A (Gentle) Introduction to Supersymmetric Localization

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Outline

Aim

- Provide an introduction to the topic of supersymmetric localization.
- Explain how to compute the exact partition function for a supersymmetric theory by reducing the infinite-dimensional path integral to a finite-dimensional integral.

Intuitive Example - Method 1

- Consider the unit sphere S^2, with metric $\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$.
- Compute the following integral

$$I = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \sin\theta e^{it\cos\theta} \ . \tag{1}$$

$$\begin{aligned} I &= 2\pi \int_{-1}^{1} \mathrm{d}(\cos\theta) e^{it\cos\theta} \\ &= \frac{2\pi}{t}\left[-ie^{it} + ie^{-it}\right] = \frac{4\pi \sin t}{t} \ . \end{aligned} \tag{2}$$

- Can we compute this integral using a second method?

Intuitive Example - Method 2

$$I = 2\pi \int_{-1}^{1} \mathrm{d}(\cos\theta) e^{it\cos\theta} \ . \tag{3}$$

- Consider the large t scenario and apply the **stationary phase approximation**.
- There are two stationary points of $\cos\theta$, the North and South poles of S^2.
- Evaluating the Gaussian integral for the quadratic fluctuations of each critical point and adding them up gives

$$I = \frac{2\pi}{t}\left[-ie^{it} + ie^{-it}\right] = \frac{4\pi \sin t}{t} \ . \tag{4}$$

- This matches the previous exact computation.

The Path Integral of a Supersymmetric QFT

- The **Euclidean partition function** of a theory is defined by

$$Z = \int_{\mathcal{M}} [\mathcal{D}X]\, e^{-S[X]} \tag{5}$$

- We require **non-perturbative techniques** to compute this object exactly.
- **Supersymmetric localization** reduces the problem to a finite-dimensional integral.
- This relies on the construction of supersymmetry in curved space.

Set-up

- What ingredients do we need in the theory?
- **Conserved supercharges** Q,

$$Q^2 = B \ . \tag{6}$$

- **Supersymmetry invariant action** $S[X]$,

$$QS[X] = 0 \ . \tag{7}$$

- The supersymmetry transformations schematically take the form

$$Q(\mathit{bosons}) = (\mathit{fermions}) \ \ \mathit{and} \ \ Q(\mathit{fermions}) = (\mathit{bosons}) \ . \tag{8}$$

What is the expectation value for a Q-exact operator $\mathcal{A}$?

- The expectation value of $\mathcal{A}$ is the observable

$$\langle \mathcal{A} \rangle = \int_{\mathcal{M}} [\mathcal{D}X]\, \mathcal{A} e^{-S[X]} \,. \tag{9}$$

- Since it is **Q-exact**, the operator can be written in the form $\mathcal{A} = Q\mathcal{O}$.

$$\langle Q\mathcal{O} \rangle = \int_{\mathcal{M}} [\mathcal{D}X]\, (Q\mathcal{O})\, e^{-S[X]} = \int_{\mathcal{M}} [\mathcal{D}X]\, Q \left(\mathcal{O} e^{-S[X]} \right) = 0 \,. \tag{10}$$

- This vanishes provided that no boundary terms exist and $e^{-S[X]} \to 0$ as $|X| \to \infty$.

Deforming the Path Integral

- A freedom exist to **deform** a path integral by a **Q-exact term**,

$$Z = \int_{\mathcal{M}} [\mathcal{D}X] \, e^{-S[X] - tS_{loc}[X]} \, . \tag{11}$$

- $S_{loc}[X]$ is the **localizing action**, given by

$$S_{loc}[X] = QV[X] \quad where \quad Q^2 V[X] = 0 \, . \tag{12}$$

- Significantly, it is possible to show that

$$\frac{d}{dt} Z = 0 \, . \tag{13}$$

- The partition function is equally described by

$$Z = \int_{\mathcal{M}} [\mathcal{D}X] \, e^{-S[X]} = \lim_{t \to \infty} \int_{\mathcal{M}} [\mathcal{D}X] \, e^{-S[X] - tS_{loc}[X]} \, . \tag{14}$$

What is $S_{loc}[X]$?

$$S_{loc}[X] = QV[X] \ . \tag{15}$$

- A canonical choice for the localizing action is

$$V[X] = \sum_{\psi} \left[(Q\psi)^\dagger \psi + \psi^\dagger \left(Q\psi^\dagger \right)^\dagger \right] \ . \tag{16}$$

- The bosonic piece is a positive sum of squares.
- The **saddle points** are the configurations

$$\psi = \psi^\dagger = 0 \quad \Rightarrow \quad \mathit{fermions} = 0 \ . \tag{17}$$

$$Q\psi = Q\psi^\dagger = 0 \quad \Rightarrow \quad Q(\mathit{fermions}) = 0 \ . \tag{18}$$

Evaluating the Path Integral

- Expand the fields X about the saddle points X_0 of $S_{loc}[X]$,

$$X = X_0 + \frac{1}{\sqrt{t}}\delta X \ . \tag{19}$$

- This means that

$$S[X] + tS_{loc}[X] \rightarrow S[X_0] + \frac{1}{2}\int \left.\frac{\delta^2 S_{loc}[X]}{\delta X^2}\right|_{X=X_0} (\delta X)^2 \ . \tag{20}$$

- Then integrate out the transverse quadratic fluctuations $(\delta X)^2$.

Localization Formula

- The **localized path integral** is given by

$$Z = \int_{\mathcal{M}_Q} [\mathcal{D}X_0]\, Z_{classical} Z_{1-loop} \ . \tag{21}$$

where

$$\mathcal{M}_Q = \{[X] \in \mathcal{M} | fermions = 0\ , Q(fermions) = 0\}\ . \tag{22}$$

- This integral can be computed to obtain an **exact** result for the partition function of a theory.
- The **classical** piece is

$$Z_{classical} = e^{-S[X_0]} \tag{23}$$

- Z_{1-loop} is the **non-perturbative** contribution.

1-loop Determinant

- The **1-loop determinant** is given by

$$Z_{1-loop} = \frac{Det\Delta_{fermion}}{Det\Delta_{boson}} \ . \tag{24}$$

- Evaluating this is often the most difficult part of the localization calculation.
- Techniques to compute Z_{1-loop} include
  - Spherical Harmonic Decomposition.
  - Eigenmode Pairing.
  - Index Theorems.

Summary

- When the path integral is deformed by a localizing action term, the semi-classical approximation is exact, and the path integral is localized to the configurations given by solving $\mathit{fermions} = 0$ and $Q(\mathit{fermions}) = 0$.
- Localization has led to many exact results for broad classes of supersymmetric field theories on curved manifolds of various dimensions.
- Localization formulae can be used to extract physical/mathematical information about a theory, compute expectation values and correlators of certain observables, and to test or infer non-perturbative dualities.

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Thank You For Listening