Bifurcations in the RG-Flow of QCD

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Renormalization group: Overview

- Renormalization group provides information about the change of physical system, when energy scales of a physical process change.
- General Method:
  1. Introduce new parameter in the QFT.
  2. Redefine theory in terms of the new parameter such that it is finite.
  3. Parameter introduces an energy scale.
  4. Impose that the physics (moments) is independent of the new energy scale.
- Objects of interest:
  1. Beta Functions $\beta(g(\mu), \mu)$.
  2. Anomalous dimensions $\gamma(g(\mu), \mu)$.

Renormalization Group: 2 different methods

Perturbative Renormalization Group

- Bare Action
- Renormalization Group or Callan-Symanzik equation
- Energy scale is usually a mass scale μ
- Perturbative beta functions

Exact/Functional Renormalization Group

- Truncated Effective Action
- Wilson, Wetterich or Polchinski equation
- Energy scale is usually a cut-off momentum scale Λ
- Exact beta functions

Bifurcation theory

A **continuous time dynamical system** can be generated by a set of ODE's.

Bifurcations: Changes in the topology of the flow generated by the dynamical system.

Bifurcations involve one or multiple limit sets of the dynamical system.

Typical limit sets:

- equilibria (continuous time) and fixed points (discrete time);
- limit cycles;
- chaotic attractors.

Bifurcations can be categorized by their codimension, (non-)degeneracy conditions and normal forms.

Bifurcations: Saddle-Node bifurcation

Codim-1 bifurcation.
Normal Form: $\dot{y} = \beta \pm y^2, \quad y, \beta \in \mathbb{R}$

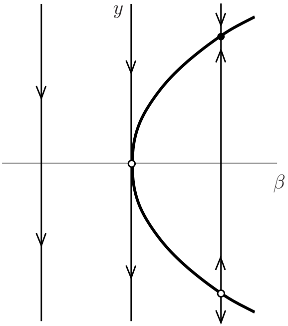

Figure: Saddle-Node bifurcation in the system $\dot{y} = \beta + y^2$

Codim-1 bifurcation in system with trivial equilibrium.
Normal Form $\dot{y} = \beta y \pm y^2 \quad y, \beta \in \mathbb{R}$

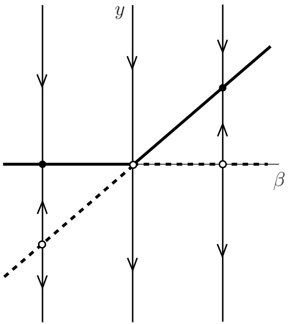

Figure: Transcritical bifurcation in the system $\dot{y} = \beta y - y^2$.

Why study Bifurcations in RG-Flows

- Recently proposed (Gukov 2017 [1]).
- Beta functions naturally generate a dynamical system.
- Fixed points (equilibria) and finite renormalized trajectories (heteroclinic orbits) are of particular interest.
- Easy to find and categorize transitions of the RG-Flow.
- Ideal to find exotic behaviour (e.g. limit cycles, homoclinic orbits, chaos), if it exists.
- Numerical tools (e.g. Matcont [2]) have been developed to study complicated systems of ODE's.

Bifurcations in RG-Flows

Three typical transitions in RG-Flows (Kaplan et al. 2009 [3]):

- Fixed point merger (saddle-node bifurcation)
  - Square root scaling of anomalous dimensions
  - Exponential (BKT/Miransky) scaling
  - Walking coupling constants close to the bifurcation
- Transcritical bifurcation
  - Linear scaling of anomalous dimensions
- Divergence of fixed points

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$$\mathcal{L} = -\frac{1}{4} G^A_{\mu\nu} G_A^{\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}_{ai} \left(i \not{D}^a_b - m_i \delta^a_b \right) \psi^{bi},$$

where

$$G^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + g f^A_{BC} [A^B_\mu, A^C_\nu]$$
$$D_\mu = \partial_\mu - i g t_A A^A_\mu .$$

$L := \psi_L := \frac{1-\gamma^5}{2}\psi$ and $R := \psi_R := \frac{1+\gamma^5}{2}\psi$
Locally invariant under $SU(N_c)$

Set $m_i = 0$
Globally invariant under $SU(N_f)_L \times SU(N_f)_R$; can be broken to $SU(N_f)_V$

QCD_4: Perturbative Beta Function

$$\begin{aligned} \beta_{\alpha_g} &= \gamma\alpha_g - b_1\alpha_g^2 - b_2\alpha_g^3 + ... \\ \gamma &= 0 \\ b_1 &= \frac{2}{3}N(11-2x) \\ b_2 &= \frac{2}{3}N^2\left(34 - 13x + 3\frac{x}{N^2}\right), \end{aligned}$$

where $\alpha_g := \frac{g^2}{(4\pi)^2}$, $x := \frac{N_f}{N_c}$, $N := N_c$.

Fixed points:

$$\begin{aligned} \alpha_g &= 0 \\ \alpha_g &= \alpha_g^{nt} = N\frac{11-2x}{13N^2x - 34N^2 - 3x} \end{aligned}$$

QCD_4: Conformal Window

Conformal Window is region where non-trivial fixed points exists.

- $x = 5.5 \Rightarrow \alpha_g^{nt} = 0$
- $x = \frac{34N^2}{13N^2-3} \approx 2.7(N=3) \Rightarrow \alpha_g^{nt} \to \infty$
  $\Rightarrow$ Perturbation theory is not applicable $\Rightarrow x_{\mathrm{crit}} =??$

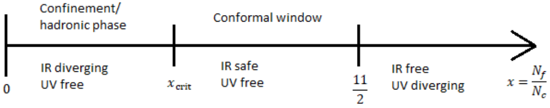

Figure: Phase diagram of QCD_4 at $T = 0$, $\mu = 0$

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Effective Lagrangian

$$\Gamma_\Lambda = \int d^4x \left(-\frac{1}{4g^2} G^A_{\mu\nu} G^{A\mu\nu} + \bar{\psi}_{ai} i \not{D} \psi^{ai} + \mathcal{L}_{4f} \right)$$

$$\mathcal{L}_{4f} = \frac{G_S}{\Lambda^{2(1+\eta)}} \mathcal{O}_S + \frac{G_V}{\Lambda^{2(1+\eta)}} \mathcal{O}_V + \frac{G_{V_1}}{\Lambda^{2(1+\eta)}} \mathcal{O}_{V_1} + \frac{G_{V_2}}{\Lambda^{2(1+\eta)}} \mathcal{O}_{V_2}$$

$$\mathcal{O}_S = 2\bar{L}_i R^j \bar{R}_j L^i = \frac{1}{2}[\bar{\psi}_i \psi^j \bar{\psi}_j \psi^i - \bar{\psi}_i \gamma_5 \psi^j \bar{\psi}_j \gamma_5 \psi^i]$$

$$\mathcal{O}_V = \bar{L}_i \gamma^\mu L^j \bar{L}_j \gamma_\mu L^i + (L \leftrightarrow R) = \frac{1}{2}[\bar{\psi}_i \gamma^\mu \psi^j \bar{\psi}_j \gamma_\mu \psi^i + \bar{\psi}_i \gamma^\mu \gamma_5 \psi^j \bar{\psi}_j \gamma_\mu \gamma_5 \psi^i]$$

$$\mathcal{O}_{V_1} = 2\bar{L}_i \gamma^\mu L^i \bar{R}_j \gamma_\mu R^j = \frac{1}{2}[(\bar{\psi}_i \gamma^\mu \psi^i)^2 - (\bar{\psi}_i \gamma^\mu \gamma_5 \psi^i)^2]$$

$$\mathcal{O}_{V_2} = (\bar{L}_i \gamma^\mu L^i)^2 + (L \leftrightarrow R) = \frac{1}{2}[(\bar{\psi}_i \gamma^\mu \psi^i)^2 + (\bar{\psi}_i \gamma^\mu \gamma_5 \psi^i)^2]$$

Model based on (Kusafuka, Terao 2011 [4]).

Rescaling of parameters

$$\alpha_g := \frac{g^2}{(4\pi)^2} \quad g_i := \frac{G_i}{4\pi^2}$$

$$x := \frac{N_f}{N_c} \quad N := N_c$$

$$t = -\ln(\Lambda/\Lambda_0)$$

$$N\alpha_g \to \alpha_g$$

$$Ng_S \to g_S$$

$$Ng_V \to g_V$$

$$N^2 g_{V_1} \to g_{V_1}$$

$$N^2 g_{V_2} \to g_{V_2},$$

Effective model: Beta functions

$$
\begin{cases}
\frac{d\alpha_g}{dt} = & \frac{2}{3}(11-2x)\alpha_g^2 + \frac{2}{3}(34-13x)\alpha_g^3 - 2xg_V\alpha_g^2 \\
& +N^{-2}\left(2x\alpha_g^3\right), \\
\frac{dg_S}{dt} = & -2g_S + 2g_S^2 - 2xg_Sg_V + 6g_S\alpha_g + \frac{9}{2}\alpha_g^2 \\
& +N^{-2}\left(-6g_Sg_{V_1} - 2g_Sg_{V_2} - 6g_S\alpha_g - 12g_{V_1}\alpha_g - 12\alpha_g^2\right), \\
\frac{dg_V}{dt} = & -2g_V - \frac{1}{4}xg_S^2 - (1+x)g_V^2 + \frac{3}{4}\alpha_g^2 \\
& +N^{-2}\left(6g_Vg_{V_2} + 6g_V\alpha_g - 6g_{V_2}\alpha_g - 6\alpha_g^2\right), \\
\frac{dg_{V_1}}{dt} = & -2g_{V_1} + \frac{1}{4}g_s^2 + g_Sg_V + xg_Sg_{V_2} - 2(1+x)g_Vg_{V_1} - 2xg_{V_1}g_{V_2} \\
& -\frac{3}{4}\alpha_g^2 + N^{-2}\left(3g_{V_1}^2 - 2g_{V_1}g_{V_2} - 6g_{V_1}\alpha_g - 3\alpha_g^2\right), \\
\frac{dg_{V_2}}{dt} = & -2g_{V_2} + 3g_V^2 + xg_{V_1}^2 - xg_{V_2}^2 + xg_Sg_{V_1} - 2(1+x)g_Vg_{V_2} \\
& -6g_V\alpha_g + \frac{9}{4}\alpha_g^2 + N^{-2}\left(2g_{V_2}^2 + 6g_{V_2}\alpha_g + 3\alpha_g^2\right).
\end{cases}
$$

Veneziano limit

$$N \to \infty$$

$$\begin{cases} \dot{\alpha_g} &= \frac{2}{3}(11-2x)\alpha_g^2 + \frac{2}{3}(34-13x)\alpha_g^3 - 2x g_V \alpha_g^2, \\ \dot{g_S} &= -2g_S + 2g_S^2 - 2x g_S g_V + 6 g_S \alpha_g + \frac{9}{2}\alpha_g^2, \\ \dot{g_V} &= -2g_V - \frac{1}{4}x g_S^2 - (1+x) g_V^2 + \frac{3}{4}\alpha_g^2. \end{cases}$$

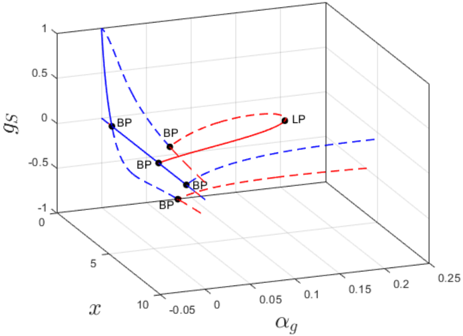

Figure: Fixed points of the RG-flow.

- Saddle-Node bifurcation at $x \approx 4.05$
- Operator crossing marginality $\mathcal{O} = 0.09\mathcal{O}_{\alpha_g} + 0.98\mathcal{O}_S - 0.15\mathcal{O}_V$
- Two points diverge at $x \approx 2.5$

Veneziano Limit: RG-Flow

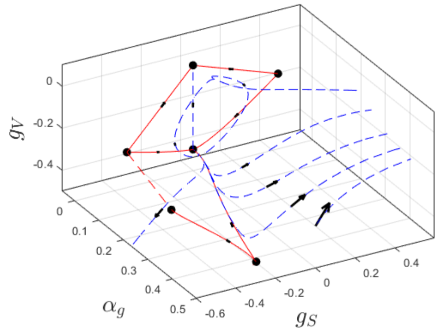

Figure: RG-flow in the Veneziano limit at $x = 4$.

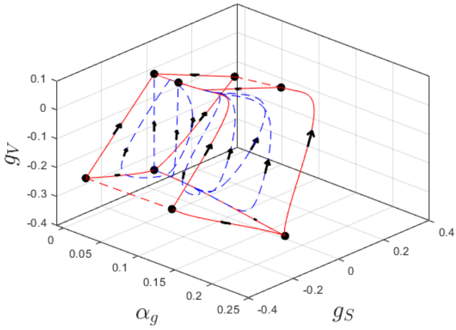

Figure: RG-flow in the Veneziano limit at $x = 5$.

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Effective Model: Complete Model

$$
\begin{cases}
\frac{d\alpha_g}{dt} = & \frac{2}{3}(11-2x)\alpha_g^2 + \frac{2}{3}(34-13x)\alpha_g^3 - 2x g_V \alpha_g^2 \\
& + N^{-2}\left(2x\alpha_g^3\right), \\
\frac{dg_S}{dt} = & -2g_S + 2g_S^2 - 2x g_S g_V + 6 g_S \alpha_g + \frac{9}{2}\alpha_g^2 \\
& + N^{-2}\left(-6 g_S g_{V_1} - 2 g_S g_{V_2} - 6 g_S \alpha_g - 12 g_{V_1}\alpha_g - 12\alpha_g^2\right), \\
\frac{dg_V}{dt} = & -2g_V - \frac{1}{4} x g_S^2 - (1+x) g_V^2 + \frac{3}{4}\alpha_g^2 \\
& + N^{-2}\left(6 g_V g_{V_2} + 6 g_V \alpha_g - 6 g_{V_2}\alpha_g - 6\alpha_g^2\right), \\
\frac{dg_{V_1}}{dt} = & -2g_{V_1} + \frac{1}{4} g_s^2 + g_S g_V + x g_S g_{V_2} - 2(1+x) g_V g_{V_1} - 2x g_{V_1} g_{V_2} \\
& -\frac{3}{4}\alpha_g^2 + N^{-2}\left(3 g_{V_1}^2 - 2 g_{V_1} g_{V_2} - 6 g_{V_1}\alpha_g - 3\alpha_g^2\right), \\
\frac{dg_{V_2}}{dt} = & -2g_{V_2} + 3 g_V^2 + x g_{V_1}^2 - x g_{V_2}^2 + x g_S g_{V_1} - 2(1+x) g_V g_{V_2} \\
& -6 g_V \alpha_g + \frac{9}{4}\alpha_g^2 + N^{-2}\left(2 g_{V_2}^2 + 6 g_{V_2}\alpha_g + 3\alpha_g^2\right).
\end{cases}
$$

Effective Model: Gauge coupling beta function

$$\frac{d\alpha_g}{dt} = \frac{2}{3}(11 - 2x)\alpha_g^2 + \frac{2}{3}(34 - 13x)\alpha_g^3 - 2xg_V\alpha_g^2 + N^{-2}\left(2x\alpha_g^3\right)$$

$$\alpha_g = 0$$

$$\alpha_g = \alpha_g^{nt} := N^2 \frac{11 - 2x - 3xg_V}{13N^2x - 34N^2 - 3x}$$

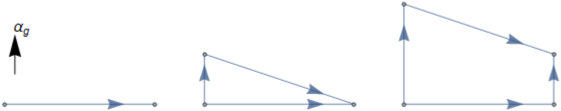


Figure: Topology of the RG-flow between 4 related fixed points. Left (Large N): the 2 non-trivial fixed points have $\alpha_g < 0$. Middle (intemediate N): The least stable non-trivial fixed points has $\alpha_g > 0$. Right (small N): Both of the non-trivial fixed points have $\alpha_g > 0$.

Flow in $\alpha_g = 0$ plane

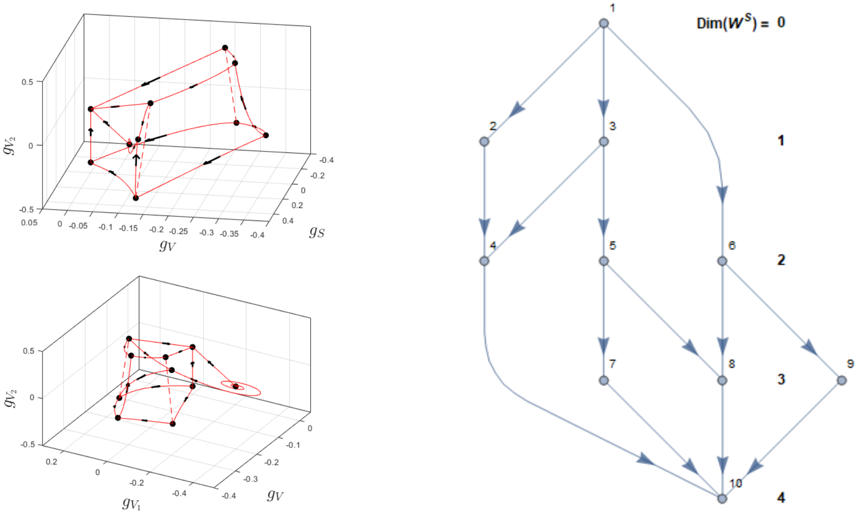
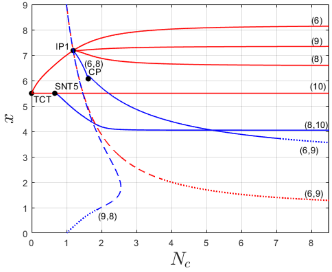
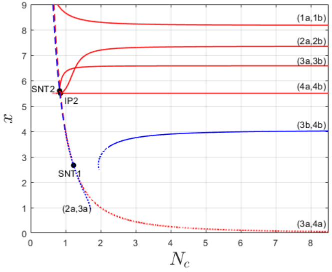
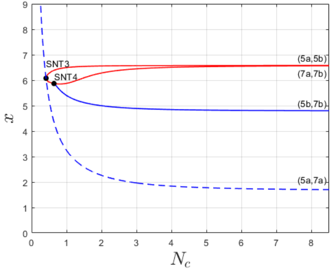


Figure: RG-flow on the invariant set $\alpha_g = 0$ at $(N_c, N_f) = (3, 15)$.

Complete Model: Bifurcation Diagrams







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Conclusions & Outlook

- Used numerical bifurcation analysis to study an effective QCD model.
- Found a rich structure of fixed points in the effective QCD model.
- Non-trivial fixed points are generated by transcritical bifurcations.
- Most non-trivial fixed points disappear through a saddle-node bifurcation.
- A few fixed points still disappear through divergence.
- Conformal window closes through saddle-node bifurcation at $x(N) \approx 4$.
- Multiple degenerate bifurcation points are found.
- Effective operators can have complex scaling dimensions and hence spiraling orbits.
- Hopf bifurcations and limit cycles exist just outside the physical regime of the model.

Outlook

- Extension of the model.
- Study new fixed points and exotic flows in more detail and connect to lattice or holographic studies.
- Apply numerical bifurcation analysis to different models with multiple variables and parameters.

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