Renormalization Group Properties of the Conformal Mode of a Torus

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Introduction

This work is based on two papers:

- Renormalization group properties in the conformal sector: towards perturbatively renormalizable quantum gravity, Tim R. Morris - arXiv:1802.04281
- Renormalization group properties of the conformal mode of a torus, Matthew P. Kellett and Tim R. Morris - arXiv:1803.00859

This talk will focus on the more phenomenological aspects of the latter paper; the former gives the relevant background and development.

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- But then the partition function from the Einstein-Hilbert Lagrangian $\mathcal{L}_{\mathsf{EH}} = -\frac{2}{\kappa^2}\sqrt{g}R$ is divergent:

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- We can, however make sense of this. If we write the metric as

$$g_{\mu\nu} = \delta_{\mu\nu}\left(1 + \frac{\kappa}{2}\varphi\right) + \kappa h_{\mu\nu}$$

where $h_{\mu\nu}$ is traceless, then the kinetic terms left are

$$\mathcal{L}_{\text{EH}} = \frac{1}{2}(\partial_\lambda h_{\mu\nu})^2 - \frac{1}{2}(\partial_\lambda \varphi)^2$$

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- Discarding the traceless part and focusing only on the conformal sector, we write the Wilsonian effective action as

$$S^{\text{tot},\Lambda}[\varphi] = S^{\Lambda}[\varphi] - \frac{1}{2}\varphi \cdot (\Delta^{\Lambda})^{-1} \cdot \varphi$$

where $\Delta^{\Lambda} = \frac{C^{\Lambda}(p)}{p^2}$ is the massless propagator regularised by the cutoff profile C^{Λ}.

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  where $\Delta^{\Lambda} = \frac{C^{\Lambda}(p)}{p^2}$ is the massless propagator regularised by the cutoff profile C^{Λ}.
- The interactions then satisfy the Wilson/Polchinski equation

$$\frac{\partial}{\partial\Lambda} S^{\Lambda} = -\frac{1}{2}\frac{\delta S^{\Lambda}}{\delta\varphi} \cdot \frac{\partial\Delta^{\Lambda}}{\partial\Lambda} \cdot \frac{\delta S^{\Lambda}}{\delta\varphi} + \frac{1}{2}\mathrm{tr}\left(\frac{\partial\Delta^{\Lambda}}{\partial\Lambda} \cdot \frac{\delta^2 S^{\Lambda}}{\delta\varphi\delta\varphi}\right)$$

Gaussian fixed point

- Linearising (hence, not AS) around the Gaussian fixed point $S^\Lambda = 0$ gives

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- If we write $S^\Lambda = -\epsilon\int d^4x\, V(\varphi(x),\Lambda)$, then this is solved provided that $\partial_t V = -\Omega_\Lambda \partial_\varphi^2 V$.
- Here, $t = \ln(\mu/\Lambda)$ and
$$\Omega_\Lambda = |\langle\varphi(x)\varphi(x)\rangle| = \int\frac{d^4p}{(2\pi)^4}\Delta^\Lambda$$
is the regularised tadpole integral.

New features

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- The equation for V is very much reminiscent of a backwards diffusion equation, which means that the direction of well-posedness is reversed.
- Therefore, the flow will generally break down at some point towards the IR. This will be a crucial effect, and is the reason the wrong-sign kinetic term is important.
- The eigenspectrum is actually degenerate in this case, leading to a continuum of eigenoperators. We recover a discrete set if we impose the following on the bare potential:

$$\int_{-\infty}^{\infty} d\varphi\, V^2(\varphi,\Lambda)\exp\left(\frac{\varphi^2}{2\Omega_\Lambda}\right) < \infty$$

Physical quantities

- The Hilbert space of solutions is spanned by

$$\delta_\Lambda^{(n)}(\varphi) = \frac{1}{\sqrt{2\pi\Omega_\Lambda}} \frac{\partial^n}{\partial\varphi^n} \exp\left(-\frac{\varphi^2}{2\Omega_\Lambda}\right) = \exp\left(\frac{1}{2}\Omega_\Lambda \frac{\partial^2}{\partial\varphi^2}\right) \delta^{(n)}(\varphi)$$

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- This scale marks the point at which the IR evolved potential leaves the Hilbert space.

Evolution on Manifolds

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- However, when the eigenoperators are evolved from $\Lambda = \Lambda_0$ (bare operators) to the IR scale k we find that they become

$$\delta^{(n)}_{k,\Lambda_0}(\varphi) = \exp\left(\frac{1}{2}\Omega_{k,\Lambda_0}(x)\frac{\partial^2}{\partial\varphi^2}\right)\delta^{(n)}(\varphi)$$

where $\Omega_{k,\Lambda_0}(x) = |\langle\varphi(x)\varphi(x)\rangle|_{\mathbb{R}^4} - |\langle\varphi(x)\varphi(x)\rangle|_{\mathcal{M}}$.

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- To get the physical Ω we need to remove the regulators

$$\Omega_p(x) = \lim_{\substack{\Lambda_0\to\infty \\ k\to 0}} \Omega_{k,\Lambda_0}(x)$$

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  where L is some characteristic scale and $\mathcal{S}(x)$ is some dimensionless shape function.
- This modifies the behaviour of the physical potential at large φ:

$$V_p(\varphi(x), x) \sim \exp\left(-\frac{\varphi^2(x)}{\Lambda_p^2 + 2\Omega_p(x)}\right)$$

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- This effect on the physical potential means that the flow exists to $k \to 0$, and thus the QFT exists, if and only if

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- Thus, if $\mathcal{S}_{\mathrm{min}}$ is negative, we have a minimum size of the manifold given by

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- $\mathcal{S}$ decreases the more inhomogeneous a manifold gets, and this would have significant application in cosmology.

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- We use a four-torus as a toy model, then look at $\mathbb{T}^3 \times \mathbb{R}$, with the three-torus recognised as the spatial submanifold.
- We then looked at twisted versions of these tori.

Four-torus

- In the case of the four-torus we find that

$$\mathcal{S}_4(l_\mu) = 2 - s_4(l_\mu) - s_4(1/l_\mu)$$

where

$$s_4(l_\mu) = \int_0^1 \frac{dt}{t^2} \left(\prod_{\mu=1}^{4} \Theta\left(\frac{l_\mu^2}{t}\right) - 1 \right)$$

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- Here, $l_\mu = \frac{L_\mu}{L}$, where $L = V_4^{\frac{1}{4}}$ and L_μ are the lengths of the fundamental loops of the torus. Θ is the third Jacobi theta function, which we take as

$$\Theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}$$

Four-torus: Numerical values

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- For $l_1 = l_2 = 1$ and $l_3 = 3$, $l_4 = \frac{1}{3}$, we have $\mathcal{S}_4 = -7.149$, so we see indeed that $\mathcal{S}_4$ can be negative, and thus the constraint on Ω_p applies.
- In the case of $\mathbb{T}^4$, this constraint is

$$V > \frac{\mathcal{S}_4(l_\mu)^2}{4\pi^2\Lambda_p^4}$$

Cosmological Implications

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- Smaller universes are constrained to be very symmetric, whereas larger universes are permitted to be more anisotropic.
- This could provide a solution for the "Why now?" problem and why the initial conditions for inflation must be so incredibly smooth.

Further Work

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- We want to apply some of this thinking more general manifolds, but we do not yet have the full theory developed so it is not entirely clear how one would do this.
- Equally, it is not entirely clear how many of these effects will survive a full theory of quantum gravity, however the cosmological effects so far studied certainly merit further investigation.

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- Alex Mitchell (who is speaking tomorrow) is pursuing a slightly different route involving BRST invariance.

Thank you for listening!

Any questions?