be repeated for the case of $J\bar{T}$ deformations. In this case we assume that the worldsheet CFT has a holomorphic current $K(z)$. As shown in GKS (1998), this means that the boundary CFT has a holomorphic current $J(x)$, described by the vertex operator

$$J(x) = -\frac{1}{k}\int d^2 z K(z)\overline{J}_{\rm SL}(\overline{x};\overline{z})\Phi_1(x,\overline{x};z,\overline{z})$$

Associated with it is the dimension (1,2) single trace operator

$$A(x,\overline{x}) = \int d^2 z K(z)(\partial_{\overline{x}}\overline{J}_{\rm SL}\partial_{\overline{x}}\Phi_1 + 2\partial_{\overline{x}}^2\overline{J}_{\rm SL}\Phi_1)$$

This operator has the same quantum numbers as the operator $J\bar{T}$ in the boundary CFT, but as in the discussion of $T\bar{T}$ above, it is a single trace operator, while $J\bar{T}$ is a double trace operator.

Adding the operator $A(x,\bar{x})$ to the Lagrangian of the boundary CFT corresponds to adding to the worldsheet Lagrangian the term $K(z)\bar{J}^{-}(\bar{z})$, since one can show that

$$\int d^2x A(x,\overline{x}) \simeq \int d^2z K(z)\overline{J}^{-}_{\mathrm{SL}}(\overline{z})$$

As before, an irrelevant deformation of the boundary CFT corresponds to a marginal, current-current, deformation of the worldsheet one, and similar comments apply.

Adding the above single trace operators to the action corresponds to a modification of the geometry. In the $T\bar{T}$ case with t>0, the AdS_3 is replaced by a background $\mathcal{M}_3$ described by

$$
\begin{aligned}
ds_3^2 &= f_1^{-1} l_s^2 d\gamma d\bar{\gamma} + R_5^2 d\phi^2 \ , \\
e^{2\Phi} &= \frac{v}{p} e^{-2\phi} f_1^{-1} \ , \\
dB &= 2ie^{-2\phi} f_1^{-1} \epsilon_3 = d\left[i f_1^{-1} d\gamma \wedge d\bar{\gamma} \right] \ ,
\end{aligned}
$$

$$
f_1 = 1 + \frac{1}{k} e^{-2\phi} \ , \qquad e^{\phi} = \frac{l_s r}{R_1 R_5} = \sqrt{\frac{v}{pk}} \frac{r}{g_s l_s}
$$

where $(\phi, \gamma, \bar{\gamma})$ are the radial and boundary coordinates, and the various constants have an interpretation in terms of branes.

This geometry interpolates between AdS_3 in the infrared region $\phi \to -\infty$, and a linear dilaton spacetime $R_\phi \times R_t \times S^1$ near the boundary at $\phi \to \infty$. This is in nice correspondence with the behavior of the spectrum of $T\bar{T}$ deformed CFT mentioned above, since the asymptotically linear dilaton spacetime is related by holography to Little String Theory, which has a Hagedorn high energy spectrum.

One can think of the background $\mathcal{M}_3$ as follows. We start with the linear dilaton near-horizon geometry of fivebranes, which corresponds to a vacuum of LST, and add to it a large number of fundamental strings.

The infrared AdS_3 describes the near-horizon geometry of both the strings and the fivebranes, while the UV linear dilaton region describes the geometry in the near-horizon geometry of the fivebranes but far from the strings.

Thus, adding the single trace deformation $\lambda D(x,\bar{x})$ allows one to connect the dynamics in the near-horizon region of the strings to that far from them.

For $T\bar{T}$ with t<0 the harmonic function f_1 is replaced by

$$f_1 = -1 + \frac{1}{k} e^{-2\phi}$$

In the IR ($\phi \to -\infty$) it still corresponds to AdS_3 , but now there is a singularity at a finite value of ϕ, and behind it the roles of space and time are interchanged. In particular, there are closed timelike curves.

For the J$\bar{T}$ case, the geometry one finds can be summarized by the sigma model

$$S(\epsilon) = \frac{k}{2\pi}\int d^2z \left(\partial\phi\bar{\partial}\phi + e^{2\phi}\partial\bar{\gamma}\bar{\partial}\gamma + 2\epsilon e^{2\phi}\partial y\bar{\partial}\gamma + \frac{1}{k}\partial y\bar{\partial}y\right)$$

For $\epsilon = 0$ this is $AdS_3 \times S^1$, with the current J associated with left-moving momentum on S^1.

Dimensionally reducing the deformed geometry on S^1 gives the geometry

$$ds^2 = k\left(d\phi^2 + e^{2\phi} d\gamma d\overline{\gamma} - \epsilon^2 e^{4\phi} d\gamma^2\right)$$

some gauge field and B-field.

This geometry appeared before in the discussion of Schroedinger spacetimes

- D. T. Son, 0804.3972.
- K. Balasubramanian, J. Mcgreevy, 0804.4053.

and in the context of the Kerr/CFT correspondence. It is non-singular, but has closed timelike curves beyond some critical value of the radial coordinate ϕ.

A natural question is what is the relation between the single trace deformations of string theory on AdS_3 that give rise to the above geometries, and the $T\bar{T}$ and $J\bar{T}$ deformations of CFT_2.

The full answer to this question is not known, mainly because we do not have complete control over the original AdS_3/CFT_2 duality, but one element in the story is the following.

By thinking about AdS_3/CFT_2 duality as arising from the low energy dynamics of strings and fivebranes, it is reasonable to expect that the Ramond sector of the CFT_2 is described by a symmetric product, M^N/S_N , where N is the number of strings creating the background. Indeed, there is a lot of evidence that this is the case.

The single trace deformations of AdS_3 correspond in the symmetric product to $T\bar{T}$ and J$\bar{T}$ deformations of the CFT $M.$

Thus one can use results from string theory on AdS_3 to learn about $T\bar{T}$ and J$\bar{T}$ deformed CFT, and vice versa.

The string theory systems one encounters in the process are of independent interest. One may hope to learn more about holography in asymptotically linear dilaton spacetimes (LST), systems realizing Schroedinger symmetries, the Kerr/CFT correspondence, and the physics associated with singularities and closed timelike curves.

And, the study of deformations may teach us about the original AdS_3/CFT_2 duality.

Torus partition sum: I

In the rest of this talk, we will discuss the torus partition sum of the deformed CFT's discussed above. We will see that we can derive the partition sums (and thus spectra) of these theories by imposing certain general constraints, and discuss some of their features.

Consider first the $T\bar{T}$ deformed theory. Thinking about it abstractly, one can view it as a theory satisfying the following conditions:

- It is a theory with one scale, associated with the coupling t.

- It is modular invariant, if we assign to the dimensionless coupling $\lambda \sim \frac{t}{R^2}$ a non-trivial transformation property dictated by its scaling dimension:

$$\mathcal{Z}\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\middle| \frac{\lambda}{|c\tau+d|^2}\right) = \mathcal{Z}(\tau,\bar{\tau}|\lambda)$$

- The energies of states in the deformed theory depend only on the energies and momenta of the corresponding states in the undeformed theory and on λ.

We will show that these three assumptions uniquely determine the partition sum (and thus the spectrum) to be that of the $T\bar{T}$ deformation of the theory with $\lambda = 0$, to all orders in λ. Non-perturbatively, some interesting issues arise.

Before doing that, some comments on the assumptions.

- The first assumption essentially means that the theory exists. If we think of it as a CFT deformed by some irrelevant operator with coupling t, this is the statement that physical observables are well behaved in the limit where the UV cutoff $\Lambda \to \infty$. Incidentally, in my presentation of the analysis, t will be taken to have dimension -2, but it is easy to generalize to arbitrary dimension.

- The second assumption essentially means that the theory can be formulated on the torus. If this is the case for the original CFT, it is also the case for the deformed theory, at least to all orders in t.

- The third assumption is, of course, highly non-trivial. It is satisfied by the spectrum of $T\bar{T}$ deformed CFT_2 , but it is a qualitative assumption. The result of our analysis is that $T\bar{T}$ deformed CFT_2 is the only theory that satisfies this assumption (and the other two).

We now move on to proving the above result. The hero of our story is going to be the partition sum

$$\mathcal{Z}(\tau,\bar{\tau}|\lambda) = \sum_n e^{2\pi i \tau_1 R P_n - 2\pi \tau_2 R \mathcal{E}_n}$$

At $\lambda = 0$ it approaches the usual CFT partition sum

$$Z_0(\tau,\bar{\tau}) = \mathrm{Tr}\,\left[e^{2\pi i \tau (L_0 - \frac{c}{24})} e^{-2\pi i \bar{\tau}(\bar{L}_0 - \frac{c}{24})}\right] = \sum_n e^{2\pi i \tau_1 R P_n - 2\pi \tau_2 R E_n}$$

When we turn on λ, the energies are deformed,

$$E_n \mapsto \mathcal{E}_n(E_n, P_n, \lambda),$$

The deformed energies can be Taylor expanded

$$\mathcal{E}_n(E_n, P_n, \lambda) = \sum_{k=0}^{\infty} \mathrm{E}_n^{(k)} \lambda^k = \mathrm{E}_n^{(0)} + \lambda\, \mathrm{E}_n^{(1)} + \lambda^2\, \mathrm{E}_n^{(2)} + \cdots$$

where $E_n^{(k)}$ are functions of $E_n (= E_n^{(0)})$ and P_n , to be determined.

Plugging in the Taylor expansion of the energies into the partition sum gives:

$$\mathcal{Z}(\tau, \bar{\tau}|\lambda) = \sum_{k=0}^{\infty} Z_k \, \lambda^k = Z_0 + Z_1 \, \lambda + Z_2 \, \lambda^2 + \cdots$$

The coefficients Z_p are modular forms of weight (p, p),

$$Z_p \left(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \right) = (c\tau + d)^p (c\bar{\tau} + d)^p \, Z_p(\tau, \bar{\tau}).$$

We next show that they are uniquely determined by the above assumptions.

To see that, consider the first few Z_p:

$$Z_1 = \sum_n \left(-2\pi R\tau_2 \mathrm{E}_n^{(1)}\right) e^{2\pi i\tau_1 RP_n - 2\pi\tau_2 RE_n},$$

$$Z_2 = \sum_n \left(\frac{\tau_2^2}{2}(2\pi R\mathrm{E}_n^{(1)})^2 - 2\pi R\tau_2 \mathrm{E}_n^{(2)}\right) e^{2\pi i\tau_1 RP_n - 2\pi\tau_2 RE_n},$$

$$Z_3 = \sum_n \left(-\frac{\tau_2^3}{6}(2\pi R\mathrm{E}_n^{(1)})^3 + (2\pi R\tau_2)^2 \mathrm{E}_n^{(1)}\mathrm{E}_n^{(2)} - 2\pi R\tau_2 \mathrm{E}_n^{(3)}\right) e^{2\pi i\tau_1 RP_n - 2\pi\tau_2 RE_n}$$

$E_n^{(k)}$ are functions of the unperturbed energies and momenta, E_n and P_n . Therefore, in the expressions for Z_p we can replace

$$2\pi RE_n \mapsto -\partial_{\tau_2} = \frac{1}{i}(\partial_\tau - \partial_{\bar{\tau}}), \qquad 2\pi iRP_n \mapsto \partial_{\tau_1} = \partial_\tau + \partial_{\bar{\tau}}$$

This leads to an expression for Z_p that takes the form

$$Z_p = \left[\tau_2^p \widehat{\mathcal{O}}_1^{(p)}(\partial_\tau, \partial_{\bar\tau}) + \tau_2^{p-1} \widehat{\mathcal{O}}_2^{(p)}(\partial_\tau, \partial_{\bar\tau}) + \cdots + \tau_2 \widehat{\mathcal{O}}_p^{(p)}(\partial_\tau, \partial_{\bar\tau})\right] Z_0(\tau, \bar\tau)$$

where $\widehat{\mathcal{O}}_j^{(p)}(\partial_\tau, \partial_{\bar\tau})$ are differential operators that encode the information about the energy shifts $E_n^{(k)}$. Using this structure, one can show that given $Z_0(\tau, \bar\tau)$, there is a unique $Z_p(\tau, \bar\tau)$ that satisfies all the constraints (see paper for details).

The solution satisfies the following recursion relation:

$$Z_{p+1} = \frac{\tau_2}{p+1}\left(\mathsf{D}_\tau^{(p)}\mathsf{D}_{\bar\tau}^{(p)} - \frac{p(p+1)}{4\tau_2^2}\right) Z_p$$

where

$$\mathsf{D}_\tau^{(k)} = \partial_\tau - \frac{ik}{2\tau_2}, \qquad \mathsf{D}_{\bar\tau}^{(k)} = \partial_{\bar\tau} + \frac{ik}{2\tau_2}$$

are covariant derivatives acting on modular forms.

One way to make contact with the spectrum mentioned above is to note that the recursion relation above is equivalent to a differential equation satisfied by the partition sum:

$$\partial_\lambda \mathcal{Z}(\tau,\bar{\tau}|\lambda) = \left[\tau_2 \partial_\tau \partial_{\bar{\tau}} + \frac{1}{2}\left(i\left(\partial_\tau - \partial_{\bar{\tau}}\right) - \frac{1}{\tau_2}\right)\lambda \partial_\lambda\right] \mathcal{Z}(\tau,\bar{\tau}|\lambda)$$

Plugging the definition of the partition sum into this equation and comparing the coefficients of a given exponential on the left and right hand sides gives a differential equation for the energies, which is precisely the inviscid Burgers equation obtained in the original papers on $T\bar{T}$ deformed CFT_2.

So far, our discussion has been perturbative in the coupling λ. It is natural to ask what happens when we go beyond perturbation theory, and in particular work at small but finite λ. It is not completely clear how to define the theory in that case.

We will take the approach that the differential equation satisfied by the partition sum should be valid non-perturbatively, and ask whether it defines a unique function $\mathcal{Z}(\tau, \bar{\tau}|\lambda)$ given the initial condition

$$\mathcal{Z}(\tau, \bar{\tau}|0) = Z_0(\tau, \bar{\tau})$$

The answer turns out to be the following.

- For $\lambda > 0$ the differential equation has a unique solution.
- For $\lambda < 0$ one encounters a non-perturbative ambiguity. This ambiguity is associated with the fact that in that case the differential equation is consistent with the existence of states whose energies diverge like $\frac{1}{\lambda}$ as $\lambda \to 0$.

One can think of these states as having energies of the same form as the ones above, but with the opposite sign in front of the square root.

As is clear from the formula for the energies, for $\lambda > 0$ such states have energies that go to $-\infty$ as $\lambda \to 0$, and therefore they are not allowed (e.g. because their contribution to the partition sum does not satisfy the boundary condition at $\lambda = 0$).

For $\lambda < 0$ these states have energies that go to $+\infty$, and therefore they are allowed. Moreover, these states can be the negative branch states of a different (modular invariant) CFT, unrelated to the original one, with partition sum $Z_0(\tau)$.

The fact that for $\lambda > 0$ we find a unique partition sum is in agreement with the fact that the spectrum of the theory is real in this case, and that the dual asymptotically linear dilaton dual geometry is well behaved.

The non-perturbative ambiguity we find for $\lambda < 0$ is likely related to the breakdown of unitarity associated with the complex energies, and with the fact that the dual geometry has singularities and closed timelike curves. However, more work is needed to understand this better.

Another interesting comment is the following. As mentioned above, the theory with $\lambda > 0$ has a Hagedorn spectrum, which means that the partition sum is only convergent when the modular parameter satisfies the constraint

$$\tau_2 > \tau_2^{\mathrm{H}}(\lambda),$$

$$\tau_2^{\mathrm{H}}(\lambda) = \sqrt{\frac{\pi c \lambda}{6}}$$

Another constraint on the coupling comes from the requirement that low lying states in the deformed theory have real energies. This leads to the constraint $\pi c\lambda < 6$. The two constraints are related by the modular transformation

$$\mathcal{Z}(\tilde{\tau}_2, \tilde{\lambda}) = Z(\tau_2, \lambda), \quad \text{with} \;\; \tilde{\tau}_2 = \frac{1}{\tau_2}, \;\; \tilde{\lambda} = \frac{\lambda}{\tau_2^2}$$

They can be summarized as the statement that on a square torus, the partition sum is well defined only when both sides of the torus are larger than $2\pi\sqrt{\frac{2\pi ct}{3}}$.

- This is clearly a manifestation of the non-locality of the theory. The non locality scale is proportional to $\sqrt{ct}$, as expected from other considerations.

- It is reminiscent of the usual relation between the high energy density of states and the mass of the lowest lying state winding around Euclidean time familiar from free critical string theory.

Torus partition sum: II

The discussion above is easy to generalize to the case of J$\overline{T}$ deformed CFT. Using the same logic as before, we define the theory by the following conditions:

- It is a theory with one scale, associated with the coupling μ.

- The partition sum with a chemical potential for the U(1) charge,

$$\mathcal{Z}(\tau,\bar{\tau},\nu|\widehat{\mu}) = \sum_n e^{2\pi i\tau_1 R P_n - 2\pi\tau_2 R\mathcal{E}_n + 2\pi i\nu\mathcal{Q}_n}$$

is modular covariant, if we assign to the dimensionless coupling $\hat{\mu} \sim \frac{\mu}{R}$ a non-trivial transformation property dictated by its scaling dimension:

$$\mathcal{Z}\left(\frac{a\tau+b}{c\tau+d},\frac{a\bar{\tau}+b}{c\bar{\tau}+d},\frac{\nu}{c\tau+d}\middle|\,\frac{\widehat{\mu}}{c\bar{\tau}+d}\right) = \exp\left(\frac{i\pi kc\nu^2}{c\tau+d}\right)\mathcal{Z}(\tau,\bar{\tau},\nu|\widehat{\mu})$$

- The energies and charges of states in the deformed theory depend only on the energies, momenta and charges of the corresponding states in the undeformed theory and on $\hat{\mu}$.

Note that the transformation properties of ν encode the fact that the current J(x) is holomorphic in the deformed theory, and its anomaly, k.

We can now run the same program as before. Expanding the partition sum,

$$\mathcal{Z}(\tau,\bar{\tau},\nu|\widehat{\mu}) = \sum_{p=0}^{\infty} Z_p \widehat{\mu}^p = Z_0 + Z_1\widehat{\mu} + Z_2\widehat{\mu}^2 + \cdots$$

we find that the expansion coefficients Z_p, which are Jacobi forms of weight (0,p) and rank k, are uniquely determined and satisfy the recursion relation

$$Z_p = \frac{\tau_2}{p}\left[\mathsf{D}_\nu \mathsf{D}_{\bar{\tau}}^{(p-1)} - \frac{i\pi k\nu(p-1)}{2\tau_2^2}\right] Z_{p-1} - \frac{i\pi k}{2p}\sum_{j=0}^{p-2}\left(\frac{\pi\nu k}{2i\tau_2}\right)^j \mathsf{D}_{\bar{\tau}}^{(p-j-2)} Z_{p-j-2}$$

where

$$\mathsf{D}_\nu \equiv \partial_\nu + \frac{\pi k\nu}{\tau_2}$$

is a covariant derivative acting on Jacobi forms.

This recursion relation is equivalent to the differential equation

$$\left(1+\frac{i\pi k\widehat{\mu}\nu}{2\tau_2}\right)\partial_{\widehat{\mu}}\mathcal{Z}=\tau_2\mathsf{D}_\nu\mathcal{D}_{\bar{\tau}}\mathcal{Z}-\frac{i\pi k\widehat{\mu}}{2}\frac{1}{1+\frac{i\pi k\widehat{\mu}\nu}{2\tau_2}}\mathcal{D}_{\bar{\tau}}\mathcal{Z}$$

From this equation one can read off the flow equations for the (dimensionless) energies and charges of states as a function of the coupling,

$$\begin{aligned}\mathbb{E}_n'(\widehat{\mu})\left[1+\pi\widehat{\mu}\mathcal{Q}_n(\widehat{\mu})\right]&=\pi\left[\mathbb{P}_n-\mathbb{E}_n(\widehat{\mu})\right]\mathcal{Q}_n(\widehat{\mu}),\\ \mathcal{Q}_n'(\widehat{\mu})\left[1+\pi\widehat{\mu}\mathcal{Q}_n(\widehat{\mu})\right]&=\frac{\pi k}{2}\left[\mathbb{P}_n-\mathbb{E}_n(\widehat{\mu})\right],\end{aligned}$$

These are the analog of the Burgers equation of the $T\bar{T}$ case.

There is again a non-perturbative ambiguity in the partition sum, this time for all values of the coupling, whose structure is similar to that of the $T\bar{T}$ theory with negative coupling.

This is consistent with the fact that the energies of highly excited states are complex, and the dual geometry has closed timelike curves, though no curvature singularities.

Discussion

There is clearly a lot to do on this subject.

- On the field theory side, it seems that $T\bar{T}$ deformed CFT with $t > 0$ is a well defined theory with a Hagedorn spectrum. It would be nice to understand other observables, and provide a non-perturbative definition of the theory. There was some work on this subject. E.g. it was observed early on that for $c = 24$ the spectrum of $T\bar{T}$ deformed CFT is the same as that of string theory on a background that includes the CFT and an additional $R{\times}S^1$, in a sector with winding one. However, it is not clear whether/why the two are the same. For general c there is a proposal of JT gravity coupled to matter. It would be nice to understand these proposals better.

- For $T\bar{T}$ with $t < 0$ and J$\bar{T}$ we found complex energies and non-perturbative ambiguities. It would be nice to understand whether the theory makes sense, if it is unitary, and if so how is unitarity reconciled with modular invariance on the torus.

- On the gravity side, it would be nice to develop the description of the double trace deformations to the point that all the phenomena we discussed in the field theory picture become manifest.

- For single trace deformations, it would be nice to understand their interpretation in the QFT better, and relate the phenomena seen in the QFT to the deformed geometry more precisely.

- Developing the last point might help understand the implications of the recent results for LST, Kerr/CFT, Schroedinger backgrounds and other systems with these symmetries.

- The string theory construction suggests that we should be able to solve exactly more general theories, involving combinations of couplings, such as $T\bar{T}$, J$\bar{T}$, $T\bar{J}$, $J\bar{J}$. It would be interesting to generalize the discussion to this case.