Supergravity on a 3-Torus:

Quantum Linearisation Instabilities with a Supergroup

Lasse Schmieding

December 19, 2018

Linearisation Instabilities

- Interesting equations in physics are hard to solve. Typically we resort to perturbing around a few known solutions.
- Linearised equations determining first order perturbations around a given background are often easier to solve while still containing interesting physics.

Linearisation Instabilities

- Interesting equations in physics are hard to solve. Typically we resort to perturbing around a few known solutions.
- Linearised equations determining first order perturbations around a given background are often easier to solve while still containing interesting physics.
- *Linearisation instabilities*: Not all solutions to the linearised equations can be extended to solutions of the non-linear equations.

Linearisation Instabilities: An Example

- Let $(x, y) \in \mathbb{R}^2$ and consider the equation

$$x(x^2 + y^2) = 0, \tag{1}$$

this has solutions $(0, y)$, for any y.

Linearisation Instabilities: An Example

- Let $(x, y) \in \mathbb{R}^2$ and consider the equation
$$x(x^2 + y^2) = 0, \tag{1}$$
this has solutions $(0, y)$, for any y.
- Making a perturbation $(x_0 + \delta x, y_0 + \delta y)$ we obtain
$$(3x_0^2 + y_0)\delta x + 2x_0 y_0 \delta y = 0. \tag{2}$$
Given a solution (x_0, y_0), this determines $(\delta x, \delta y)$.

Linearisation Instabilities: An Example

- Let $(x, y) \in \mathbb{R}^2$ and consider the equation

$$x(x^2 + y^2) = 0, \tag{1}$$

this has solutions $(0, y)$, for any y.
- Making a perturbation $(x_0 + \delta x, y_0 + \delta y)$ we obtain

$$(3x_0^2 + y_0)\delta x + 2x_0 y_0 \delta y = 0. \tag{2}$$

Given a solution (x_0, y_0), this determines $(\delta x, \delta y)$.
- If we choose $(0, 0)$ as the background, $(\delta x, 0)$ is a possible solution, but can not be a linearisation of an exact solution.

Linearisation Instabilities: Electrodynamics

- Take a closed universe: Cauchy surface Σ with no boundary. Think: Pac-Man: flat $\mathbb{R} \times \mathbb{T}^n$ or de Sitter $\mathbb{R} \times \mathbb{S}^3$.

Linearisation Instabilities: Electrodynamics

- Take a closed universe: Cauchy surface Σ with no boundary. Think: Pac-Man: flat $\mathbb{R} \times \mathbb{T}^n$ or de Sitter $\mathbb{R} \times \mathbb{S}^3$.
- Equip the universe with scalar electrodynamics. The total charge Q has to vanish as we can express it as a boundary integral

$$Q = -\int_{\Sigma} \star J = \int_{\partial\Sigma} \star F = 0. \tag{3}$$

- This charge is quadratic in the scalar field Φ.

Linearisation Instabilities: Electrodynamics

- Start with an empty universe, start perturbatively filling it:

$$\Phi = \Phi^{(1)} + \Phi^{(2)} + \dots \tag{4}$$

- The total charge must vanish order by order in the perturbation. At second order

$$Q^{(2)} = ie \int_{\Sigma} \star \left(\Phi^{(1)*} \mathrm{d}\Phi^{(1)} - \Phi^{(1)} \mathrm{d}\Phi^{(1)*} \right) = 0. \tag{5}$$

- This imposes a quadratic constraint on $\Phi^{(1)}$, which does not follow from the linearised theory alone.

Linearisation Instabilities: Einstein Gravity

- In Einstein gravity, initial data (g, π) on Σ must obey the Hamiltonian $\mathcal{H} = 0$, and momentum $\mathcal{H}_i = 0$, constraints.
- Smearing against a vector field X allows us to consider

$$\Psi_X = \int_\Sigma \mathrm{d}^3\vec{x}\, (X_\perp \mathcal{H} + X_\parallel^i \mathcal{H}_i). \tag{6}$$

Linearisation Instabilities: Einstein Gravity

- In Einstein gravity, initial data (g, π) on Σ must obey the Hamiltonian $\mathcal{H} = 0$, and momentum $\mathcal{H}_i = 0$, constraints.
- Smearing against a vector field X allows us to consider

$$\Psi_X = \int_\Sigma \mathrm{d}^3\vec{x}\,(X_\perp \mathcal{H} + X_\parallel^i \mathcal{H}_i). \tag{6}$$

- Now perturb around a background $(g^{(0)}, \pi^{(0)})$ as

$$g = g^{(0)} + h^{(1)} + h^{(2)} + \dots, \tag{7}$$

$$\pi = \pi^{(0)} + p^{(1)} + p^{(2)} + \dots. \tag{8}$$

Linearisation Instabilities: Einstein Gravity

- When the background has Killing Vectors X, first order term in Ψ_X vanishes for all perturbations.

Linearisation Instabilities: Einstein Gravity

- When the background has Killing Vectors X, first order term in Ψ_X vanishes for all perturbations.
- Consequently, the second order term in Ψ_X imposes a quadratic constraint on the perturbations $(h^{(1)}, p^{(1)})$.

Linearisation Instabilities: Einstein Gravity

- When the background has Killing Vectors X, first order term in Ψ_X vanishes for all perturbations.
- Consequently, the second order term in Ψ_X imposes a quadratic constraint on the perturbations $(h^{(1)}, p^{(1)})$.
- The imposed constraint is precisely that the conserved generator Q_X of the symmetry vanishes.
- The vanishing of these generators must be imposed as *linearisation stability conditions* on the linear theory

[Brill & Deser, Moncrief, Fisher & Marsden]

Linearisation Instabilities: Supergravity

- Linearisation Instabilities in the presence of a Supergroup.

Linearisation Instabilities: Supergravity

- Linearisation Instabilities in the presence of a Supergroup.
- Expanding around a toroidal background

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2, \quad 0 \leq x, y, z \leq L, \tag{9}$$

- At linear level have a graviton $h_{\mu\nu}$ and a gravitino $\Psi_{\mu\alpha}$

$$S[h, \Psi] = S_{FP}[h] + S_{RS}[\Psi], \tag{10}$$

related by a supersymmetry transformation.

Linearisation Instabilities: Supergravity

- Linearisation Instabilities in the presence of a Supergroup.
- Expanding around a toroidal background

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2, \quad 0 \leq x, y, z \leq L, \tag{9}$$

- At linear level have a graviton $h_{\mu\nu}$ and a gravitino $\Psi_{\mu\alpha}$

$$S[h, \Psi] = S_{FP}[h] + S_{RS}[\Psi], \tag{10}$$

related by a supersymmetry transformation.
- From the non-linear theory get linearisation stability conditions:

$$H = \vec{P} = Q_\alpha = 0. \tag{11}$$

Canonical Quantisation with Constraints

- Now quantise the system subject to the linearisation stability constraints.
- In Dirac quantisation, the constraints Q imposed as conditions on physical states:

$$Q\,|\text{phys}\rangle = 0. \tag{12}$$

Canonical Quantisation with Constraints

- Now quantise the system subject to the linearisation stability constraints.
- In Dirac quantisation, the constraints Q imposed as conditions on physical states:

$$Q\,|\text{phys}\rangle = 0. \tag{12}$$

- This can make construction of the physical Hilbert space non-trivial.
- For linearised gravity in de Sitter, this would mean the only physical state is the vacuum.

Refined Algebraic Quantisation: Example

- As Flatlanders, we naturally consider Quantum Mechanical states $\psi(x,y) \in \mathcal{L}^2(\mathbb{R}^2)$:

$$\int \mathrm{d}x\mathrm{d}y \, |\psi(x,y)|^2 < \infty. \tag{13}$$

Refined Algebraic Quantisation: Example

- As Flatlanders, we naturally consider Quantum Mechanical states $\psi(x,y) \in \mathcal{L}^2(\mathbb{R}^2)$:

$$\int \mathrm{d}x\mathrm{d}y \, |\psi(x,y)|^2 < \infty. \tag{13}$$

- To consider a one dimensional system, do not care about position of the system in the y direction. Impose constraint

$$p_y \psi_{\text{phys}}(x,y) = 0. \tag{14}$$

Refined Algebraic Quantisation: Example

- As Flatlanders, we naturally consider Quantum Mechanical states $\psi(x,y) \in \mathcal{L}^2(\mathbb{R}^2)$:

$$\int \mathrm{d}x\mathrm{d}y \, |\psi(x,y)|^2 < \infty. \tag{13}$$

- To consider a one dimensional system, do not care about position of the system in the y direction. Impose constraint

$$p_y \psi_{\text{phys}}(x,y) = 0. \tag{14}$$

- But there are no non-trivial $\psi_{\text{phys}} \in \mathcal{L}^2(\mathbb{R}^2)$.
- How can we systematically get to the physical Hilbert space - $\mathcal{L}^2(\mathbb{R})$ for Quantum Mechanics on a line?

Refined Algebraic Quantisation:

- To obtain a physical Hilbert space in presence of symmetry group: [Higuchi, Moncrief]

1. Make invariant, non-normalisable states $|\psi)$ from non-invariant normalisable $|\phi\rangle$ by integrating

$$|\psi) = \int \mathrm{d}\mathbf{c}\, U(\mathbf{c})\, |\phi\rangle. \tag{15}$$

2. Redefine inner product on the physical states:

$$(\psi_1|\psi_2) = \int \mathrm{d}\mathbf{c}\, \langle\phi_1|\, U(\mathbf{c})\, |\phi_2\rangle. \tag{16}$$

Refined Algebraic Quantisation: Example

- Returning to our example, the invariant states are

$$\psi(x) = \int \mathrm{d}y \, e^{-ip_y y} \phi(x,y). \tag{17}$$

- The averaged inner product on the new states is

$$\begin{aligned}(\psi_1, \psi_2) &= \int \mathrm{d}x \mathrm{d}y_1 \mathrm{d}y_2 \; \phi_1(x, y_1)^* e^{-ip_{y_2} y_2} \phi_2(x, y_2) \\ &= \int \mathrm{d}x \; \psi_1(x)^* \psi_2(x),\end{aligned} \tag{18}$$

which we recognize as the correct inner product for $\mathcal{L}^2(\mathbb{R})$.

Linearised Quantum Gravity

- If the invariance under the de Sitter group was imposed directly on the Hilbert space of linearised gravity, only the Euclidean vacuum state would survive for de Sitter.

Linearised Quantum Gravity

- If the invariance under the de Sitter group was imposed directly on the Hilbert space of linearised gravity, only the Euclidean vacuum state would survive for de Sitter.
- This procedure can be carried out for linearised gravity on a torus and for de Sitter space, to obtain non-trivial Hilbert spaces of physical states. [Higuchi]

Linearised Quantum Supergravity

- In the supersymmetric case, have to incorporate an additional fermionic constraint into the theory

$$Q_\alpha \,|\text{phys}\rangle = 0. \tag{19}$$

Linearised Quantum Supergravity

- In the supersymmetric case, have to incorporate an additional fermionic constraint into the theory

$$Q_\alpha \left|\text{phys}\right\rangle = 0. \tag{19}$$

- We can obtain a modified inner product on the physical Hilbert space by averaging over the supergroup of background symmetries.

Linearised Quantum Supergravity

- In the supersymmetric case, have to incorporate an additional fermionic constraint into the theory

$$Q_\alpha \, |\text{phys}\rangle = 0. \tag{19}$$

- We can obtain a modified inner product on the physical Hilbert space by averaging over the supergroup of background symmetries.

- We can explicitly construct non-trivial physical states, obeying the constraints.

Thank You

- Thank You!