LOOPS WITHOUT LOOPS and the FEYNMAN TREE

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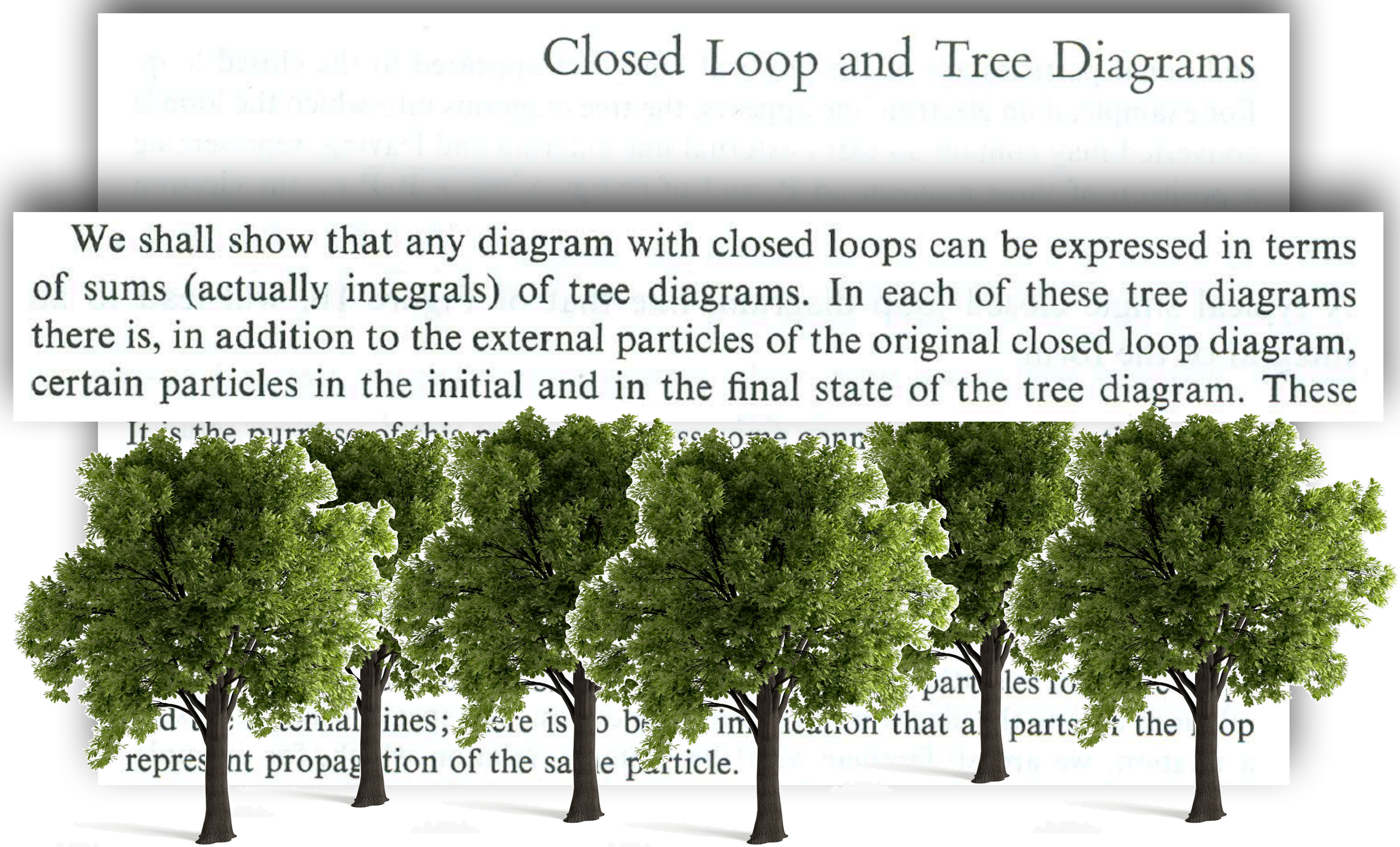
R. P. Feynman, Closed Loop And Tree Diagrams. (talk),
Magic Without Magic: J. A. Wheeler, A Collection of Essays in Honor of his Sixtieth Birthday. Edited by John R. Klauder, **1972**, p.355
In *Brown, L.M. (ed.): Selected papers of Richard Feynman* 867-887


Closed Loop and Tree Diagrams

We shall show that any diagram with closed loops can be expressed in terms of sums (actually integrals) of tree diagrams. In each of these tree diagrams there is, in addition to the external particles of the original closed loop diagram, certain particles in the initial and in the final state of the tree diagram. These

WEAKNESSES OF QFT

- **SM/BSM** extrapolated to **infinite energy** (**zero distance**) in loop corrections $\gg M_{\text{Plank}}$
- Quantum state with *N* partons $\neq$ quantum state with **zero energy emission** of extra patrons
- Partons can be emitted in **exactly the same direction**

soft singularities (IR)

collinear singularities (IR)

Ultraviolet singularities (UV)

and **threshold** singularities,
integrable but numerically unstable

in **four** space-time dimensions

LOCAL SUBTRACTION IN THE UV

UNSUBTRACTION IN THE IR

- **LTD:** open loops to trees
- **FDU:** mapping of V ▶R kinematics

- **Integrand cancellation** of singularities in d=4 space-time dimensions
- **V+R simultaneous:**
  - More efficient event generators
  - LTD suitable for amplitudes, FDU aimed at $d\sigma$

IT´S ALL ABOUT THE TINY +I0 FROM

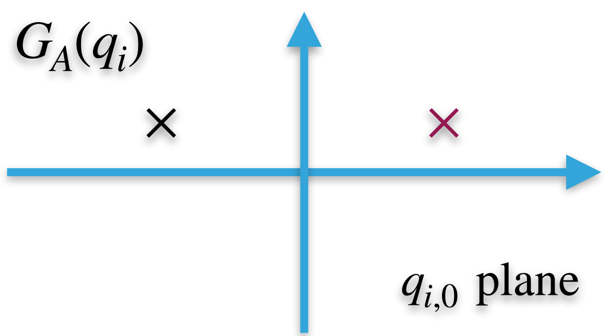
PROPAGATORS

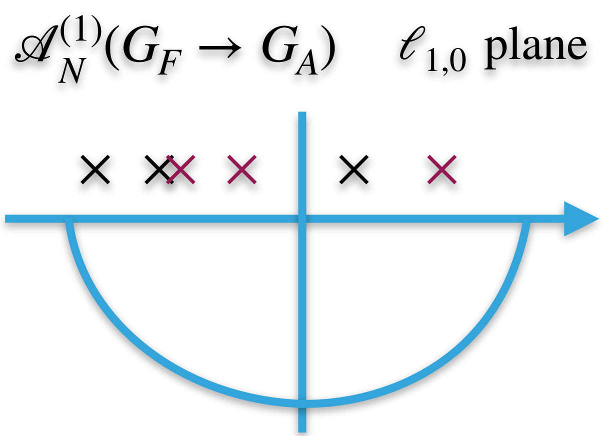
$$G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0}$$

- MATH: the +i0 is a small quantity usually ignored, assuming that the **analytical continuation** to the physical kinematics is well defined
- PHYS: the +i0 encodes **CAUSALITY | positive frequencies are propagated forward in time, and negative backward**

THE FEYNMAN'S TREE THEOREM (FTT)

$$G_A(q_i) = \frac{1}{q_i^2 - m_i^2 - i0q_{i,0}}$$





- **Advanced (Retarded) propagator:** Both poles are displaced above (below) the real axis, independently of the sign of the energy
- An amplitude with all the Feynman propagators replaced by Advanced propagators **vanishes**

$$\int_{\ell_1} \mathcal{N}(\ell_1) \prod G_A(q_i) = 0$$

- Related to the Feynman propagator by a delta function

$$\frac{1}{x \pm i0} = PV\left(\frac{1}{x}\right) \mp i\pi\delta(x)$$

$$G_A(q_i) = G_F(q_i) + \tilde{\delta}(q_i)$$

THE FEYNMAN'S TREE THEOREM (FTT)

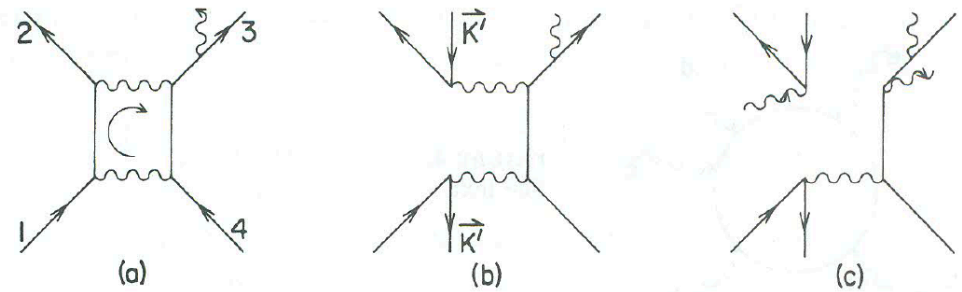


FIGURE 2.
A closed loop diagram (*a*) and some of the tree diagrams (*b*), (*c*) into which it can be broken up.

$$\int \frac{d^4k}{(2\pi)^4} N(k) \prod_i \left[I_+^{(i)}(k) - \frac{\pi}{\sqrt{(\mathbf{K}-\mathbf{P}_i)^2 + m_i^2}} \times \delta\left(\omega - E_i + \sqrt{(\mathbf{K}-\mathbf{P}_i)^2 + m_i^2}\right) \right] = 0. \quad (10)$$

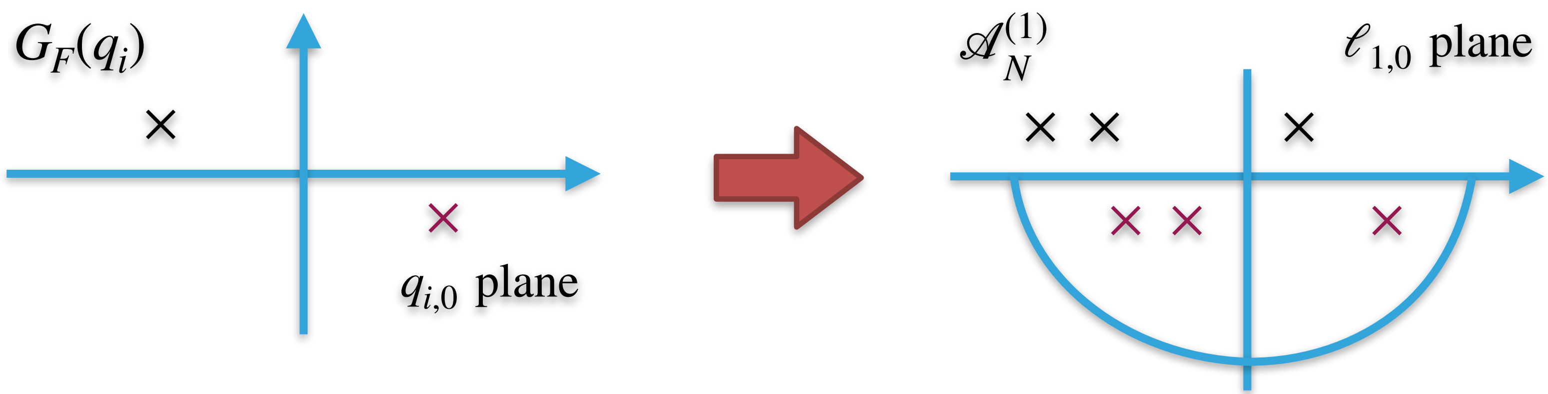
- The loop amplitude is the sum of **multiple-cut (includes disjoint trees)** contributions
- In four-dimensions: maximum 4 cuts
- The double-cut contribution from the FTT is different from the unitarity cut that gives the imaginary part due to the different positive-energy flow of the internal lines

 [Catani et al. 2008]

THE LOOP-TREE DUALITY (LTD)

Cauchy residue theorem

in the loop energy complex plane



Feynman Propagator +i0:
positive frequencies are propagated forward in time, and negative backward

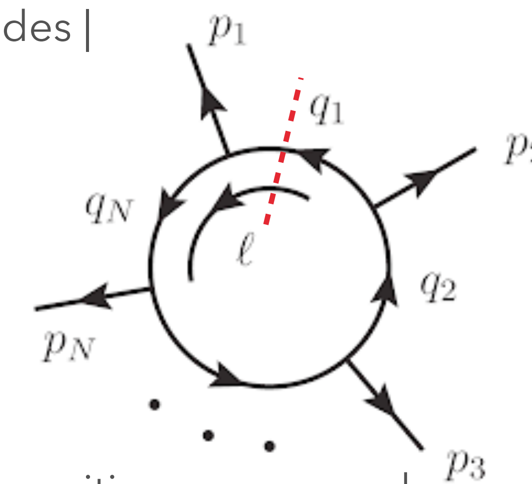
$$G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0}$$

selects residues with definite **positive energy and negative imaginary part** (indeed in any other coordinate system)

 [Catani et al. 2008]

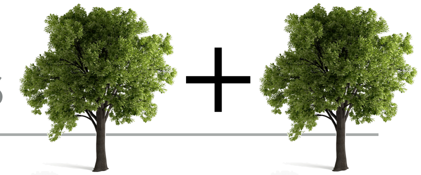
THE LOOP-TREE DUALITY (LTD)

One-loop amplitudes in any relativistic, local and unitary QFT represented as a linear combination of N **single-cut phase-space/dual** amplitudes | **no disjoint trees** (at higher orders: number of cuts equal to the number of loops)

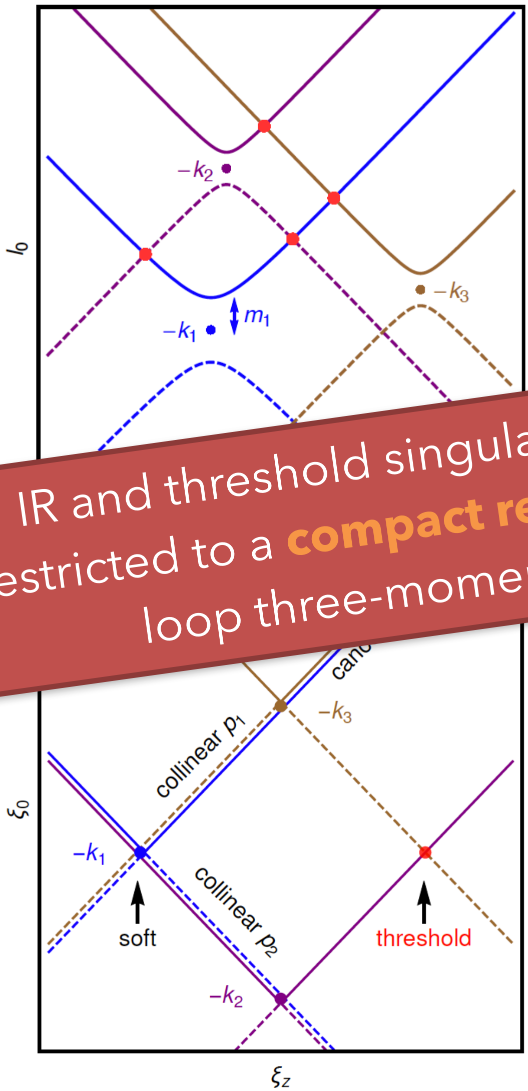


$$\int_{\ell_1} \mathcal{N}(\ell_1) \prod G_F(q_i) = - \int_{\ell_1} \mathcal{N}(\ell_1) \otimes \sum \tilde{\delta}(q_i) \prod_{i \neq j} G_D(q_i; q_j)$$

- $\tilde{\delta}(q_i) = i\, 2\pi\, \theta(q_{i,0})\, \delta(q_i^2 - m_i^2)$ sets internal line on-shell, positive energy mode
- $G_D(q_i; q_j) = \dfrac{1}{q_j^2 - m_j^2 - i0\, \eta\, k_{ji}}$ **dual propagator**, $k_{ji} = q_j - q_i$
- best choice $\eta^\mu = (1, \mathbf{0})$: energy component integrated out, remaining integration in **Euclidean space**
- LTD realised by **modifying the customary +i0 prescription** of the Feynman propagators (only the sign matters), it encodes in a compact way the effect of **multiple-cut** contributions that appear in the **Feynman's Tree Theorem**



THE FEYNMAN'S FOREST



IR and threshold singularities are restricted to a **compact region** of the loop three-momentum

Integration along the forward on-shell hyperboloids (light-cones for massless) or positive energy modes

- **LTD:** $G_D(q_i; q_j)\, \tilde{\delta}(q_i) + (i \leftrightarrow j)$
  - **Time-like distance (causally connected):** generates ... al threshold singularities: always **+i0**

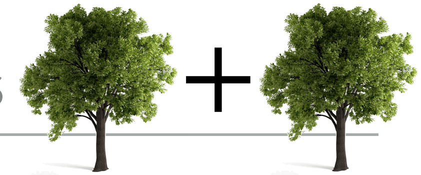
$$\frac{1}{x + i0}$$

  - **Space-like distance**: there is a perfect cancellation of singularities, due to the dual **+i0** prescription

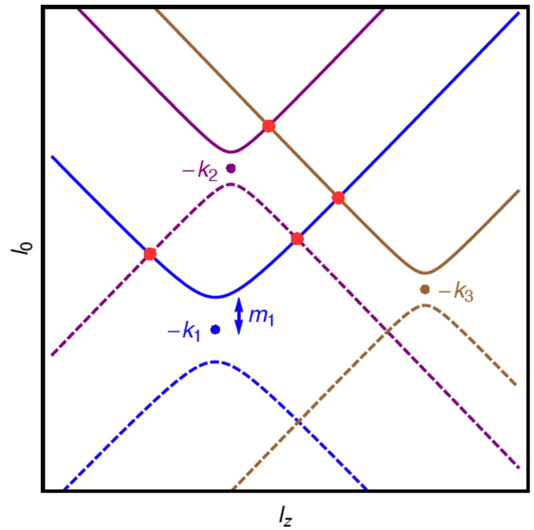
$$\frac{1}{x + i0} + \frac{1}{-x - i0}$$

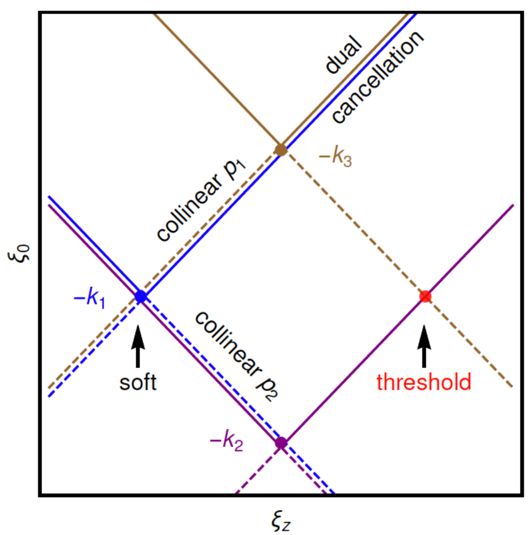
  - **Light-like distance:** both singular configurations, partial cancellation, IR singularities remain

G. Rodrigo, IPPP 2018



THE FEYNMAN'S FOREST





Integration along the forward on-shell hyperboloids (light-cones for massless) or positive energy modes

- **FTT:** $G_F(q_j)\,\tilde{\delta}(q_i) + (i \leftrightarrow j)$
  - **Time-like distance (causally connected):** physics does not depend on the FTT or LTD representation

$$\frac{1}{x+i0}$$

  - **Space-like distance**: there is mismatch in the **+i0** prescription

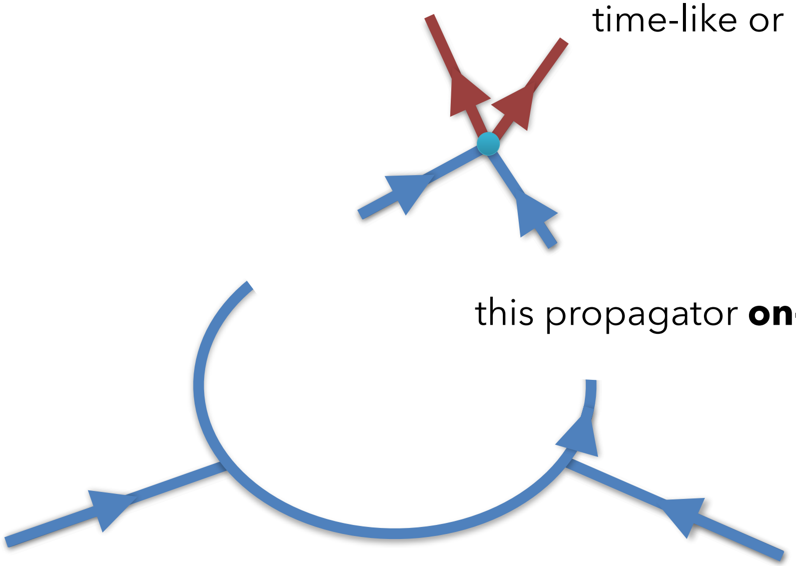
$$\frac{1}{x+i0} + \frac{1}{-x+i0}$$

  - needs to be compensated by **multiple cuts**

SINGULARITIES OF SINGLE CUT TREES

energy of the **on-shell** propagator smaller than the energy of the emitted particles

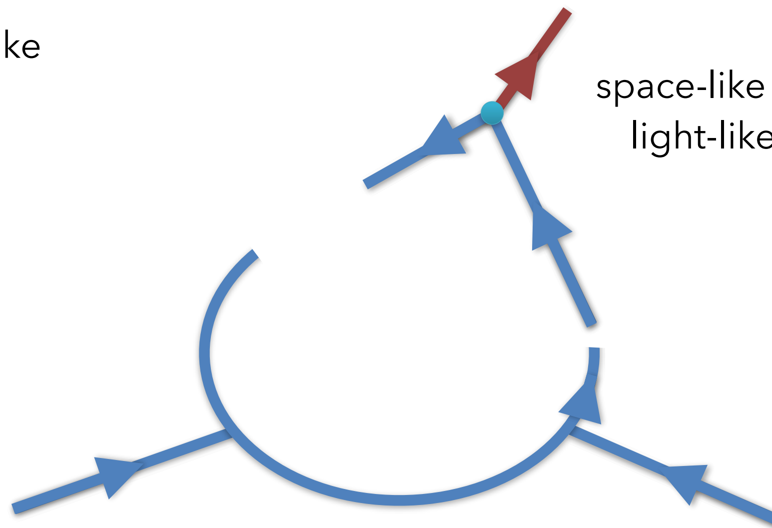
time-like or light-like



this propagator **on-shell**

energy of the **on-shell** propagator larger than the energy of the emitted particle

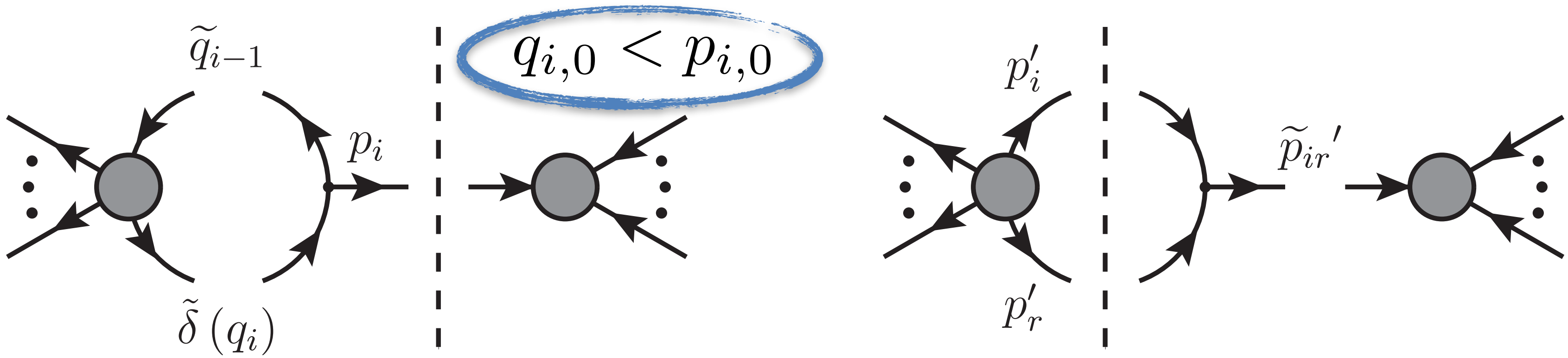
space-like or light-like



- **Threshold** singularities occur when a second propagator gets on-shell: consistent with **Cutkosky**
- It becomes **collinear (soft)** when a single massless particle is emitted

- Virtual particle emitted and absorbed
- Potential **threshold and IR singularities cancel** in the sum of single-cut trees
- Configurations at very large energies expected to be **suppressed.** If not sufficiently suppressed, we **renormalise**
- **The bulk of the physics** is in the **"low" energy** region of the loop momentum

MOMENTUM MAPPING: MULTI-LEG



- Motivated by the **factorisation properties of QCD**: assuming q_i^μ on-shell, and close to collinear with p_i^μ ,we define the momentum mapping

$$p_r'^\mu = q_i^\mu \ ,$$

$$p_i'^\mu = p_i^\mu - q_i^\mu + \alpha_i \, p_j^\mu \ , \qquad \alpha_i = \frac{(q_i - p_i)^2}{2p_j \cdot (q_i - p_i)} \ ,$$

$$p_j'^\mu = (1-\alpha_i) \, p_j^\mu \ , \qquad p_k'^\mu = p_k^\mu \ , \qquad k \neq i, j$$

- All the primed momenta (real process) on-shell and momentum conservation: p_i^μ is the **emitter**, p_j^μ the **spectator** needed to absorb momentum recoil
- **Quasi-collinear configurations** can also be conveniently mapped such that the massless limit is smooth

UV RENORMALISATION: LOCAL SUBTRACTION

- Expand propagators and numerators around a UV propagator [Weinzierl et al.]

$$G_F(q_i) = \frac{1}{q_{\rm UV}^2 - \mu_{\rm UV}^2 + i0}\left[1 - \frac{2q_{\rm UV}\cdot k_i + k_i^2 - m_i^2 + \mu_{\rm UV}^2}{q_{\rm UV}^2 - \mu_{\rm UV}^2 + i0} + \frac{(2q_{\rm UV}\cdot k_i)^2}{(q_{\rm UV}^2 - \mu_{\rm UV}^2 + i0)^2}\right] + \ldots$$

$$q_{\rm UV} = \ell + k_{\rm UV} \qquad k_i = q_i - q_{\rm UV}$$

- and adjust **subleading** terms to subtract only the pole ($\overline{MS}$ **scheme**), or to define any other renormalisation scheme. For the scalar two point function

$$I_{\rm UV}^{\rm cnt} = \int_\ell \frac{1}{(q_{\rm UV}^2 - \mu_{\rm UV}^2 + i0)^2}$$

- dual representation needs to deal with **multiple poles** [Bierenbaum et al.]

$$I_{\rm UV}^{\rm cnt} = \int_\ell \frac{\tilde{\delta}(q_{\rm UV})}{2\left(q_{\rm UV,0}^{(+)}\right)^2}$$

$$q_{\rm UV,0}^{(+)} = \sqrt{\mathbf{q}_{\rm UV}^2 + \mu_{\rm UV}^2 - i0}$$

Hernández-Pinto, Sborlini, GR, JHEP **1602,** 044

- Integration on the UV on-shell hyperboloid: loop three-momentum unconstrained, but **loop contributions suppressed** for loop energies larger than $\mu_{\rm UV}$

LTD / FDU: MULTI-LEG @ NLO

- The **dual representation** of the renormalised loop cross-section: one single integral in the loop three-momentum

$$\int_N d\sigma_{\rm V}^{(1,{\rm R})} = \int_N \int_{\vec{\ell}_1} 2\,{\rm Re}\,\langle \mathcal{M}_N^{(0)}|\left(\sum_i \mathcal{M}_N^{(1)}(\tilde{\delta}(q_i))\right) - \mathcal{M}_{\rm UV}^{(1)}(\tilde{\delta}(q_{\rm UV}))\rangle$$

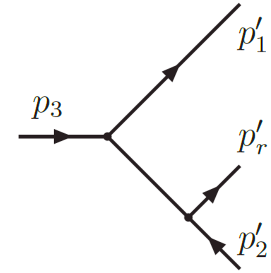
- A **partition** of the real phase-space

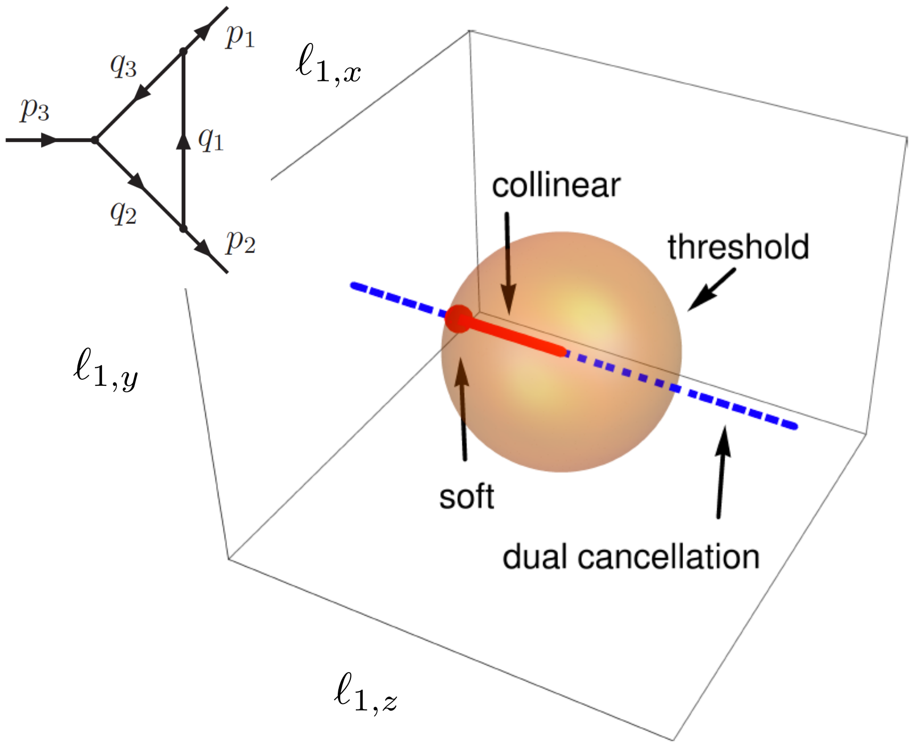
$$\sum_i \mathcal{R}_i(\{p'_j\}_{N+1}) = 1$$

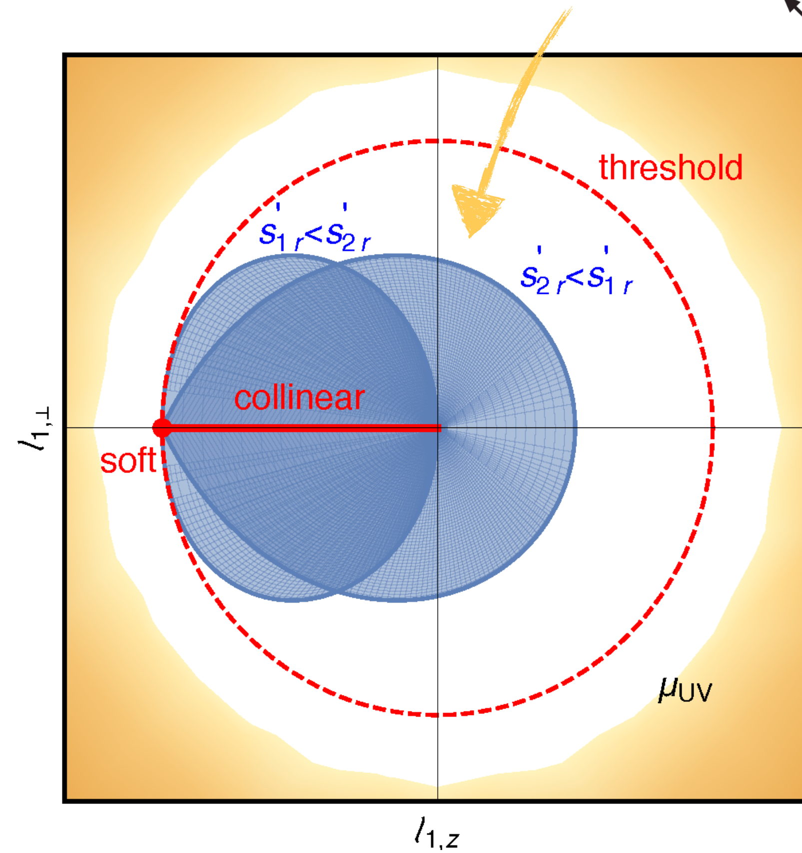
- The real contribution **mapped** to the Born kinematics + loop three-momentum

$$\int_{N+1} d\sigma_{\rm R}^{(1)} = \int_N \int_{\vec{\ell}_1} \sum_i \mathcal{J}_i(q_i)\,\mathcal{R}_i(\{p'_j\})\,|\mathcal{M}_{N+1}^{(0)}(\{p'_j\})|^2\Big|_{\{p'_j\}_{N+1}\to(q_i,\{p_k\}_N)}$$

LTD / FDU IN THE LOOP THREE-MOMENTUM SPACE







- The real contribution **mapped** to the **Born kinematics + loop three-momentum** (mappings inspired by the factorization properties of QCD)
- **UV** subtracted locally
- integrand cancellation of IR singularities: works in ***d*=4** space-time dimensions **without subtractions**

LTD / FDU @ NNLO

- At **NNLO**

$$\sigma^{\rm NNLO} = \int_N d\sigma_{\rm VV}^{(2,{\rm R})} + \int_{N+1} d\sigma_{\rm VR}^{(2,{\rm R})} + \int_{N+2} d\sigma_{\rm RR}^{(2)}$$

where the **VV** contribution reads

$$d\sigma_{\rm VV}^{(2)} = \int_{\vec\ell_1} \int_{\vec\ell_2} \sum_{i,j} \Big[2\,{\rm Re}\, \langle \mathcal{M}_N^{(0)} | \mathcal{M}_N^{(2)}(\tilde\delta(q_i,q_j)) \rangle + \langle \mathcal{M}_N^{(1)}(\tilde\delta(q_i)) | \mathcal{M}_N^{(1)}(\tilde\delta(q_j)) \rangle \Big] \, \mathcal{O}(\{p_k\})$$

- Need the **VR** and **RR** contributions **mapped** to the Born kinematics + the two independent loop three-momenta

- **Known two-loop amplitudes** not suitable: requires LTD unintegrated representation

FTT AT TWO-LOOPS (AND BEYOND)

If there is more than one loop in the original diagram, the loops may be opened in succession. Choose any one loop; that is, integration over any one virtual momentum k, leaving the others to integrate later. Then this loop can be opened. What results is a diagram sum and integral over diagrams with extra particles, but which still has loops remaining in it. However, there is now one less loop, and in each remaining loop all the propagators are I_+ (if equation 10 is used). Therefore, a remaining loop may be treated in the same way, thus reducing the number of loops still further, until there are none left.

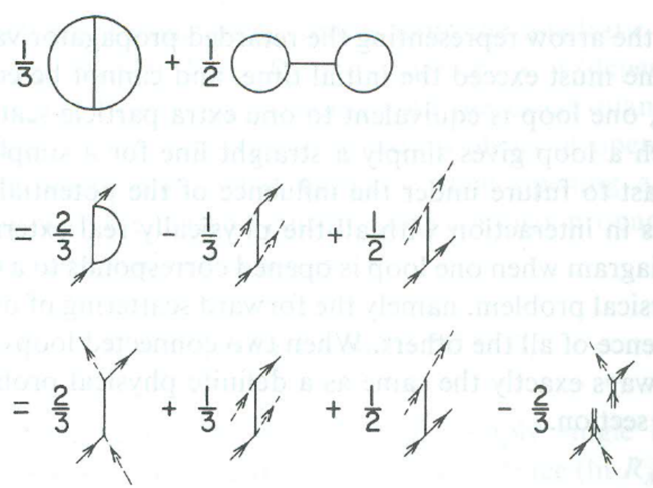


FIGURE 8.
Reduction of the Figure 6 diagrams to trees.

LTD AT TWO-LOOPS (AND BEYOND)

$$G_F(\alpha_k) = \prod_{i\in\alpha_k} G_F(q_i) \qquad G_D(\alpha_k) = \sum_{i\in\alpha_k} \tilde{\delta}(q_i) \prod_{j\neq i} G_D(q_i; q_j)$$

- At one loop:

$$\int_{\ell_1} G_F(\alpha_1) = -\int_{\ell_1} G_D(\alpha_1)$$

LTD AT TWO-LOOPS (AND BEYOND)

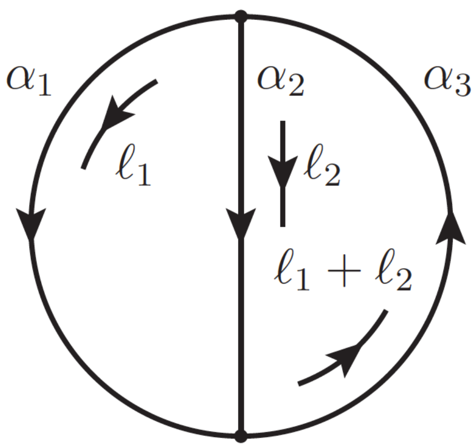
$$G_F(\alpha_k) = \prod_{i\in\alpha_k} G_F(q_i) \qquad G_D(\alpha_k) = \sum_{i\in\alpha_k} \tilde{\delta}(q_i) \prod_{j\neq i} G_D(q_i;q_j)$$

- At one loop:

$$\int_{\ell_1} G_F(\alpha_1) = -\int_{\ell_1} G_D(\alpha_1)$$

- At two-loops:

$$\int_{\ell_1}\int_{\ell_2} G_F(\alpha_1\cup\alpha_2\cup\alpha_3)$$
$$= -\int_{\ell_1}\int_{\ell_2} G_F(\alpha_1)\, G_D(\alpha_2\cup\alpha_3)$$



$$G_D(\alpha_2)\, G_D(\alpha_3) + G_D(\alpha_2)\, G_F(\alpha_3) + G_F(\alpha_2)\, G_D(\alpha_3)$$

two cuts ✔

$$-G_D(\alpha_1\cup\alpha_3)$$

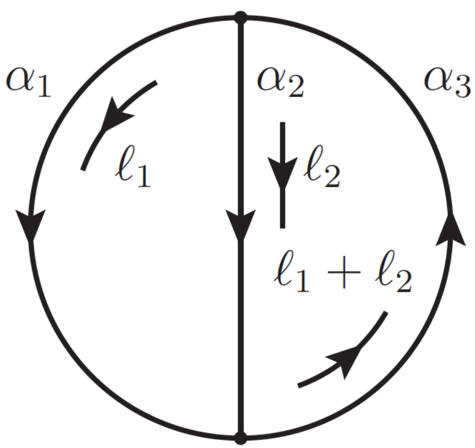
$$-G_D(-\alpha_2\cup\alpha_1)$$

rearrangement of imaginary prescriptions, similar to relation with Advanced propagators

LTD AT TWO-LOOPS (AND BEYOND)

- At two-loops:

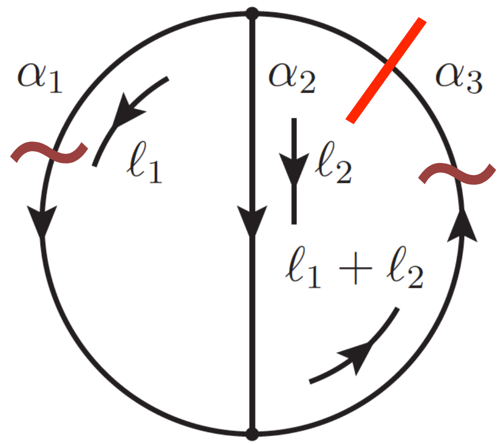
$$\int_{\ell_1}\int_{\ell_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3) = \int_{\ell_1}\int_{\ell_2} \Big\{ G_D(\alpha_2)\, G_D(\alpha_1 \cup \alpha_3)$$
$$G_F(-\alpha_2 \cup \alpha_1)\, G_D(\alpha_3) - G_F(\alpha_1)\, G_D(\alpha_2)\, G_D(\alpha_3) \Big\}$$



With a **number of cuts equal to the number of loops** the loop amplitude opens to a **non-disjoint**  level like object

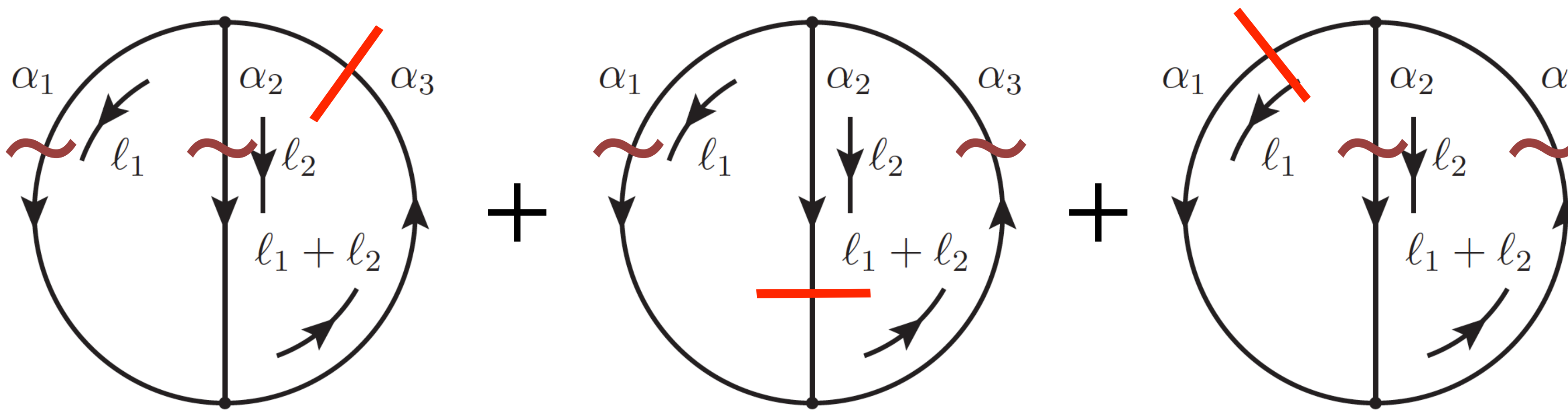
THE TWO-LOOP FOREST

- One propagator gets on-shell in the same line where there is a cut propagator: equivalent to the **one-loop** case



THE TWO-LOOP FOREST

- The genuine two-loop case occurs when the singularity is generated in another loop line

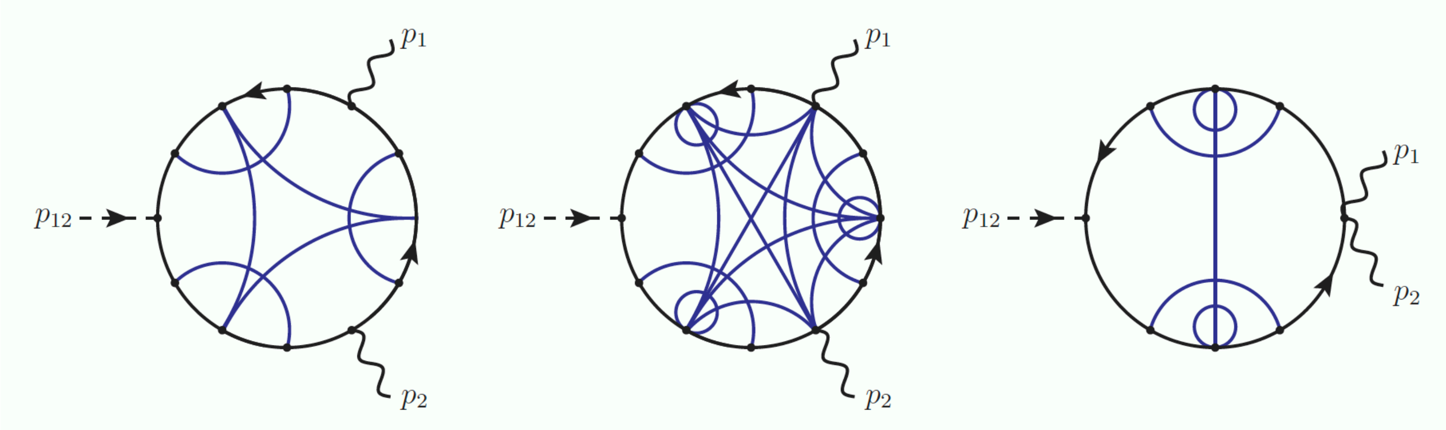


- There are **four singular configurations**: two of them cancel in the sum

[Driencourt-Mangin, Sborlini, Torres, GR, in preparation]

Driencourt-Mangin, Sborlini, Torres-Bobadilla, GR

DUAL AMPLITUDE FOR $H \to \gamma\gamma$ AT TWO-LOOPS



- Simplest two loop amplitude: **proof of concept** for other amplitudes with higher multiplicities
- Well known numerically/analytically, however, **known amplitude not suitable** within LTD/FDU, requires unintegrated amplitude
- **IBP** would modify the **local behaviour** of the integrand: not suitable
- Dual propagators are **linear in the loop momenta:** tensor reduction simpler (reduction to master integrals not necessary)
- **Universality** also holds at two-loops ?

DUAL AMPLITUDE FOR $H \to \gamma\gamma$ AT ONE-LOOP

- **Universality** and **compactness** of the dual representation. In four space-time dimensions after **local renormalization**

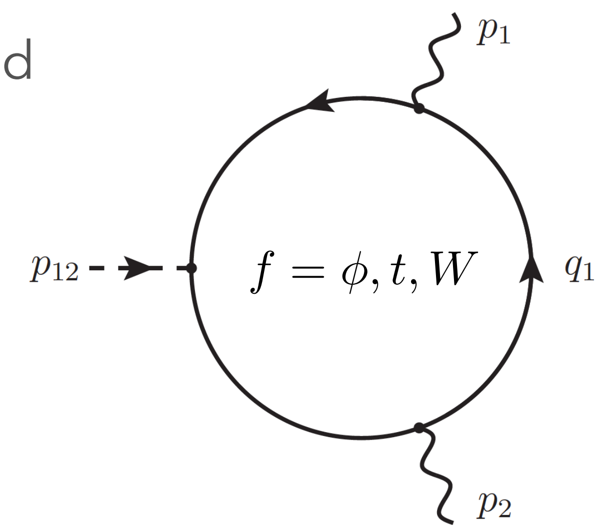
$$\mathcal{A}^{(1,f)}_{1,\mathrm{R}}\Big|_{d=4} = g_f s_{12} \int_\ell \Bigg[\frac{1}{2\ell_0^{(+)}} \left(\frac{\ell_0^{(+)}}{q_{1,0}^{(+)}} + \frac{\ell_0^{(+)}}{q_{2,0}^{(+)}} + \frac{2(2\ell\cdot p_{12})^2}{s_{12}^2 - (2\ell\cdot p_{12} - \imath 0)^2} \right) \times \frac{M_f^2}{(2\ell\cdot p_1)(2\ell\cdot p_2)} c_1^{(f)} + \frac{3\mu_{\mathrm{UV}}^2}{4(q_{\mathrm{UV},0}^{(+)})^5} \hat{c}_{23}^{(f)} \Bigg]$$

$$q_{i,0}^{(+)} = \sqrt{\mathbf{q}_i^2 + m_i^2 - i0}$$

- The flavour of the internal particles is encoded by **two scalar coefficients**

$$c_1^{(f)} = \left(2, -4 + \frac{s_{12}}{M_t^2}, 6 - \frac{3 s_{12}}{M_W^2}\right)$$

$$\hat{c}_{23}^{f} = \frac{c_{23}^{(f)}}{d-4}\Big|_{d=4} = \left(1, -2, 3 + \frac{s_{12}}{2M_W^2}\right)$$



FDM, Sborlini, Torres, Rodrigo, in preparation

UNIVERSALITY OF THE DUAL AMPLITUDES

- The 22 dual double cuts can be written with 9 generators, for instance

$$\mathcal{A}_1^{(2,f)}(q_i,q_4) = g_f^{(2)} \int_{\ell_1}\int_{\ell_2} \tilde{\delta}(q_i,q_4) \Bigg\{ -\frac{r_f\, c_1^{(f)}}{D_3\, D_{12}} \Big(G(D_{\bar{i}},\kappa_i,c_{4,u}^{(f)})(1+H(D_3\,D_{12},\kappa_i)) + F(D_{\bar{i}},\kappa_4/\kappa_i) \Big)$$
$$+ \left(c_7^{(f)} \left(\frac{1}{D_{\bar{i}}} - \frac{1}{D_{\bar{3}}} \left(1 - \frac{D_3}{D_{12}} \left(1 - \frac{D_{\overline{12}}}{D_{\bar{i}}} \right)\right)\right)\right) + \frac{1}{D_3}\left(c_8^{(f)} \left(\frac{1}{D_{\bar{3}}} - \frac{1}{D_{\bar{i}}} \right) - \frac{1}{D_{\overline{12}}} \left(c_9^{(f)} - c_{10}^{(f)} \frac{D_{\bar{3}}}{D_{\bar{i}}} \right)\right)$$
$$+ 2\, r_f \Bigg[\frac{1}{D_3\, D_{12}} \left(c_1^{(f)} \left(\frac{1}{D_3\, D_{\bar{3}}} + \frac{1}{D_{\bar{i}}} \left(\frac{1}{D_{\bar{3}}} - \frac{1}{D_3} \right)\right)\right) + \frac{c_{14}^{(f)}}{D_{\bar{3}}} + \frac{c_{20}^{(f)}}{D_{\bar{i}}} - c_{16}^{(f)}$$
$$+ c_{17}^{(f)} \left(\frac{D_{\bar{i}} - D_{\overline{12}}}{D_{\bar{3}}} + \frac{D_{\bar{3}}}{D_{\bar{i}}} \right)\Bigg) - \frac{1}{D_{\bar{i}}\, D_{\bar{3}}} \left(\frac{c_7^{(f)}}{D_{12}} + c_{18}^{(f)} \right) \Bigg] + \{3 \leftrightarrow 12\} \Bigg) \Bigg\}$$

- The $c_i^{(f)}$ are scalar coefficients and depend only on the reduced mass $r_f = \dfrac{s_{12}}{M_f^2}$ and the dimension d, while the D_i are normalized propagators

CONCLUSIONS

- Already in 1963-1972 Feynman developed the idea of **opening loops to trees** (motivated by the difficulties to describe gravity), nowadays called Generalised Unitarity.
- Low number of citations !!!, even if he outlined already the two-loop case.
- The **forest is less singular than the individual trees**: more suitable to predict physical observables: essential feature for LTD/FDU. Potential advantages also for amplitudes.
- First attempt towards **LTD/FDU at NNLO**
- All arises from the tiny

LTD/FDU REFERENCES

- S. Catani, T. Gleisberg, F. Krauss, G. Rodrigo and J. C. Winter, "*From loops to trees by-passing Feynman's theorem*," JHEP **0809** (2008) 065 [arXiv:0804.3170 [hep-ph]].
- I. Bierenbaum, S. Catani, P. Draggiotis and G. Rodrigo, "*A Tree-Loop Duality Relation at Two Loops and Beyond*," JHEP **1010** (2010) 073 [arXiv:1007.0194 [hep-ph]].
- I. Bierenbaum, S. Buchta, P. Draggiotis, I. Malamos and G. Rodrigo, "*Tree-Loop Duality Relation beyond simple poles*," JHEP **1303** (2013) 025 [arXiv:1211.5048 [hep-ph]].
- S. Buchta, G. Chachamis, P. Draggiotis, I. Malamos and G. Rodrigo, "*On the singular behaviour of scattering amplitudes in quantum field theory*," JHEP **1411** (2014) 014 [arXiv:1405.7850 [hep-ph]].
- S. Buchta, "*Theoretical foundations and applications of the Loop-Tree Duality in Quantum Field Theories*," **PhD thesis**, Universitat de València, 2015, arXiv:1509.07167 [hep-ph].
- S. Buchta, G. Chachamis, P. Draggiotis and G. Rodrigo, "*Numerical implementation of the Loop-Tree Duality method*," EPJC **77** (2017) 274 [arXiv:**1510**.00187 [hep-ph]].
- R. J. Hernández-Pinto, G. F. R. Sborlini and G. Rodrigo, "***Towards gauge theories in four dimensions,***" JHEP **1602** (2016) 044 [arXiv:**1506.04617** [hep-ph]].
- G. F. R. Sborlini, F. Driencourt-Mangin, R.J. Hernández-Pinto and G. Rodrigo, "***Four dimensional unsubtraction from the loop-tree duality,***"JHEP **1608** (2016) 160 [arXiv:**1604.06699** [hep-ph]].
- G. F. R. Sborlini, F. Driencourt-Mangin and G. Rodrigo, "***Four dimensional unsubtraction with massive particles,***" JHEP **1610** (2016) 162 [arXiv:**1608.01584** [hep-ph]].
- F. Driencourt-Mangin, G. Rodrigo and G.F.R. Sborlini, "***Universal dual amplitudes and asymptotic expansions for gg->H and H->gamma gamma,***" EPJC **78** (2008) 231 [arXiv:**1702.07581** [hep-ph]].

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