Towards a Classification of Supersymmetric Spacetimes

Ross Grassie

University of Edinburgh

YTF Durham, December 2018

Motivation

Aim: Classify the geometric models of spacetime

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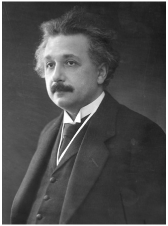

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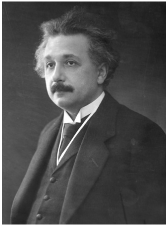

GR Postulate: Spacetime is described by a four-dimensional Lorentzian manifold

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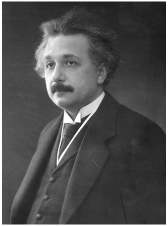

GR Postulate: Spacetime is described by a four-dimensional Lorentzian manifold

Consequence: Spacetime is determined by its relativity group

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1+1-dimensional Minkowski space:

$$\mathcal{M} = \mathbb{R}^{1,1}$$

Relativity group:

$$\mathcal{G} = O^{+}_{1,1}(\mathbb{R}) \ltimes \mathbb{R}^{1,1}$$

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$$p \in \mathcal{M}, \quad \mathcal{H} = \{g \in \mathcal{G} : g \cdot p = p\} = O^+_{1,1}(\mathbb{R})$$

$$\boxed{\mathcal{M} \cong \mathcal{G}/\mathcal{H}}$$

Where are we now?

Aim: Classify the geometric models of spacetime

Found: Spacetime is determined by its relativity group

Therefore: Classify the relativity groups

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Question: What do we allow as a relativity group?

Kinematical Lie Groups

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$$\mathfrak{g} = s \oplus 2V \oplus \mathbb{R}, \tag{1}$$

where $s \cong \mathfrak{so}(D)$. $\mathfrak{h}$ Lie subalgebra of $\mathfrak{g}$.

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \delta_{bc} J_{ad} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac} + \delta_{ad} J_{bc} \\
[J_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b \\
[J_{ab}, B_c] &= \delta_{bc} B_a - \delta_{ac} B_b \\
[J_{ab}, H] &= 0.
\end{aligned} \tag{2}$$

Let $\mathfrak{a} := 2V \oplus \mathbb{R}$, such that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$.

Method of Classification

A Lie algebra $\mathfrak{g}$ is defined by the action of its bracket $[-,-]$ on a basis $\{X_A\}$

$$[X_a, X_b] = f_{ab}{}^c X_c. \tag{3}$$

Note, $[-,-] \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

Therefore, if we want to classify Kinematical Lie algebras (KLAs), we need to explore the space $\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ subject to our kinematical constraints.

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Define $C^p(\mathfrak{g};\mathfrak{g}) = \Lambda^p\mathfrak{g}^* \otimes \mathfrak{g}$.

Let $\mu = \alpha \otimes X \in C^p(\mathfrak{g};\mathfrak{g})$, and $\nu = \beta \otimes Y \in C^q(\mathfrak{g};\mathfrak{g})$.

Define the product, $\bullet : C^p(\mathfrak{g};\mathfrak{g}) \times C^q(\mathfrak{g};\mathfrak{g}) \to C^{p+q}(\mathfrak{g};\mathfrak{g})$ such that

$$(\alpha \otimes X) \bullet (\beta \otimes Y) := (\alpha \wedge \iota_X \beta) \otimes Y. \tag{4}$$

Graded Lie Superalgebra

Define the Nijenhuis–Richardson bracket,

$$[\![\mu, \lambda]\!] := \mu \bullet \lambda - (-1)^{pq} \lambda \bullet \mu. \tag{5}$$

The space $C^\bullet(\mathfrak{g}; \mathfrak{g})$ satisfies the corresponding super-Jacobi identity

$$[\![\lambda, [\![\mu, \nu]\!]]\!] = [\![[\![\lambda, \mu]\!], \nu]\!] + (-1)^{pq} [\![\mu, [\![\lambda, \nu]\!]]\!]. \tag{6}$$

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Let $\mu_0 \in C^2(\mathfrak{g};\mathfrak{g})$ define a Lie algebra. Then,

$$\partial : C^p(\mathfrak{g};\mathfrak{g}) \to C^{p+1}(\mathfrak{g};\mathfrak{g}); \qquad \mu \mapsto \partial\mu = (-1)^p [\![\mu_0, \mu]\!], \tag{7}$$

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is a differential operator. Space of p-cochains: $C^p(\mathfrak{g}; \mathfrak{g})$

$$0 \xrightarrow{\partial} C^1(\mathfrak{g}; \mathfrak{g}) \xrightarrow{\partial} C^2(\mathfrak{g}; \mathfrak{g}) \xrightarrow{\partial} C^3(\mathfrak{g}; \mathfrak{g}) \xrightarrow{\partial} \dots$$

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$$\mu = \mu_0 + \varphi. \tag{9}$$

Imposing the condition that μ must be a Lie algebra, we get

$$\partial\varphi = [\![\varphi,\varphi]\!]. \tag{10}$$

This may be solved by writing $\varphi = \sum_{n\geq 1} t^n \varphi_n$, and solving order by order in t.

$$\partial\varphi_n = \sum_{m=1}^{n-1} [\![\varphi_m,\varphi_{n-m}]\!]. \tag{11}$$

Cohomology

Let $n = 1$ in the equation (11).

$$\partial \varphi_1 = 0. \tag{12}$$

Therefore, the first-order (infinitesimal) deformation is in $Z^2(\mathfrak{g};\mathfrak{g})$.

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Therefore, the deformations of interest are in $H^2(\mathfrak{g}; \mathfrak{g})$.

Using the Hochschild-Serre factorisation: $H^2(\mathfrak{g}; \mathfrak{g}) \cong H^2(\mathfrak{a}; \mathfrak{g})^{\mathfrak{s}}$. This allows us to preserve the kinematical brackets.

Classification

Label	D	Nonzero Lie brackets in addition to $[\boldsymbol{J},\boldsymbol{J}]=\boldsymbol{J}$, $[\boldsymbol{J},\boldsymbol{B}]=\boldsymbol{B}$, $[\boldsymbol{J},\boldsymbol{P}]=\boldsymbol{P}$					Comments
S1	≥ 1	$[H,\boldsymbol{B}]=-\boldsymbol{P}$		$[\boldsymbol{B},\boldsymbol{B}]=\boldsymbol{J}$	$[\boldsymbol{B},\boldsymbol{P}]=H$		Minkowski
S2	≥ 2	$[H,\boldsymbol{B}]=-\boldsymbol{P}$	$[H,\boldsymbol{P}]=-\boldsymbol{B}$	$[\boldsymbol{B},\boldsymbol{B}]=\boldsymbol{J}$	$[\boldsymbol{B},\boldsymbol{P}]=H$	$[\boldsymbol{P},\boldsymbol{P}]=-\boldsymbol{J}$	de Sitter
S3	≥ 1	$[H,\boldsymbol{B}]=-\boldsymbol{P}$	$[H,\boldsymbol{P}]=\boldsymbol{B}$	$[\boldsymbol{B},\boldsymbol{B}]=\boldsymbol{J}$	$[\boldsymbol{B},\boldsymbol{P}]=H$	$[\boldsymbol{P},\boldsymbol{P}]=\boldsymbol{J}$	anti-de Sitter
S4	≥ 1	$[H,\boldsymbol{B}]=\boldsymbol{P}$		$[\boldsymbol{B},\boldsymbol{B}]=-\boldsymbol{J}$	$[\boldsymbol{B},\boldsymbol{P}]=H$		Euclidean
S5	≥ 1	$[H,\boldsymbol{B}]=\boldsymbol{P}$	$[H,\boldsymbol{P}]=-\boldsymbol{B}$	$[\boldsymbol{B},\boldsymbol{B}]=-\boldsymbol{J}$	$[\boldsymbol{B},\boldsymbol{P}]=H$	$[\boldsymbol{P},\boldsymbol{P}]=-\boldsymbol{J}$	sphere
S6	≥ 1	$[H,\boldsymbol{B}]=\boldsymbol{P}$	$[H,\boldsymbol{P}]=\boldsymbol{B}$	$[\boldsymbol{B},\boldsymbol{B}]=-\boldsymbol{J}$	$[\boldsymbol{B},\boldsymbol{P}]=H$	$[\boldsymbol{P},\boldsymbol{P}]=\boldsymbol{J}$	hyperbolic
S7	≥ 1	$[H,\boldsymbol{B}]=-\boldsymbol{P}$					Galilean
S8	≥ 1	$[H,\boldsymbol{B}]=-\boldsymbol{P}$	$[H,\boldsymbol{P}]=-\boldsymbol{B}$				galilean de Sitter $(\mathrm{dS\,G})_{\gamma=-1}$
S9	≥ 1	$[H,\boldsymbol{B}]=-\boldsymbol{P}$	$[H,\boldsymbol{P}]=\gamma\boldsymbol{B}+(1+\gamma)\boldsymbol{P}$				torsional Galilean de Sitter $(\mathrm{dS\,G})_{\gamma\in(-1,1]}$
S10	≥ 1	$[H,\boldsymbol{B}]=-\boldsymbol{P}$	$[H,\boldsymbol{P}]=\boldsymbol{B}$				galilean anti de Sitter $(\mathrm{AdS\,G})_{\chi=0}$
S11	≥ 1	$[H,\boldsymbol{B}]=-\boldsymbol{P}$	$[H,\boldsymbol{P}]=(1+\chi^2)\boldsymbol{B}+2\chi\boldsymbol{P}$				torsional Galilean anti-de Sitter $(\mathrm{AdS\,G})_{\chi>0}$
S12	2	$[H,\boldsymbol{B}]=-\boldsymbol{P}$	$[H,\boldsymbol{P}]=(1+\gamma)\boldsymbol{P}-\chi\widetilde{\boldsymbol{P}}+\gamma\boldsymbol{B}-\chi\widetilde{\boldsymbol{B}}$				$\gamma\in[-1,1),\chi>0$
S13	≥ 2				$[\boldsymbol{B},\boldsymbol{P}]=H$		Carrollian
S14	≥ 2		$[H,\boldsymbol{P}]=-\boldsymbol{B}$		$[\boldsymbol{B},\boldsymbol{P}]=H$	$[\boldsymbol{P},\boldsymbol{P}]=-\boldsymbol{J}$	Carrollian de Sitter
S15	≥ 2		$[H,\boldsymbol{P}]=\boldsymbol{B}$		$[\boldsymbol{B},\boldsymbol{P}]=H$	$[\boldsymbol{P},\boldsymbol{P}]=\boldsymbol{J}$	Carrollian anti-de Sitter
S16	≥ 1	$[H,\boldsymbol{B}]=\boldsymbol{B}$	$[H,\boldsymbol{P}]=-\boldsymbol{P}$		$[\boldsymbol{B},\boldsymbol{P}]=H+\boldsymbol{J}$		non-reductive Carrollian
S17	1	$[H,B]=-P$			$[B,P]=-H-2P$		
S18	1	$[H,B]=H$			$[B,P]=-P$		
S19	1	$[H,B]=(1+\chi)H$			$[B,P]=(1-\chi)P$		$\chi>0$
S20	1	$[H,B]=-P$			$[B,P]=-(1+\chi^2)H-2\chi P$		$\chi>0$
A21	≥ 0						static (S)
A22	≥ 1		$[H,\boldsymbol{P}]=\boldsymbol{P}$				torsional static (TS)
$\mathrm{A23}_{+1}$	≥ 2					$[\boldsymbol{P},\boldsymbol{P}]=-\boldsymbol{J}$	$\mathbb{R}\times\mathbb{S}^D$
$\mathrm{A23}_{-1}$	≥ 2					$[\boldsymbol{P},\boldsymbol{P}]=\boldsymbol{J}$	$\mathbb{R}\times\mathbb{H}^D$
A24	2					$[\boldsymbol{P},\boldsymbol{P}]=H$	

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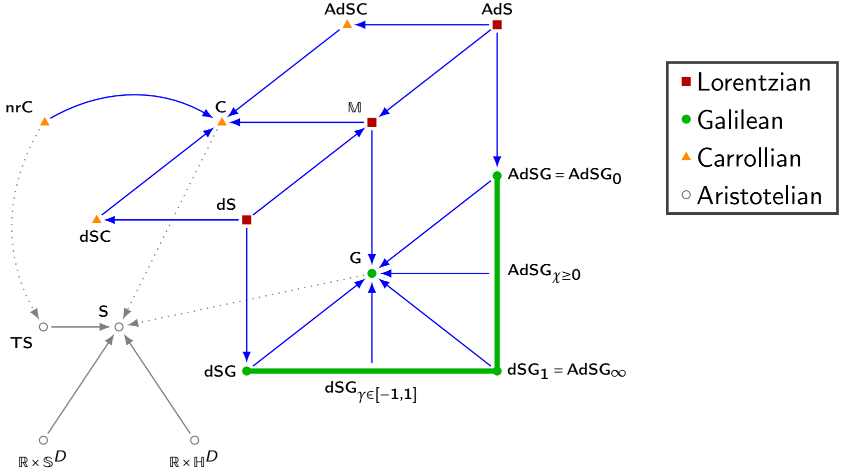

Figure: Homogeneous spacetimes in dimension $D + 1 \geq 4$ and their limits.

Related Papers

José M. Figueroa-O'Farrill.
Kinematical Lie algebras via deformation theory.
J. Math. Phys., 59(6):061701, 2018.

José M. Figueroa-O'Farrill.
Higher-dimensional kinematical Lie algebras via deformation theory.
J. Math. Phys., 59(6):061702, 2018.

Tomasz Andrzejewski and José Miguel Figueroa-O'Farrill.
Kinematical lie algebras in $2 + 1$ dimensions.
J. Math. Phys., 59(6):061703, 2018.

José M. Figueroa-O'Farrill.
Conformal Lie algebras via deformation theory.
2018.

José Figueroa-O'Farrill and Stefan Prohazka.
Spatially isotropic homogeneous spacetimes.
2018.

Read this!

Coming soon...

GEOMETRY OF SPATIALLY ISOTROPIC HOMOGENEOUS SPACETIMES

JOSÉ FIGUEROA-O'FARRILL, ROSS GRASSIE, AND STEFAN PROHAZKA

ABSTRACT. Simply-connected homogeneous spacetimes for kinematical and Aristotelian Lie algebras (with space isotropy) have recently been classified in all dimensions. In this paper, we continue the study of these spacetimes by investigating their local geometry. For each such spacetime and relative to exponential co-ordinates, we calculate the (infinitesimal) action of the kinematical symmetries, paying particular attention to the action of the boosts, showing in almost all cases that they act with generic non-compact orbits. We also calculate the soldering form, the associated vielbein and any invariant Lorentzian, Galilean (i.e., Newton–Cartan) or Carrollian structures, as well as determining the Lie algebra of automorphisms of the invariant structure, which in the case of Galilean and Carrollian structures are typically infinite-dimensional and reminiscent of BMS Lie algebras. We also determine the space of invariant

Kinematical Lie Superalgebras

Define the "static" Kinematical Lie superalgebra (KLS) in $3+1$-dimensions,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \tag{13}$$

where

$$\mathfrak{g}_0 = \mathfrak{s} \oplus 2V \oplus \mathbb{R}, \qquad \mathfrak{g}_1 = S, \tag{14}$$

and S is an irreducible spinor module of $\mathfrak{s}$.

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Method is identical to before, but with more basis elements and additional structure.

Thank you