Resurgence and Non-Perturbative Physics

Philip Glass

Durham University

YTF 2018

Contents

- Divergent asymptotic series are everywhere and are important.
- How do we sum divergent series?
- What can we learn from analytic continuation of the coupling constants?
- Summary of what you can do with resurgence.
- Resurgence in SUSY Localizable theories.

Asymptotic Series are Ubiquitous in Physics.

- Without some miracle, e.g. SUSY Localizability or Integrability, in almost every interesting theory in physics observables are calculated by way of perturbation theory.
- Dating back to 1952, Dyson showed that the perturbative expansion of QED was asymptotic. This is also the case with perturbative expansions of the Standard Model.
- String scattering is currently defined in terms of a perturbative expansion, summing over all possible topologies of string scattering. These expansions are also asymptotic.

The basic problem

We expand our partition function/correlation function/... as

$$\mathbb{I} = \sum_i e^{-S_i} \sum_{n=0}^{\infty} a_{i,n} h^n \tag{1}$$

around some non-perturbative background with action S_i. We find that each of the sums

$$\sum_{n=0}^{\infty} a_{i,n} h^n \tag{2}$$

has **zero** radius of convergence!

Implications

$$\mathbb{I} = \sum_i e^{-S_i} \sum_{n=0}^{\infty} a_{i,n} h^n \quad (3)$$

- The asymptoticity of QED/Standard Model/most QFT's is generally attributed to the factorial growth of the number of Feynman diagrams.
- Theories defined in terms of asymptotic series are NOT rigorously defined. The equals sign shouldn't be there as the RHS isn't defined anywhere.
- The series expansions have zero radius of convergence, and thus simply computing higher order Feynman diagrams will not eternally lead to greater accuracy.

Summation of Divergent Series

Suppose we have some machine $\mathcal{S}$ which sums our divergent series. We ask it to have 2 properties:

- Summation: $\mathcal{S}(a_0 + a_1 + a_2 + ...) = a_0 + \mathcal{S}(a_1 + a_2 + ...)$
- Linearity $\mathcal{S}((\alpha + \beta)a_0 + (\alpha + \beta)a_1 + (\alpha + \beta)a_2 + ...) = \alpha\mathcal{S}(a_0 + a_1 + a_2 + ...) + \beta\mathcal{S}(a_0 + a_1 + a_2 + ...)$

Note we could demand different things, notably zeta summation doesn't obey these requirements.
Summation machines which obey these requirements give the same answers where they can do the sums.

Euler Summation

- We take the series $a_0 + a_1 + a_2 + ...$ and make the series $a_0 + a_1 x + a_2 x^2 + ...$
- Next evaluate the series where it converges.
- Analytically continue the answer to $x = 1$.

Eg. $1 + 2 + 4 + 8 + ... \rightarrow$

$1 + 2x + 4x^2 + 8x^3 + ... = \dfrac{1}{1 - 2x}$

Thus we have $1 + 2 + 4 + 8 + ... \sim -1$

Eg.2 $1 - 1 + 1 - 1 + ... \sim \dfrac{1}{2}$

Borel Summation

- We take the series $\sum_{n=0}^{\infty} a_n$ and multiply each term by $1 = \frac{1}{n!}\int_0^{\infty} x^n e^{-x} dx$, giving $\sum_{n=0}^{\infty}\left(\frac{a_n}{n!}\int_0^{\infty} x^n e^{-x} dx\right)$.
- Next swap the sum and the integral, $\int_0^{\infty}\sum_{n=0}^{\infty}\frac{a_n}{n!}x^n e^{-x} dx$
- Compute the sum, and then the integral.

Eg. $1 - 1 + 1 - 1 + ... \rightarrow$

$$\int_0^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n e^{-x} dx = \int_0^{\infty} e^{-2x} dx = \frac{1}{2}$$

Borel Summation 2

- We typically don't have access to all the terms in the sum, thus we need to use something like Padé approximants to approximate the answer.
- For almost all QFT and Strings applications, we find that there are branch cuts and poles in the plane of integration, thus the answer is ambiguous, depending on the contour we take...

Anayltic Continuation into the Complex Plane

Consider the Hamiltonian for a two state system:

$$H = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \epsilon \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \qquad (4)$$

This has the eigenvalues $E_{\pm} = \dfrac{(a+b) \pm \sqrt{(a-b)^2 + 4(\epsilon c)^2}}{2}$.

Take only the root E_+ and consider E to be a function of complex ϵ.

We have branch points at $\epsilon = \pm \dfrac{i(a-b)}{2}$, and smoothly continuing ϵ round one of them through the branch cut will bring us back to E_-

Analytic continuation of coupling constant in Borel Plane

After the Borel Transform,

$$Z(g) = \sum_{n=0}^{\infty} a_n g^{-n-1} \rightarrow \hat{Z}(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \tag{5}$$

we compute the now convergent sum. The function has poles and branch cuts in the complex z-plane. To get back to a resummed Z we take the Laplace transform

$$S(Z)(g) = \int_0^{\infty} dz e^{-zg} \hat{Z}(z) \tag{6}$$

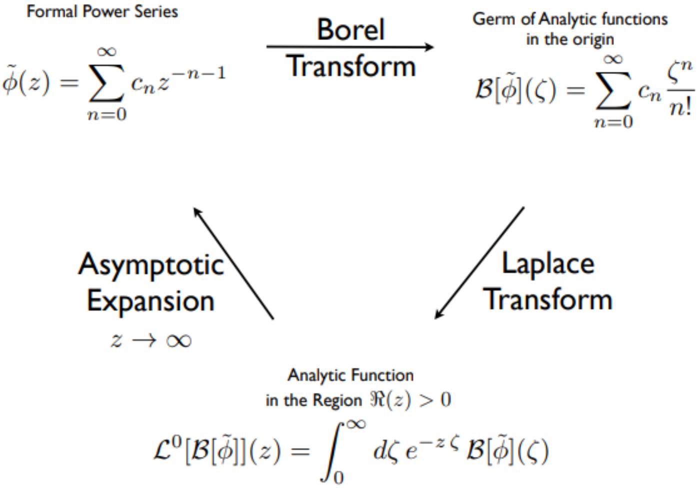
Computing the discontinuities across the cuts of $e^{-zg}\hat{Z}(z)$ around all the branch cuts and poles in the Borel Plane gives us all the non-perturbative data!!!

Resurgence

The Procedure

- Take series (say the expansion around the 0-instanton/ perturbative sector) and Borel transform.
- Compute the Sum
- Find the discontinuities of the laplace transform of $e^{-zg}\hat{Z}(z)$ around all the poles and branch cuts, giving you the borel sums of all the non-perturbative sectors as well.
- Fix contributions of non-perturbative parts from boundary conditions of problem, and sum to get final finite unambiguous answer with all non-perturbative data included.

Schematic


Formal Power Series
$\tilde{\phi}(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$
Borel Transform
Germ of Analytic functions in the origin
$\mathcal{B}[\tilde{\phi}](\zeta) = \sum_{n=0}^{\infty} c_n \frac{\zeta^n}{n!}$
Laplace Transform
Asymptotic Expansion
$z \to \infty$
Analytic Function in the Region $\Re(z) > 0$
$\mathcal{L}^0[\mathcal{B}[\tilde{\phi}]](z) = \int_0^{\infty} d\zeta \, e^{-z\zeta} \, \mathcal{B}[\tilde{\phi}](\zeta)$

Current Status of Resurgence

- Rigorous and very useful in solving differential equations and in Quantum Mechanics.
- Increasing evidence it still holds up in QFT, but not concrete at the moment.

QM Double Well Example

$V(x) = x^2(1 - gx)^2$

- Perturbation theory around one of the vacuums finds $E_0 \sim \sum_{n=0} c_n g^{2n}$ where $c_n \sim -3^n n!$
- Borel Transform. Branch cut in complex plane. Ambiguity in laplace transform given by $\mathrm{Im}(E_0) \sim \mp\pi e^{\frac{-1}{3g^2}}$
- Demanding the contributions from the non-perturbative sector cancel this, we get the contribution from the Instanton-Anti-Instanton.

QM Double Well Example II

From the perturbative sector we can read off all the non-perturbative data, and the final answer is of the form

$$E = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{q=0}^{k-1} c_{n,k,q} g^{2n} \left[\exp\left(\frac{-S}{g^2} \right) \right]^k \left[\log\left(\frac{-1}{g^2} \right) \right]^q \tag{7}$$

This is called a *Trans-Series*.

Cheshire Cat Resurgence in SUSY-Localizable Field Theories

The grin of Cheshire cat resurgence from supersymmetric localization.
Dorigoni, Glass
arXiv:1711.04802

$\mathcal{N} = (2,2)$ CP^{N-1} model on 2-Sphere

We can get the exact result for the partition function from localization (Benini, Cremonesi, arXiv:1206.2356):

$$Z_{\mathbb{CP}^{N-1}} = \sum_{B \in \mathbb{Z}} e^{-i\theta B} \int_{-\infty}^{+\infty} \frac{d\sigma}{2\pi} e^{-4\pi i \xi \, \sigma} \left(\frac{\Gamma(-i\sigma - B/2)}{\Gamma(1 + i\sigma - B/2)} \right)^N \tag{8}$$

Includes sum over instanton sectors, but can we get the non-perturbative data from the perturbative data only?

Deform the Theory

The Integrand of the path integral takes the form

$$\tilde{Z}_{matter}(\sigma) = \frac{(\det \mathcal{O}_\psi)^{N_f}}{(\det \mathcal{O}_\phi)^{N_b}} \tag{9}$$

The SUSY theory has $N_f = N_b$. We make the formal deformation $N_b \to N_b - \Delta$, so the theory is no longer SUSY.

$$\tilde{Z}_{matter}(\sigma) = \frac{(\det \mathcal{O}_\psi)^{N_f}}{(\det \mathcal{O}_\phi)^{N_b}} (\det \mathcal{O}_\phi)^{\Delta} \tag{10}$$

We can do a weak coupling expansion in the non-perturbative sector, everything becomes asymptotic, and we can do Resurgence.