What do knots, quantum computation and field theories have in common?

Rajath Krishna Radhakrishnan

Centre for Research in String Theory, Queen Mary University of London

Young Theorists' Forum 11, Durham University
20$^{\text{th}}$ December 2018

Knots

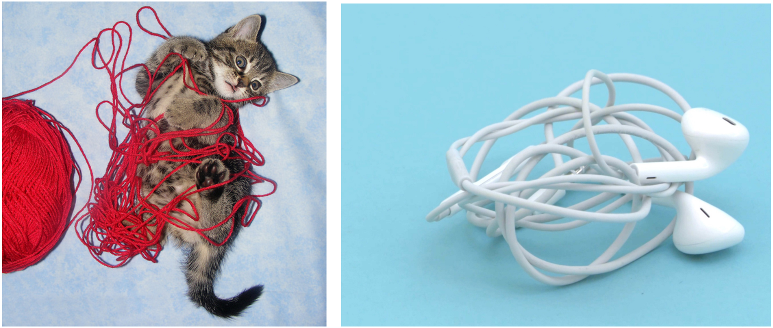


Figure: Knots[1]

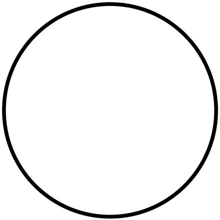
[1] Pictures taken from http://forums.nrvnqsr.com/showthread.php/2452-The-Cat-Thread/page65 and https://www.businessinsider.com/iphone-headphones-not-get-tangled-instructions-video-2016-9?r=UKIR=T respectively.

Definition

Knots are embeddings of a circle in $\mathbb{R}^3$.

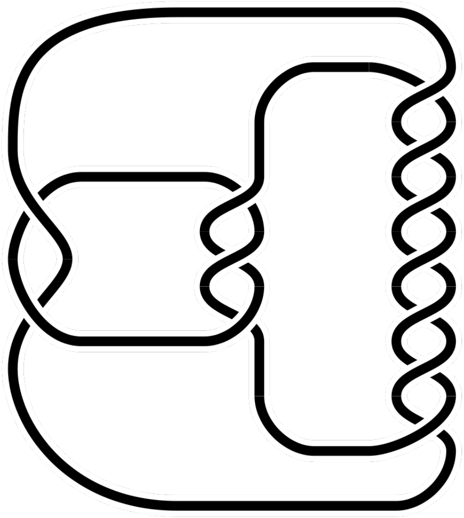
$$ý : w^1 \quad ! \quad \mathbb{R}^3$$

The Unknot



The Unknot is the simplest knot.

A complicated knot



Checking whether to knots are the same is a hard problem. While it is easier to check that the following two knots are the same

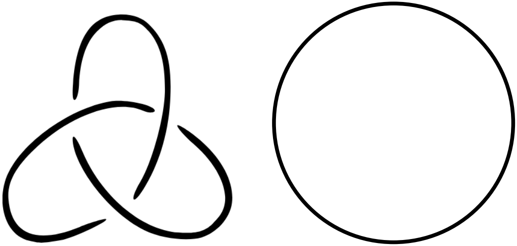
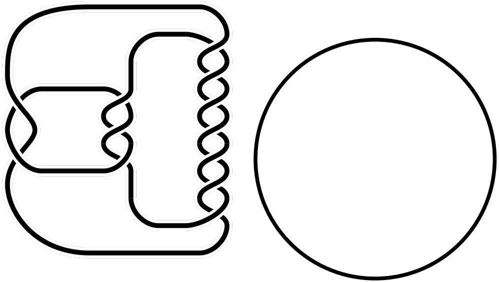


Figure: Trefoil knot and the Unknot

it is much harder with these two knots.



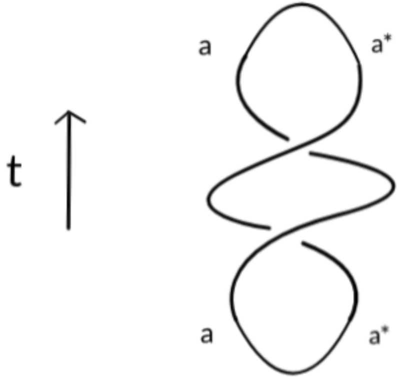
Knots invariants are functions on knots which helps us to distinguish between them. If $ý$ is a knot, let $\mathbb{H}(ý)$ be the knot-invariant of $ý$. Function $\mathbb{H}$ satisfies the property

$$\mathbb{H}(ý_1) \neq \mathbb{H}(ý_2) \Rightarrow ý_1 \nsim ý_2$$

$ý_1$ is not isotopic to $ý_2$.

Knots and particle worldlines

Knots can be interpreted as particle worldlines in 3 spacetime dimensions.



- But, where does such knotty wordlines arise in nature? Topological phases of matter, for example, in the fractional quantum hall effect experiment.

- But, where does such knotty wordlines arise in nature? Topological phases of matter, for example, in the fractional quantum hall effect experiment.

- Note that we stick to 3 spacetimes dimensions instead of 4 because in 4 dimensions, knots are trivial.

- But, where does such knotty wordlines arise in nature? Topological phases of matter, for example, in the fractional quantum hall effect experiment.

- Note that we stick to 3 spacetimes dimensions instead of 4 because in 4 dimensions, knots are trivial.

- It turns out that motion of (quasi) particles in topological phases of matter can be modelled using a mathematical object called the modular tensor category.

Modular Tensor Category

Category

A category is a collection of objects and maps between them. The objects in the category will label our particles. We will denote them as *Ä; Ð; : : :*

We also require the category to have a special object 1 whch denotes the vacuum of the theory.

Monoidal Category

We would like to describe the fusion of two particles to give a third particle. This is modelled by adding a monoidal structure to a category.

We will denote the fusion of *Ä* and *Đ* as *Ä* *Đ*.

If *Ñ* = *Ä* *Đ* then we say that the charge of the composite system of anyons with charge a and b is *Ñ*.

Since the total charge should not depend on the ordering of the particles, should be a commutative operation.

Fusion Rules

Once we have a structure for combining two particles, we can write down a set of rules for our model called fusion rules.

$$a \quad Đ = \mathrm{L}_{\tilde{N}} \wedge^{\tilde{N}}_{\ddot{A}Đ} \; \tilde{N}$$

where $\wedge^{\tilde{N}}_{\ddot{A}Đ}$ are non-negative integers and the sum is over the complete set of labels. $\wedge^{\tilde{N}}_{\ddot{A}Đ}$ denotes the number of ways in which $\tilde{N}$ can be obtained by fusion particles $\ddot{A}$ and $Đ$.

Example: Fibonacci Anyon Model

Labels $f1; \quad g$

Fusion Rules: $1 \quad 1 = 1 \quad 1 \qquad = \qquad\qquad = 1$

Rigid Monoidal Category

To model anti-particles, we need to add a structure called rigidity to the category. An object $Ä^*$ is said to be the dual of $Ä$ if there exists maps

$$Ð_{Ä} : 1 \; ! \; Ä \quad Ä^*$$

$$×_{Ä} : Ä^* \quad Ä \; ! \; 1$$

If every object of the category has a dual, then the category is said to be **rigid**.

Rigid Braided Monoidal Category

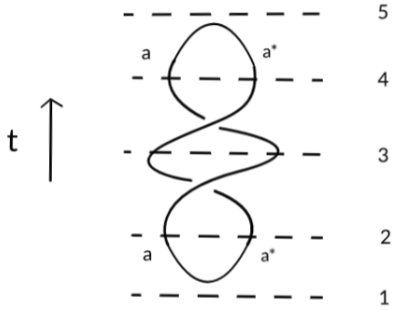
To model the braiding of particle worldlines, we need to introduce a new function $\tilde{N}_{Ä,Ð} : Ä \quad Ð \;\; ! \;\; Ð \quad Ä$.

MTC $=$ Rigid Braided Monoidal Category $+$ Ribbon Structure $+$ Modularity Constraint.

An MTC is the mathematical object which models particle kinematics in topological phases of matter.

Knot Invariant from MTC

Using the structure of an MTC, we can associate an amplitude for a particle worldline.



The above diagram corresponds to the map

$$\times_{\ddot{A}} \quad \tilde{N}_{\ddot{A}^*\ddot{A}} \quad \tilde{N}_{\ddot{A}\ddot{A}^*} \quad Ð_{\ddot{A}}$$

MTC in Quantum Field Theory

Topological Quantum Field Theory

- TQFTs are QFTs which does not depend on the metric of the manfold on which the theory is defined.
- The observables of the theory compute topological invariants of the manifold.
- An example is Chern-Simons theory (for a choice of gauge group G and 3d manifold M) with the following action

$$w = \int \frac{ý}{4} \wedge \times + \frac{2}{3} \wedge \wedge$$

MTC and TQFT

- TQFTs are a special class of QFTs which have the mathematical maturity of General Relativity. A 3-dimensional TQFT can be vaguely understood as associating a vector space to every 3-dimensional manifold.

- It turns out that given an MTC, one can always construct such a map between manifolds and vector spaces. MTCs contain the algebraic data which defines a 3D TQFT.

MTC and CFT

- By definition, a rational conformal field theory has only a finite number of primary fields. Let us denote them as $d_{\hat{\imath}}$. The OPE between these fields can be written symbolically as an algebra

$$d_{\hat{\imath}} d_{\acute{u}} = \sum_{\acute{y}} \wedge^{\acute{y}}_{\hat{\imath}\acute{u}} d_{\acute{y}}$$

- By definition, a rational conformal field theory has only a finite number of primary fields. Let us denote them as $d_{\hat{\imath}}$. The OPE between these fields can be written symbolically as an algebra

$$d_{\hat{\imath}} d_{\acute{u}} = \sum_{\acute{y}} \wedge^{\acute{y}}_{\hat{\imath}\acute{u}} d_{\acute{y}}$$

- If we choose the primary fields as the objects in a category, the above algebra corresponds to the fusion rules. Moreover, one can construct braid maps using the crossing symmetry of 4-point correlation functions.

- By definition, a rational conformal field theory has only a finite number of primary fields. Let us denote them as $d_{\hat{\imath}}$. The OPE between these fields can be written symbolically as an algebra

$$d_{\hat{\imath}} d_{\acute{u}} = \sum_{\acute{y}} \wedge^{\acute{y}}_{\hat{\imath}\acute{u}} d_{\acute{y}}$$

- If we choose the primary fields as the objects in a category, the above algebra corresponds to the fusion rules. Moreover, one can construct braid maps using the crossing symmetry of 4-point correlation functions.
- It turns out that one can construct the whole structure of an MTC from any given RCFT. An MTC encodes many crucial (but not all!) information in a CFT.

Topological Quantum Computation

Quantum Computation

- Quantum computation involves a set of techniques to manipulate quantum systems to perform computations.
- At least in certain specific situations the computations can be done much faster than a classical computer.
- Superpositions and Entanglement are the most crucial ingredients responsible for this speed-up.

- However, superposition and entanglement are notoriously fragile and the environment is a continuous observer.

- In standard realizations of quantum computers, information is encoded locally in the system. For example, in the spin of electrons.

- Locally encoded information is easier to be corrupted by noise.

Quantum computation involves

1. initialization of states, usually a set of n qubits
   $|0\rangle \quad |0\rangle \qquad\qquad |0\rangle$.

2. some n-qubit unitaries £ act on this state. These are called quantum gates.

3. measuring the system in a suitable basis.

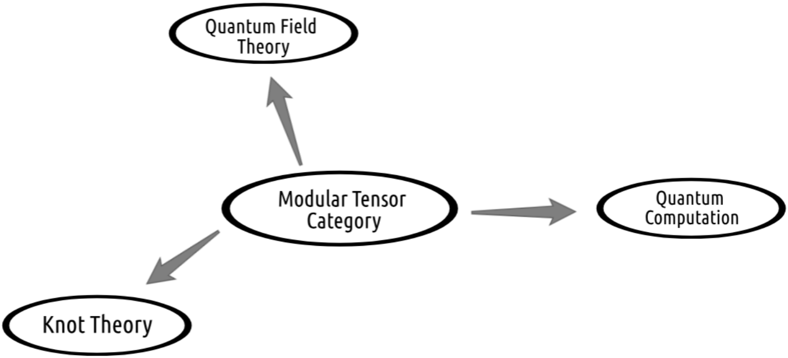
Topological Quantum Computation

Topological Quantum Computation (TQC) is a way to encode information non-locally in a quantum system which greatly reduces the effect of noise. Topological quantum computation involves

1. initialization of states. This involves producing many particle-antiparticle pairs.
2. unitaries acting on the particles. Quantum gates are represented by braid operations $\tilde{N}_{\ddot{A}\text{Đ}}$ on the particles.
3. measuring the system. This involves the annihilation of particle pairs and checking whether the resulting particle has trivial charge or not.

The idea is to encode data in the braidings of particles in a topologically ordered material. Braiding is a non-local process. **It is easier to change the spin of a particle than to unbraid.**

Conclusion


Quantum Field Theory
Modular Tensor Category
Quantum Computation
Knot Theory

Questions?

Crossing Symmetry in CFT

Consider the four point function

$$@(ń; \bar{n}) = h \;\; {}_{î}(ń_1; \bar{n}_1) \;\; {}_{ú}(ń_2; \bar{n}_2) \;\; {}_{ý}(ń_3; \bar{n}_3) \;\; {}_{Ā}(ń_4; \bar{n}_4) i$$

It can only depend on the crossing ratios

$$ľ = \frac{ń_{12} ń_{34}}{ń_{13} ń_{24}}; \quad Ť = \frac{\bar{n}_{12} \bar{n}_{34}}{\bar{n}_{13} \bar{n}_{24}}$$

We can use the OPE for ${}_{î} \;\; {}_{ú}$ and ${}_{ý} \;\; {}_{Ā}$ to reduce the four point function to

$$@(ń; \overline{(}ń)) = \underset{Ě}{\mathrm{X}} +^{Ě}_{îú} +^{Ě}_{ýĀ} ?^{ýĀ}_{îú}(Ěj ľ) \overline{?}^{ýĀ}_{îú}(Ěj Ť)$$

where $?^{ýĀ}_{îú}$ are called conformal blocks.

Alternatively, one can use the OPE for ú ý and î Ā. Also, one can use the OPE ú Ā and î ý. Since the order of fields is irrelevant in the correlation function the four point function obtained from these choices should be the same. This is called the crossing symmetry.

For a rational conformal field theory, the conformal blocks form a finite dimensional vector space and the crossing symmetry boils down to a basis change on these vector spaces.

There exists matrices ? and * which performs this basis change.