STFC HEP Summer School
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High Energy Physics Summer School 2019
Pre-School Notes and Problems

Quantum Field Theory
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Introduction to QED and QCD
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Solutions for the problems will be made available via the School website

1 Basics

The aim of these notes is to review some concepts which will be assumed for the school, and also provide some example problems with which you can test your understanding. Throughout, we will work in natural units so $\hbar = c = 1$.

2 Quantum Field Theory

We will build on ideas already encountered by the reader, where the assumed structure of this knowledge can be summarised as in figure 1. On the left-hand side, we have classical mechanics, in which there are two types of entity - *particles* and *fields*. The former obey Newton's laws, but can also be described by the Lagrangian or Hamiltonian formalisms. The power of the latter approach is that one can also describe fields using such a framework, as we will see. In the upper right-hand corner, we have the quantum mechanics of particles. As the reader is no doubt aware, there is a well-defined prescription for taking any classical particle theory and progressing to the appropriate quantum theory (i.e. moving from left to right in the upper row of the figure). It is also possible to describe quantum particles which interact with a classical field (e.g. a quantum charged particle in an electromagnetic field). However, it is perhaps not yet clear how to fill in the box in the lower-right-hand corner. That is, how to make a quantum theory of fields. We'll see how this works in September - for now let us go over the other boxes in figure 1, beginning with the classical column on the left-hand side.

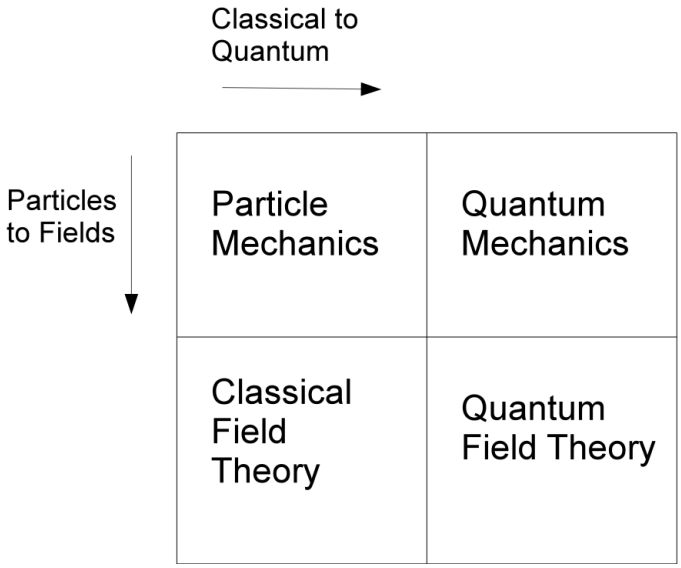

Figure 1: Schematic representation of the relationship between particles and fields, in both a classical and quantum setting. The QFT course concerns the lower right-hand corner.

2.1 Classical Mechanics

2.1.1 Particles

In this section, we review the Lagrangian formalism for classical point particle mechanics. A given particle has a kinetic energy T, and a potential energy V. From these one may form the *Lagrangian*

$$L = T - V. \tag{1}$$

As an example, a particle moving in one dimension x with time coordinate t has

$$L = \frac{1}{2}m\dot{x}^2 - V(x), \tag{2}$$

where the dot denotes differentiation with respect to t. In general, V and T will depend on more than one space coordinate, and these may not necessarily be orthogonal. The coordinates may also have dimensions other than that of length (e.g. a dimensionless angle θ is the most convenient way to describe a pendulum).

Now consider a particle which follows a path between fixed positions x_1 and x_2, at times t_1 and t_2 respectively. There are infinitely many paths connecting these positions. However, a classical particle follows only one path, namely that which extremises the *action*

$$S = \int_{t_1}^{t_2} L(\dot{x}, x)dt. \tag{3}$$

Here we have continued to assume a one-dimensional system, but the generalisation to more complicated cases is straightforward. This is known as the *principle of least action*, as S is usually minimised, and we can use eq. (3) to find an equation for the path followed by the particle, as follows.

Let $\tilde{x}(t)$ be the path followed by the particle, defined above as the path which extremises the action. We may then perturb the path by a small amount, writing

$$x(t) = \tilde{x}(t) + \epsilon(t) \tag{4}$$

for some small correction $\epsilon(t)$, where $\epsilon(t_1) = \epsilon(t_2) = 0$ due to the fixed endpoints. This in turn will perturb the action, and substituting eq. (4) into eq. (3) one finds

$$\begin{aligned} S &= \int_{t_1}^{t_2} L(\dot{\tilde{x}} + \dot{\epsilon}(t), \tilde{x}(t) + \epsilon(t))dt \\ &= \int_{t_1}^{t_2} \left[\left(\frac{\partial L}{\partial \dot{x}} \right) \dot{\epsilon}(t) + \left(\frac{\partial L}{\partial x} \right) \epsilon \right] dt + S[\tilde{x}(t)], \end{aligned} \tag{5}$$

where we have Taylor expanded L as a function of its arguments, and used the chain rule. One may perform an integration by parts in the first term to rewrite eq. (5) as

$$S = S[\tilde{x}(t)] + \delta S, \tag{6}$$

where

$$\delta S = \left[\epsilon(t) \left(\frac{\partial L}{\partial \dot{x}} \right) \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) + \left(\frac{\partial L}{\partial x} \right) \right] \epsilon(t)dt = 0, \tag{7}$$

where the final equality follows from the fact that the action is extremised by the path $\tilde{x}(t)$. In fact, the first term in eq. (7) vanishes due to the fact that $\epsilon(t) = 0$ at the endpoints, so that the principal of least action dictates that the second term vanishes, for arbitrary perturbations $\epsilon(t)$. Thus one has

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}. \tag{8}$$

This is the *Euler-Lagrange equation*, and substituting eq. (2) we see that it is equivalent (in our one dimensional example) to

$$\frac{d}{dt}(m\dot{x}) = -\frac{\partial V}{\partial x}, \tag{9}$$

which is Newton's Second Law. The Lagrangian method does a lot more, however, than allow us to rederive Newton's law. In particular:

1. Lagrangian methods greatly simplify the analyses of systems in many dimensions, and where the *generalised coordinates* may not be simple position variables (one may check that the Euler Lagrange equation has the same form for any generalised coordinate).

2. One may show quite generally in the Lagrangian approach that symmetries of a system (e.g. temporal or spatial translation invariance, rotational invariance) correspond to conserved quantities.

3. One may easily extend the Lagrangian approach to fields as well as particles, which we outline in the next section.

Before considering field theory, it is convenient to also introduce a few additional concepts. Firstly, given a Lagrangian L involving a coordinate x, one may define the *canonical momentum*

$$p = \frac{\partial L}{\partial \dot{x}}, \tag{10}$$

where p is said to be *conjugate* to x[1]. We may then define the *Hamiltonian*

$$H(p, x) = p\dot{x}(p, x) - L((\dot{x}(p, x), x), \tag{11}$$

where we have used the fact that $\dot{x}$ may be regarded as a function of x and p. Above, for example, $\dot{x} = p/m$ and one finds

$$H(x, p) = \frac{p^2}{2m} + V(x), \tag{12}$$

which is the total energy $(T + V)$ of the particle. When more than one coordinate is involved, the analogue of eq. (11) is

$$H = \sum_n p_n \dot{q}_n - L(q_n, p_n), \tag{13}$$

where the $\{q_n\}$ are the relevant generalised coordinates, and the canonical momenta are given by

$$p_n = \frac{\partial L}{\partial \dot{q}_n}. \tag{14}$$

One may again interpret H as the total energy of the system. Furthermore, one may replace the Euler-Lagrange equations with a more symmetric-looking set of first order equations. Differentiating H at constant p, one finds

$$\begin{aligned}\left(\frac{\partial H}{\partial x}\right)_p &= p\frac{\partial \dot{x}}{\partial x} - \frac{\partial L}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial x} - \frac{\partial L}{\partial x} \\ &= \frac{\partial \dot{x}}{\partial x}\left(p - \frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \\ &= -\dot{p},\end{aligned} \tag{15}$$

where we have used eq. (11) in the first line and eq. (8) in the second. Similarly, one may find

$$\left(\frac{\partial H}{\partial p}\right)_x = \dot{x}. \tag{16}$$

Equations (15) and (16) constitute *Hamilton's equations*, and they are particularly useful in relating symmetries to conservation laws. For example, if H does not depend on the position x,

[1]For our above example, $p = m\dot{x}$ and corresponds to the familiar definition of momentum. This is not necessarily the case for more complicated generalised coordinates.

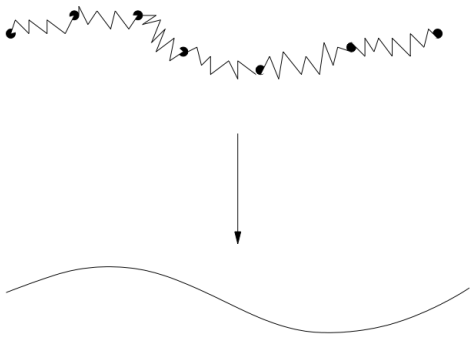

Figure 2: The transverse displacement $f(x,t)$ along a string.

then one finds that momentum is conserved directly from eq. (15). As another example, consider differentiating H with respect to t:

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p} \\ &= \frac{\partial H}{\partial t},\end{aligned} \tag{17}$$

where we have used Hamilton's equations in the second line. It follows that the left-hand side vanishes if H does not explicitly depend on t. That is, energy is conserved in a system which is invariant under time translations.

2.1.2 Fields

In classical mechanics, one does not only have particles. There may also be fields, which create forces on particles e.g. charged particles in an electromagnetic field, massive particles in a gravitational field. A particle is specified by its trajectory in space $x_i(t)$, where $\{x_i\}$ are the coordinates. A field, on the other hand, is specified by its value in space and time. For example, a given electric field may be written as $\boldsymbol{E}(\boldsymbol{x},t)$, denoting explicitly the dependence on all space and time coordinates. The quantity $\boldsymbol{E}$ here is a vector. However, fields in general may be scalar, vector or more generalised types of mathematical object such as spinors and tensors. We will consider only the case of scalar fields in the QFT course, although more complicated examples will occur in the QED / QCD course.

As a simple example of a scalar field, let us consider an example of a field in one space dimension, namely the transverse displacement $f(x,t)$ of a guitar string. This depends on the position along the string, x, as well as the time t and thus is indeed a field. How do we describe such a system? Just as in the case of point particles, we may construct the Lagrangian $L = T - V$. The kinetic energy at time t is given by

$$T = \int dx \frac{1}{2}\rho \dot{f}^2(x,t). \tag{18}$$

Here ρ is the density (mass per unit length) of the string, and $\dot{f}(x,t)$ the transverse speed of the string at position x. One then integrates over all positions to obtain the total energy. Similarly, the total potential energy is

$$V = \int dx \frac{1}{2}\sigma (f')^2(x,t), \tag{19}$$

where σ is the tension of the string, and the prime denotes differentiation with respect to x rather than t. One may then write

$$L = \int dx \mathcal{L}(f, \dot{f}, f', t), \tag{20}$$

where

$$\mathcal{L} = \frac{1}{2}\rho \dot{f}^2 - \frac{1}{2}\sigma (f')^2, \tag{21}$$

is the *Lagrangian density*, which we may also write as $\mathcal{T} - \mathcal{V}$, with $\mathcal{T}$ and $\mathcal{V}$ the kinetic and potential energy densities respectively. This structure is analogous to the point particle case, but note that the field, and not the coordinate x, appears in the Lagrangian density. The principle of least action states that the action

$$S = \int dt \int dx \mathcal{L}(f, \dot{f}, f', t) \tag{22}$$

should be extremised. This leads to the field theory version of the Euler-Lagrange equation:

$$\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t f)} + \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x f)} = \frac{\partial \mathcal{L}}{\partial f}. \tag{23}$$

In the course, we will be concerned with applying the above ideas in a relativistic context, and the appropriate generalisation of the Euler-Lagrange equation is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu f)} = \frac{\partial \mathcal{L}}{\partial f}, \tag{24}$$

where $\partial_\mu = \partial/\partial x^\mu$.

In the QFT course, we will give a fuller exposition of classical field theory. For now, let us note that, analogously to point particles, one may define the canonical momentum conjugate to the field f:

$$\pi(x, t) = \frac{\partial \mathcal{L}}{\partial \dot{f}}. \tag{25}$$

One may then construct the *Hamiltonian density*

$$\mathcal{H}(f, \pi) = \pi \dot{f} - \mathcal{L}, \tag{26}$$

whose integral over x gives the total energy carried by the field.

2.2 Quantum Mechanics

In the last section, we reviewed the Lagrangian treatment of classical mechanics, for point particles and for fields. In this section, we consider quantum mechanics. More specifically, we consider the quantum theory of point particles. We will not consider the quantum theory of fields here (i.e. that is the point of the course!). However, we will later see that some of the concepts introduced here are relevant to QFT.

To turn a classical point particle theory into a quantum theory, one replaces position and momentum variables by operators. To be more specific, let us take the one-dimensional theory whose Hamiltonian is given by eq. (12). Considering the position-space wavefunction $\Psi(x, t)$ for the particle, the position, momentum and energy operators are given by

$$\hat{x} = x, \quad \hat{p} = -i\frac{\partial}{\partial x}, \tag{27}$$

from which it is easily shown that the position and momentum operators satisfy

$$[\hat{x}, \hat{p}] = i. \tag{28}$$

The wavefunction satisfies the Schrödinger equation

$$\hat{H}\Psi(x,t) = i\frac{\partial \Psi}{\partial t}, \tag{29}$$

where $\hat{H}$ is the Hamiltonian operator (obtained straightforwardly from eq. (12))

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{30}$$

Inserting eq. (27), one obtains the explicit differential equation

$$\left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi(x) = i\frac{\partial \Psi}{\partial t}. \tag{31}$$

In the next section, we consider a particular system which will later be useful for QFT, namely the quantum harmonic oscillator.

2.2.1 The Quantum Harmonic Oscillator

The harmonic oscillator is defined by the Hamiltonian of eq. (12) with

$$V(x) = \frac{1}{2}m\omega^2 x^2, \tag{32}$$

where m is the mass of the particle in the harmonic potential well, and ω is the period of oscillation. In the quantum theory, one may use the fact that $\hat{H}$ does not depend explicitly upon t to set $\Psi(x,t) = \psi(x)\Theta(t)$ in eq. (29). Dividing also by Ψ, one finds

$$\frac{1}{\psi(x)}\hat{H}\psi(x) = \frac{i}{\Theta(t)}\frac{d\Theta(t)}{dt}. \tag{33}$$

The left-hand and right-hand sides are functions only of x and t respectively, thus must be separately equal to a constant E. The left-hand side then gives

$$\hat{H}\psi(x) = E\psi(x), \tag{34}$$

which is the *time-independent Schrödinger equation*. Given that $\hat{H}$ represents the total energy operator of the system, it follows that solutions of eq. (34) are energy eigenfunctions with energy E. The right-hand side can be solved easily to give the time dependence of the energy eigenfunctions, and need not be considered in what follows.

One may show that, as expected for a confining potential well, there is a discrete spectrum of energy eigenvalues $\{E_n\}$. One way to show this is to explicitly solve eq. (31) as a differential equation with the appropriate form of $V(x)$. However, there is a simpler way to obtain the energy spectrum which will be useful later on when we consider QFT.

Let us define the operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\hat{x} + i\sqrt{\frac{1}{m\omega}}\hat{p}\right), \tag{35}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\hat{x} - i\sqrt{\frac{1}{m\omega}}\hat{p}\right). \tag{36}$$

It is straightforward to check from eq. (28) that

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{37}$$

Furthermore, the Hamiltonian operator may be written in terms of $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{H} = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \omega\left(\hat{a}\hat{a}^\dagger - \frac{1}{2}\right) = \frac{\omega}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right), \tag{38}$$

from which it follows that

$$[\hat{H}, \hat{a}] = -\omega\hat{a}, \quad [\hat{H}, \hat{a}^\dagger] = \omega\hat{a}^\dagger. \tag{39}$$

Now consider the set of energy eigenstates $\{|n\rangle\}$. That is,

$$\hat{H}|n\rangle = E_n|n\rangle. \tag{40}$$

Acting on the left with $\hat{a}^\dagger$ and using eq. (39), one finds

$$\hat{a}^\dagger\hat{H}|n\rangle = (\hat{H}\hat{a}^\dagger - \omega\hat{a}^\dagger)|n\rangle = E_n\hat{a}^\dagger|n\rangle, \tag{41}$$

and thus

$$\hat{H}(\hat{a}^\dagger|n\rangle) = (E_n + \omega)(\hat{a}^\dagger|n\rangle). \tag{42}$$

We therefore see that $\hat{a}^\dagger|n\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue E_{n+1}, i.e.

$$\hat{a}^\dagger|n\rangle \propto |n+1\rangle. \tag{43}$$

Likewise, one may show that

$$\hat{H}(\hat{a}|n\rangle) = (E_n - \omega(\hat{a}|n\rangle). \tag{44}$$

and thus that

$$\hat{a}|n\rangle \propto |n-1\rangle. \tag{45}$$

Based on eqs. (43) and (45), $\hat{a}^\dagger$ and $\hat{a}$ are referred to as *raising* and *lowering* operators respectively, and collectively as *ladder operators*.

Let us now show that there is a minimum energy state. This follows from the fact that the norm of any state $|\phi\rangle$ must be non-negative i.e.

$$\langle\phi|\phi\rangle \geq 0. \tag{46}$$

Consider the state $|\phi\rangle = \hat{a}|n\rangle$. The conjugate of this state, as can be found from eqs.(35) and (36), is $\langle n|\hat{a}^\dagger$, so that

$$\langle n|\hat{a}^\dagger\hat{a}|n\rangle \geq 0. \tag{47}$$

However, using eq. (38) we may rewrite this as

$$\begin{aligned}\langle n|\left(\frac{\hat{H}}{\omega} - \frac{1}{2}\right)|n\rangle &= \langle n|\left(\frac{E_n}{\omega} - \frac{1}{2}\right)|n\rangle \\ &= \frac{E_n}{\omega} - \frac{1}{2} \geq 0,\end{aligned} \tag{48}$$

where we have assumed the normalisation $\langle n|n\rangle$ in the second line. We thus find that the minimum energy state $|0\rangle$ has energy $E_0 = \omega/2$. The first excited state of the system is

$$|1\rangle = \hat{a}^\dagger|0\rangle \tag{49}$$

and has energy $(1+1/2)\omega$. The n_{th} excited state has energy $(n+1/2)\omega$, and is given by

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle, \tag{50}$$

where the prefactor guarantees that $\langle n|n\rangle = 1$. Thus we may write the action of the Hamiltonian (eq. (38)) as

$$\hat{H}|n\rangle = \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\omega|n\rangle = \left(\hat{n} + \frac{1}{2}\right)|n\rangle, \tag{51}$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$ is the *number operator*. That is, $|n\rangle$ is an eigenstate of $\hat{n}$, which has energy eigenvalue $(n+1/2)\omega$ so that the number n represents the number of quanta with energy ω which have been introduced to the ground state $|0\rangle$ of the system. We may think of $\hat{a}^\dagger$ and $\hat{a}$ as creating and annihilating quanta respectively, i.e. as *creation* and *annihilation* operators.

2.2.2 The Schrödinger, Heisenberg and Interaction Pictures

QM is usually taught at undergraduate level in terms of what is called the "Schrödinger picture". In this view, the particle state (i.e. the wavefunction) is a function of time, while the operators are constant (unless they have an intrinsic time dependence). Specifically, between measurements, the time dependence of the particle state $|\psi(t)\rangle$ is given by

$$i\frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle \tag{52}$$

which has a general solution for time-independent Hamilton operators of

$$|\psi_S(t)\rangle = e^{-it\hat{H}}|\psi_{0,S}\rangle \tag{53}$$

where $|\psi_{0,S}\rangle$ is the state at the last measurement, take as $t=0$ for convenience. We keep the subscript "S" to highlight that the state is defined in the Schrödinger picture. If at a later time we measure a quantity corresponding to a (Schrödinger) operator $\hat{O}_S$, we find

$$\langle\hat{O}_S(t)\rangle = \langle\psi_S(t)|\hat{O}_S|\psi_S(t)\rangle = \langle\psi_{0,S}|e^{it\hat{H}}\hat{O}_S e^{-it\hat{H}}|\psi_{0,S}\rangle\,. \tag{54}$$

The last step makes clear that the way we set up QM is not unique! Instead of working in the "Schrödinger picture", we can introduce the "Heisenberg picture" which puts all the time dependence into the operators,

$$\hat{O}_H(t) = e^{it\hat{H}}\hat{O}_S e^{-it\hat{H}} \tag{55}$$

and the above equation becomes

$$\langle\hat{O}_H(t)\rangle = \langle\psi_H|\hat{O}_H(t)|\psi_H\rangle\,. \tag{56}$$

with the particle state being manifestly constant over time $|\psi_H\rangle = |\psi_{0,S}\rangle$.

We are only shuffling the time dependence between states and operators. More precisely we perform a basis transformation as a function of time. From the above equation we can see that the transformation matrix is given by

$$\hat{O}_H(t) = e^{it\hat{H}}\hat{O}_S e^{-it\hat{H}} = \hat{U}^\dagger(t)\hat{O}_S\hat{U}(t) \tag{57}$$

with $\hat{U}(t) = e^{-it\hat{H}}$ being the time evolution operator (in the Schrödinger picture). The time evolution can be calculated by taking the time derivative of the above equation:

$$\frac{\partial}{\partial t}\hat{O}_H = i[\hat{H}, \hat{O}_H] \tag{58}$$

Physical quantities are unaffected by how we choose our basis. This means we could have dropped $\langle\hat{O}_S\rangle = \langle\hat{O}_H\rangle = \langle\hat{O}\rangle$: the expectation value of an observable depends only on the physics encoded in the operator and initially prepared state and not on how we choose our basis. Since $\hat{H}$ commutes with itself, the Hamiltonian operator has no time dependence so $\hat{H}_H = \hat{H}_S$ at all times and the subscripts are usually suppressed (as we have done in this section throughout).

It is convenient to use the Schrödinger picture for single particle quantum mechanics as in this case the particle position is an observable. The initial wavefunction (in the spatial representation) is $|\psi_0(x)\rangle$, so that the state at a later time is $|\psi(x,t)\rangle$. This keeps the space and time components together, making the transition to a relativistic single particle QM much easier. In contrast, a field does not have a measurable position, but has an amplitude at all points in space. Hence position is no longer an observable but position becomes a label, e.g. we observe the amplitude of the field at a particular position. Therefore, when quantising a (non-interacting) field in QFT, then the operator for an observable depends on position, i.e. $\hat{O}_S(x)$. If we used the Schrödinger picture, we would still have the time dependence in the wavefunction. Hence, by changing to the Heisenberg picture when quantising fields, we have $\hat{O}_H(x,t)$ so we again keep the space and time components together.

There is a third view, the "Interaction picture" (or "Dirac picture"), which is useful when working with perturbation theory, as we will do in QFT. In these cases, the Hamiltonian has a piece which can be solved analytically $\hat{H}_0$ and an "interaction" piece $\hat{H}_1$ which we handle as a (hopefully) small perturbation on the solutions of $\hat{H}_0$:

$$\hat{H} = \hat{H}_0 + \hat{H}_1 \tag{59}$$

In this picture, we choose to put the time dependence resulting from the soluble part of the Hamiltonian into the operators and the rest goes into the particle state. This means the time dependence of the state is just due to the interaction term and so this picture becomes equivalent to the Heisenberg picture as the interaction strength goes to zero. Specifically

$$\hat{O}_I(t) = e^{it\hat{H}_0}\hat{O}_S e^{-it\hat{H}_0} \tag{60}$$

and using invariance of physical observables under this transformation

$$\langle O\rangle = \langle\psi_S(t)|\underbrace{e^{-it\hat{H}_0}e^{it\hat{H}_0}}_{=1}\hat{O}_S\underbrace{e^{-it\hat{H}_0}e^{+it\hat{H}_0}}_{=1}|\psi_S(t)\rangle\,. \tag{61}$$

we see

$$|\psi_I(t)\rangle = e^{it\hat{H}_0}|\psi_S(t)\rangle = e^{it\hat{H}_0}e^{-it\hat{H}}|\psi_0\rangle = \tilde{U}(t)|\psi_0\rangle\,. \tag{62}$$

This implies a time evolution of the Interaction picture states

$$i\frac{\partial}{\partial t}|\psi_I(t)\rangle = e^{it\hat{H}_0}\underbrace{(H-H_0)}_{=H_1}e^{-it\hat{H}}|\psi_0\rangle = e^{it\hat{H}_0}H_1\underbrace{e^{-it\hat{H}_0}e^{+it\hat{H}_0}}_{=1}e^{-it\hat{H}}|\psi_0\rangle = H_{1,I}|\psi_I(t)\rangle\,, \tag{63}$$

i.e. the states transform only with the interaction piece of the Hamiltonian *in the Interaction basis*. The relation with the Heisenberg picture operators is

$$\hat{O}_H(t) = e^{it\hat{H}}\hat{O}_S e^{-it\hat{H}} = e^{it\hat{H}}e^{-it\hat{H}_0}e^{it\hat{H}_0}\hat{O}_S e^{-it\hat{H}_0}e^{it\hat{H}_0}e^{-it\hat{H}} = e^{it\hat{H}}e^{-it\hat{H}_0}\hat{O}_I(t)e^{it\hat{H}_0}e^{-it\hat{H}} \tag{64}$$

and hence

$$\hat{O}_H(t) = \tilde{U}^\dagger(t)\hat{O}_I(t)\tilde{U}(t) \tag{65}$$

The time evolution of an operator in the Interaction picture can be read off Eq. (60)

$$\frac{\partial}{\partial t}\hat{O}_I = i[\hat{H}_0, \hat{O}_I]\,. \tag{66}$$

Note for general (i.e. non-commuting) matrices A and B that $e^A e^B \neq e^{A+B}$, so generally $U(t) \neq e^{-it\hat{H}_1}$. However if H_0 and H_1 commute, we see the principle strategy that motivates the Interaction picture: If we collect all interactions into H_1 then the time dependence of operators in the Interaction picture is given by the free (i.e. non-interacting) theory only, the time evolution of states feels only the interaction piece.

2.3 Index notation and relativity

For the "normal" three-vectors, the indices are usually written with Roman indices, e.g. a_i for $i = 1, 2, 3$. The dot product of two vectors can be written as $\mathbf{a}.\mathbf{b} = a_i b_i$, where the Einstein summation convention is used, namely that any repeated index is implicitly being summed over, so this is equivalent to writing $\mathbf{a}.\mathbf{b} = \sum_i a_i b_i$. This means any repeated index is a dummy index and so can be relabelled at will, e.g. $a_i b_i = a_j b_j$. The vectors can be considered as column vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{67}$$

The dot product is then

$$\mathbf{b}^T \mathbf{a} = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{68}$$

This can be generalised to matrices. A rotation of a vector by a rotation matrix $\mathbf{R}$ can be written as $\mathbf{a}' = \mathbf{R}\mathbf{a}$ which in terms of column vectors and explicit matrices is

$$\begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \text{i.e.} \qquad a_i' = R_{ij} a_j \tag{69}$$

Note the order of the indices for the matrix is important since generally $R_{ij} \neq R_{ji} = (R^T)_{ij}$. The convention is that the first index gives the row number and the second index the column numbers. By the usual rules of matrix multiplication, the column index is the one which is summed with the vector. The trick to converting from index-type equations to matrix-type equations and vice versa is then simple; make sure the summed indices are next to each other, as for j above. Similarly, for the inverse equation, we would write $\mathbf{a} = \mathbf{R}^{-1}\mathbf{a}'$, which is found from the original equation by

$$(R^{-1})_{ki} a_i' = (R^{-1})_{ki} R_{ij} a_j = \delta_{kj} a_j = a_k \tag{70}$$

where $\delta_{kj} = I_{kj}$, i.e. the unit matrix. Note, the unsummed indices on both sides of any equation must match, as shown by k in the above. Another common property of a matrix is its trace, meaning the sum of the diagonal elements. Using Einstein summation, the trace is simply written as

$$\mathrm{Tr}(R) = R_{ii} \tag{71}$$

and hence any expression with the first and last index repeated means take the trace. For example, $A_{ij} B_{jk} C_{ki}$ means multiply the matrices A, B and C together and then find the trace of the result.

Matrices themselves also change under a rotation. The simple rule is that "each index is rotated independently". A vector has one index and so needs one rotation matrix to transform it, as shown above. A matrix, with two indices, therefore needs to be multiplied by two rotation matrices. Explicitly for matrix M_{ij}, then

$$M_{ij}' = R_{ik} R_{jl} M_{kl} \tag{72}$$

Note, each term in the above (which has two implied summations, of course) is a simple number, not a matrix and so they can be rearranged at will. Hence, we could also write

$$M'_{ij} = R_{ik} M_{kl} R_{jl} = R_{ik} M_{kl} (R^T)_{lj} \tag{73}$$

which now brings all the summed index pairs together and hence is in the right form to convert to a matrix-type equation, namely $\mathbf{M}' = \mathbf{R}\mathbf{M}\mathbf{R}^T$.

There is one particular tensor which we will use, namely the Levi-Civita antisymmetric tensor ϵ_{ijk}. This is defined by $\epsilon_{123} = 1$ and then every exchange of any two of the three indices changes the sign, so e.g. $\epsilon_{321} = -1$, $\epsilon_{312} = 1$, etc. If it has a repeated index, then it is zero. This should transform as

$$\epsilon'_{ijk} = R_{il} R_{jm} R_{kn} \epsilon_{lmn} \tag{74}$$

but due to the properties of a rotation matrix, this gives $\epsilon'_{ijk} = \epsilon_{lmn}$, i.e. this tensor is invariant under rotations. Note, the Kronecker delta symmetric tensor δ_{ij} has the same property of invariance also.

When we get to relativity, then the indices become a little more complicated as they can be subscripts or superscripts. Let S' (coordinates (t', x', y', z')) be an inertial frame moving with speed v along the x axis with respect to a second inertial frame S (coordinates (t, x, y, z)). The Lorentz transformation relating the two sets of coordinates is

$$t' = \gamma t - \beta\gamma x, \quad x' = \gamma x - \beta\gamma t, \quad y' = y, \quad z' = z, \tag{75}$$

where $\beta = v$, $\gamma = 1/\sqrt{1 - v^2}$ and we work in natural units such that the speed of light $c = 1$. This can be written as

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{76}$$

In general, any Lorentz transformation can be represented by a 4×4 matrix Λ as follows:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \tag{77}$$

where the convention is to use Greek indices for four-vectors, and μ, $\nu = 0$, 1, 2, 3. Vectors A^{μ} and B_{μ} that transform like x^{μ} and x_{μ} are sometimes called *contravariant* and *covariant* respectively. A simpler pair of names is *vector* and *covector*. To convert between the two, we use the metric g

$$x_{\mu} = g_{\mu\nu} x^{\nu} \tag{78}$$

where g is given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{79}$$

Hence

$$x^{\mu} = (t, \boldsymbol{r}) \qquad \text{while} \qquad x_{\mu} = (t, -\boldsymbol{r}) \tag{80}$$

Generally, you can think of g as being the way to raise (or lower) an index on any Lorentz quantity, not just a vector. A particularly important covector is obtained by letting $\partial/\partial x^{\mu}$ act on a scalar ϕ:

$$\frac{\partial \phi}{\partial x^{\mu}} \equiv \partial_{\mu} \phi \ . \tag{81}$$

Note, this means

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \qquad \text{and so} \qquad \partial^\mu = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right) \tag{82}$$

which therefore has the $\pm$ the opposite way round to usual; it is a common mistake to get this wrong. However, this is essential as then

$$\hat{p}^\mu = (\hat{E}, \hat{\boldsymbol{p}}) = i\partial^\mu = \left(i\frac{\partial}{\partial t}, -i\boldsymbol{\nabla}\right) \tag{83}$$

gives the correct sign for $\hat{\boldsymbol{p}}$, as known from QM.

The dot product of two four-vectors x and y is defined as

$$x \cdot y \equiv x^\mu y^\nu g_{\mu\nu} = x_\mu y_\nu g^{\mu\nu} = x^\mu y_\mu = x_\mu y^\mu \ . \tag{84}$$

In particular x^2 is given by

$$x^2 = x \cdot x = x^\mu x_\mu \ . \tag{85}$$

If $x \cdot y = 0$, x and y are orthogonal. A four-vector x is said to be light-like if $x^2 = 0$; time-like if $x^2 > 0$; space-like if $x^2 < 0$.

3 QFT Problems

Here are some problems to check your understanding of the above material.

3.1 Classical Mechanics

1. Show from the Euler-Lagrange equation that adding a constant to the potential energy V of a system is irrelevant i.e. does not affect the equations of motion.

2. A particle of mass m moves in the gravitational potential near the Earth's surface with height z. Write down the Lagrangian, and show that the equations of motion are what you would expect. Derive also the Hamiltonian, and show that this is indeed the total energy of the particle.

3. Write down the Lagrangian for the motion of a particle of mass m in a spherically symmetric potential $V = V(r)$. You may use the fact that the dot product in spherical polar coordinates (r, θ, ϕ) is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{pmatrix} a_r & a_\theta & a_\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \begin{pmatrix} b_r \\ b_\theta \\ b_\phi \end{pmatrix}.$$

Show that the radial equation of motion is given by

$$\ddot{r} - r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{1}{m}\frac{dV}{dr} = 0.$$

Also show that the Hamiltonian of the particle is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V(r).$$

Show explicitly that p_ϕ is the angular momentum L_z. Hence, what conserved quantity does the independence of the Hamiltonian on the azimuthal coordinate ϕ correspond to?

3.2 Classical Field Theory

4. Give examples of (a) a scalar field in three dimensions; (b) a vector field in three dimensions. Point particle mechanics can be considered as a field theory in zero space dimensions. Why?

5. For the field theory whose Lagrangian density is given by eq. (21), show that the Euler-Lagrange equation (eq. (24)) has the form

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\rho}{\sigma}\frac{\partial^2 f(x,t)}{\partial t^2}.$$

Do you recognise this equation? Write down the canonical momentum conjugate to f, and the Hamiltonian density. Verify that the integral of the Hamiltonian density is the total energy carried by the field.

6. Consider the action

$$S = \int d^4x\, \mathcal{L}(\partial_\mu \phi, \phi)$$

for a relativistic field ϕ. Show that the principle of least action implies the Euler-Lagrange equation of eq. (24). The proof is analogous to the point particle case.

3.3 Quantum Mechanics

7. Starting from the Schrödinger equation for the wave function $\psi(\boldsymbol{x}, t)$, show that the probability density $\rho = \psi^*\psi$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0 \ , \tag{86}$$

where

$$\boldsymbol{J} = \frac{1}{2im}\left[\psi^*(\boldsymbol{\nabla}\psi) - (\boldsymbol{\nabla}\psi^*)\psi\right] \ . \tag{87}$$

What is the interpretation of $\boldsymbol{J}$?

8. One may derive the Schrödinger equation for a free particle ($V = 0$) in three space dimensions by starting with the kinetic energy definition

$$E = \frac{\boldsymbol{p}^2}{2m},$$

and substituting

$$E \to i\frac{\partial}{\partial t}, \quad \boldsymbol{p} \to -i\nabla,$$

which operate on a wavefunction $\Psi(\boldsymbol{x}, t)$) to get

$$\left[-\frac{1}{2m}\nabla^2\right]\Psi = i\frac{\partial \Psi}{\partial t}.$$

Show that carrying out the same procedure on the relativistic kinetic energy relation

$$E^2 = \boldsymbol{p}^2 + m^2,$$

one gets the equation

$$\left(\Box + m^2\right)\Phi = 0 \tag{88}$$

for some wavefunction $\Phi(\boldsymbol{x}, t)$, where

$$\Box = \frac{\partial^2}{\partial t^2} - \nabla^2$$

is the d'Alembertian operator. Equation (88) is the *Klein-Gordon equation* for relativistic scalar particles.

9. Verify eqs. (37), (38) and (39).

10. Check the normalisation of the state $|n\rangle$ in eq. (50). Hint: show that if $|n\rangle$ is correctly normalised, then so is $\hat{a}^\dagger|n\rangle \propto |n+1\rangle$.

11. Show for any two Hermitian operators $\hat{A}$ and $\hat{B}$ that $i[\hat{A}, \hat{B}]$ is also a Hermitian operator.

    Using the Heisenberg operator definition

$$\hat{O}_H(t) = e^{it\hat{H}} \hat{O}_S e^{-it\hat{H}} \tag{89}$$

    show that

$$\frac{\partial \hat{O}_H}{\partial t} = i\left[\hat{H}, \hat{O}_H\right] = \left(i\left[\hat{H}, \hat{O}_S\right]\right)_H \tag{90}$$

    where the final expression denotes the Heisenberg picture operator corresponding to the Schrödinger picture operator formed from i times the commutator of $\hat{H}$ with $\hat{O}_S$.

12. This question looks at a simple equivalent to going from bosons to fermions in QFT.

    Every operator commutes with itself so, including eq. (37), we can trivially write

$$[\hat{a}, \hat{a}] = 0, \qquad [\hat{a}^\dagger, \hat{a}^\dagger] = 0, \qquad [\hat{a}, \hat{a}^\dagger] = 1 \tag{91}$$

    Also, using eq. (38), the Hamiltonian operator can be written

$$\hat{H} = \frac{\omega}{2}\left(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger\right) = \frac{\omega}{2}\{\hat{a}^\dagger, \hat{a}\}, \tag{92}$$

    let's explore what happens if we change the commutators to anticommutators and vice versa, i.e.

$$\{\hat{a}, \hat{a}\} = 0, \qquad \{\hat{a}^\dagger, \hat{a}^\dagger\} = 0, \qquad \{\hat{a}, \hat{a}^\dagger\} = 1 \tag{93}$$

    and

$$\hat{H} = \frac{\omega}{2}[\hat{a}^\dagger, \hat{a}] \tag{94}$$

    Verify the commutators of the Hamiltonian operator with $\hat{a}$ and $\hat{a}^\dagger$ are the same as previously given in eq. (39). This means $\hat{a}$ and $\hat{a}^\dagger$ still act as ladder operators, and so change the energy by $\pm\omega$.

    However, starting from eq. (47), show that we now have

$$\frac{E_n}{\omega} \geq -\frac{1}{2} \tag{95}$$

    Applying a similar argument to $\langle n|\hat{a}\hat{a}^\dagger|n\rangle$, show that

$$\frac{E_n}{\omega} \leq \frac{1}{2} \tag{96}$$

    What do you deduce about the number of possible states? Is this a surprise given that eq. (93) implies $\hat{a}\hat{a} = 0$ and $\hat{a}^\dagger\hat{a}^\dagger = 0$?

Using eqs. (35) and (36), show that $\hat{x}$ and $\hat{p}$ now satisfy

$$\{\hat{x}, \hat{x}\} = \frac{1}{m\omega}, \qquad \{\hat{p}, \hat{p}\} = m\omega, \qquad \{\hat{x}, \hat{p}\} = 0 \tag{97}$$

and that

$$\hat{H} = \frac{i\omega}{2}\,[\hat{x}, \hat{p}] \tag{98}$$

Show that the Heisenberg operators for $\hat{x}$ and $\hat{p}$ obey the classical equations of motion and so this new system does still represent a quantised harmonic oscillator.

(Clearly, the above anticommutators for $\hat{x}$ and $\hat{p}$ show that they cannot be the usual x and $-i\partial/\partial x$ operators. If you want a concrete example of these operators, then take

$$\hat{x} = \frac{1}{\sqrt{2m\omega}}\sigma_1, \qquad \hat{p} = -\sqrt{\frac{m\omega}{2}}\sigma_2 \tag{99}$$

where the σ_i are the Pauli matrices. It is instructive to evaluate $\hat{a}$, $\hat{a}^\dagger$ and $\hat{H}$ from these and check they obey all the above relations.)

3.4 Relativity

13. In Euclidean space, we do not usually distinguish between covariant and contravariant indices, whereas in Minkowski space this is mandatory. Why?

14. What is the inverse matrix g^{-1} of the metric g? Since g is used to lower a Lorentz index, then any tensor $T^{\mu\nu}$ would have both its indices lowered through

$$T_{\rho\sigma} = g_{\rho\mu}\, g_{\sigma\nu}\, T^{\mu\nu},$$

where the right-hand side can be written in matrix form as $g\,T\,g^T = g\,T\,g$, as g is symmetric. Hence, choosing $T_{\rho\sigma} = g_{\rho\sigma}$, find the matrix corresponding to $T^{\mu\nu}$, i.e. solve $g = gTg$ for T. This is clearly the definition of $g^{\mu\nu}$, which is used to raise an index.

15. If $x^\mu = (t, x, y, z)$ is the position four-vector, the general Lorentz transformation matrix $\Lambda^\mu{}_\nu$ gives

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu.$$

Take $\Lambda^\mu{}_\nu$ from eq. (76). How do $\Lambda_{\mu\nu}$, $\Lambda_\mu{}^\nu$ and $\Lambda^{\mu\nu}$ differ from this? In particular, how is $\Lambda_\mu{}^\nu$ related to $\Lambda^\mu{}_\nu$?

16. Show that $\partial_\mu\phi$ does transform like x_μ and not x^μ (i.e. it is a covector).

17. By imposing the condition

$$g_{\mu\nu}x'^\mu x'^\nu = g_{\mu\nu}x^\mu x^\nu \tag{100}$$

show that

$$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma} \quad \text{or} \quad \Lambda^T g \Lambda = g\ . \tag{101}$$

(This is the analogue of the orthogonality relation for rotations in Sec. 4.1.) Check that it works for the Λ given by eq. (76).

18. Show, by reconsidering equation (100) using $x^\mu x_\mu$, or otherwise, that

$$x'_\mu = x_\nu(\Lambda^{-1})^\nu{}_\mu\ . \tag{102}$$

19. Show that any four-vector orthogonal to a time-like four-vector is space-like.

20. State which of the following are (a) Lorentz scalars; (b) Lorentz vectors; (c) Lorentz tensors of higher rank.

    (i) $\partial_\alpha x^\alpha$;
    (ii) $\phi(x^\alpha)$;
    (iii) $x^\alpha\, x^\beta\, x_\alpha$;
    (iv) $x_\alpha x_\beta$.

21. Prove the following:

    (i) $\partial_\alpha(x^\alpha\, x^\beta) = (D+1)x^\beta$;
    (ii) $g_{\mu\nu}\, x^\mu\, x^\nu = g^{\mu\nu}\, x_\mu\, x_\nu$,

    where D is the dimension of spacetime.

22. Let $k_\mu = (E, \boldsymbol{k})$, $k'_\mu = (E', \boldsymbol{k}')$ and $x^\mu = (t, \boldsymbol{x})$ be four-vectors. Given the identity

$$\int \frac{d^3\boldsymbol{x}}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} = \delta^{(3)}(\boldsymbol{p})$$

    show that

$$\int \frac{d^3\boldsymbol{x}}{(2\pi)^3} e^{i(k-k')\cdot x} = e^{i(E-E')t}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}').$$

    What does this imply about plane waves with different momenta?

4 Introduction to QED and QCD

4.1 Rotations, Angular Momentum, Groups and Generators

A general rotation of an object in 3-dimensions needs three rotation angles, which can be considered to be around the x, y and z axes. Such a 3D rotation can be represented by a 3×3 orthogonal matrix R with determinant $+1$, where orthogonal means that $R^TR = \mathbb{1}$, i.e. $R^T = R^{-1}$. This matrix can be applied to a vector to transform it to a rotated vector. There are an infinite number of such matrices, corresponding to all possible values of the three rotation angles needed in 3D, and together these matrices form an example of a group. This group is called the special (meaning determinant $= +1$) orthogonal group, $SO(3)$, where the 3 gives the size of the matrix.

A group is a collection of objects which have particular properties, namely: when multiplied together, the result must also be a member of the group; the multiplication must be associative; there must be a unit member which does nothing when it multiplies another member; and there must be an inverse member for each member such that their product gives the unit member.

The group formed by the rotation matrices is an example of a Lie (pronounced "Lee") group. These have the property that a large rotation can always be built up out of multiple small rotations. In the infinitesimal angle limit, any rotation matrix can be built out of a unit matrix plus three (infinitesimally) small angles multiplying three other matrices

$$R = \mathbb{1} + i\delta\alpha_i L_i \tag{103}$$

where $\delta\alpha_i$ is the infinitesimal angle of rotation around axis i. The three matrices L_i are called the 'generators' of the group, since by multiple applications, they can generate a finite rotation. Since R is real and orthogonal, then the L_i must be Hermitian, meaning $L_i^\dagger = L_i$. As such, they correspond to operators of observables in QM, in this case angular momentum operators for spin-1. For a finite rotation by α_i, then the general matrix can be written as

$$R = e^{i\alpha_i L_i} \tag{104}$$

These concepts can be generalised to other matrix groups. We will be interested in unitary matrices, for which $U^\dagger = U^{-1}$, where $U^\dagger$ denotes the Hermitian conjugate, i.e. the complex conjugate and transpose; $U^\dagger \equiv U^{*T}$. (Note, for a real matrix, then unitary is equivalent to orthogonal.) The collection of unitary matrices of size $N \times N$ with determinant $+1$ also form a group, now called $SU(N)$ with "U" for unitary, and these are used to transform complex vectors with N components. As for the rotation case, an infinitesimal transformation by some real parameters $\delta\alpha_i$ can be written as

$$U = \mathbb{1} + i\delta\alpha_i X_i \tag{105}$$

in terms of some generators X_i. To ensure U is unitary, these generators must also be Hermitian and again for a finite transformation

$$U = e^{i\alpha_i X_i} \tag{106}$$

Generally, for group $SU(N)$, there are $N^2 - 1$ parameters.

One such group is $SU(2)$ which has $2^2 - 1 = 3$ parameters. Its generators can be defined in terms of the Pauli matrices σ_i

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{107}$$

such that the generators $X_i = \sigma_i/2$. The X_i are Hermitian and these generators are known to correspond to spin-1/2 operators in QM. $SU(2)$ is very closely related to $SO(3)$.

4.2 Scattering in Quantum Mechanics

The scattering of a particle of mass m by a potential $V(\boldsymbol{x})$ can be described in nonrelativistic quantum mechanics by using the Born approximation. The result for the scattering cross section in the first-order Born approximation is given by

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2} \, |\widetilde{V}(\boldsymbol{q})|^2 \ , \tag{108}$$

where $d\sigma/d\Omega$ is the differential cross section in the solid angle element, $\widetilde{V}(\boldsymbol{q})$ is the Fourier transform of the potential given by

$$\widetilde{V}(\boldsymbol{q}) = \int d^3x \ e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \ V(\boldsymbol{x}) \ , \tag{109}$$

and $\boldsymbol{q}$ is the momentum transferred in the scattering.

4.3 Relativistic Kinematics

The four-velocity is defined as

$$u^\mu = \frac{dx^\mu}{d\tau} \ , \tag{110}$$

where $dx^\mu = (dt, \ d\boldsymbol{x})$, and $d\tau$ is the proper time given by

$$d\tau = \sqrt{dx^\mu dx_\mu} \ . \tag{111}$$

The four-momentum of a particle of mass m is defined as

$$p^\mu = mu^\mu = (E \, , \ \boldsymbol{p}) \tag{112}$$

where E is the energy and $\boldsymbol{p}$ is the three-momentum. The kinetic energy of a particle of momentum $p^\mu = (E, \boldsymbol{p})$ and mass m is given by

$$T = E - m \ . \tag{113}$$

Consider the relativistic scattering of two particles with four-momenta p_1 and p_2 into two particles with four-momenta p_3 and p_4, shown in fig. 3. By four-momentum conservation,

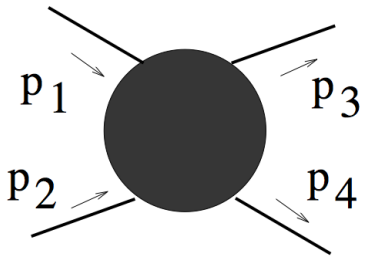

Figure 3: Relativistic $2 \to 2$ scattering.

$$p_1 + p_2 = p_3 + p_4 \ . \tag{114}$$

The particles' masses are given by

$$p_i^2 = m_i^2 \ , \quad i = 1, 2, 3, 4 \ . \tag{115}$$

We can define three quantities (called the "Mandelstam variables")

$$s = (p_1{+}p_2)^2 = (p_3{+}p_4)^2, \qquad t = (p_1{-}p_3)^2 = (p_4{-}p_2)^2, \qquad u = (p_1{-}p_4)^2 = (p_3{-}p_2)^2, \tag{116}$$

where the equalities above follow trivially from eq. (114). Being dot products of four-momenta, these are clearly Lorentz invariants.

4.4 Electromagnetism

Classical electromagnetism is summed up in the Maxwell equations, two of which are homogeneous:

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \partial_t \boldsymbol{B} = 0 \quad , \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \quad , \tag{117}$$

and two of which are inhomogeneous:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho \quad , \qquad \boldsymbol{\nabla} \times \boldsymbol{B} - \partial_t \boldsymbol{E} = \boldsymbol{J} \quad , \tag{118}$$

where we have introduced the short-hand notation $\partial_t \equiv \partial/\partial t$. The homogeneous equations are automatically solved by introducing the potentials ϕ and $\mathbf{A}$ as follows:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \partial_t \boldsymbol{A} \quad , \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \quad . \tag{119}$$

The $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are unchanged under a gauge transformation on the potentials of the form

$$\phi' = \phi + \partial_t \lambda \quad , \quad \boldsymbol{A}' = \boldsymbol{A} - \boldsymbol{\nabla}\lambda \quad , \tag{120}$$

where λ is an arbitrary function.

In relativistic notation, we define the four-potential

$$A^\mu = (\phi \ , \ \boldsymbol{A}) \tag{121}$$

and the four-current

$$j^\mu = (\rho \ , \ \boldsymbol{J}) \tag{122}$$

such that the gauge transformation is

$$A'^\mu = A^\mu + \partial^\mu \lambda \tag{123}$$

The $\boldsymbol{E}$ and $\boldsymbol{B}$ fields form the components of a second-rank antisymmetric tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{124}$$

The inhomogeneous Maxwell equations eq. (118) can then be written in relativistically covariant form as

$$\partial_\mu F^{\mu\nu} = j^\nu \quad . \tag{125}$$

Note, since $F^{\mu\nu}$ is antisymmetric that

$$\partial_\nu \partial_\mu F^{\mu\nu} = 0 = \partial_\nu j^\nu \quad . \tag{126}$$

meaning that the current must be conserved.

5 QED/QCD Problems

5.1 Rotations, Angular Momentum, Groups and Generators

1. The definition of a 3D rotation is that the length of a vector is not changed. Starting with $\boldsymbol{x}' = R\,\boldsymbol{x}$, and impose $\boldsymbol{x}' \cdot \boldsymbol{x}' = \boldsymbol{x} \cdot \boldsymbol{x}$, show that any 3D rotation matrix must be orthogonal with determinant $+1$.

2. For an *infinitesimal* rotation, write $R = \mathbb{1} + iA$ where $\mathbb{1}$ is the identity matrix and A is a matrix with infinitesimal entries. Show that A is antisymmetric and that $A^* = -A$ and hence deduce that A must be Hermitian.

3. Parameterise A as

$$A = \begin{pmatrix} 0 & -ia_3 & ia_2 \\ ia_3 & 0 & -ia_1 \\ -ia_2 & ia_1 & 0 \end{pmatrix} \equiv \sum_{i=1}^{3} a_i L_i, \tag{127}$$

where the a_i are infinitesimal. Write down explicit matrices for the L_i and verify that they satisfy the angular momentum commutation relations

$$[L_i, L_j] = i\varepsilon_{ijk} L_k. \tag{128}$$

Compute $L^2 \equiv L_1^2 + L_2^2 + L_3^2$. What is the interpretation of L^2?

4. Verify that $\frac{1}{2}\sigma_i$ are Hermitian and satisfy the same commutation relations as the L_i.

5. Verify the following identity for the Pauli σ matrices:

$$(\boldsymbol{\sigma} \cdot \mathbf{a})\ (\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\ \boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}), \tag{129}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, $\mathbf{a}$ and $\mathbf{b}$ are any two vectors, $\mathbf{a} \cdot \mathbf{b}$ is their inner product, and $\mathbf{a} \times \mathbf{b}$ is their cross product.

5.2 Scattering in Quantum Mechanics

6. Verify that for a spherically symmetric potential, $V(\boldsymbol{x}) = f(r)$ with $r = |\boldsymbol{x}|$, the integral in eq. (109) takes the form

$$\int d^3\boldsymbol{x}\ e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\ V(\boldsymbol{x}) = \frac{4\pi}{|\boldsymbol{q}|} \int_0^\infty dr\ r\ \sin(|\boldsymbol{q}|r)\ f(r)\ . \tag{130}$$

7. Using eqs. (108) and (130), show that the cross section for the scattering by a Yukawa potential of the form

$$V(\boldsymbol{x}) = C\ \frac{e^{-\mu|\boldsymbol{x}|}}{|\boldsymbol{x}|} \quad , \qquad \mu > 0\ , \tag{131}$$

is given by

$$\frac{d\sigma}{d\Omega} = \frac{4C^2m^2}{(\mu^2 + \boldsymbol{q}^2)^2}\ . \tag{132}$$

8. By making the replacements

$$C \to \alpha \quad , \qquad \mu \to 0 \tag{133}$$

in eqs. (131) and (132), obtain the cross section for the scattering off a Coulomb potential. Expressing the result in terms of the incoming particle momentum $\boldsymbol{p} = m\boldsymbol{v}$ and the scattering angle θ via the kinematic relation $\boldsymbol{q}^2 = 4\boldsymbol{p}^2 \sin^2(\theta/2)$, show that the Coulomb scattering cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Coul.}} = \frac{\alpha^2}{4\ \boldsymbol{p}^2\ \boldsymbol{v}^2\ \sin^4(\theta/2)}\ . \tag{134}$$

This equals the classical Rutherford scattering cross section.

5.3 Relativistic Kinematics

9. Show that

$$u^\mu = \left(\frac{1}{\sqrt{1-\boldsymbol{v}^2}}\,,\ \frac{\boldsymbol{v}}{\sqrt{1-\boldsymbol{v}^2}} \right) \quad , \quad \text{where } \boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} \quad , \tag{135}$$

and that

$$u^2 = u^\mu u_\mu = 1 \ . \tag{136}$$

10. Show that in the non-relativistic limit $|\boldsymbol{p}| \ll mc$, the kinetic energy

$$T = \frac{\boldsymbol{p}^2}{2m} \left[1 + \mathcal{O}\left(\frac{\boldsymbol{p}^2}{m^2} \right) \right] \ . \tag{137}$$

11. A particle of mass M comes to rest and decays into a particle of mass m and a massless particle. Show that the kinetic energy of the particle of mass m is

$$T = \frac{(M-m)^2}{2M} \ . \tag{138}$$

12. A particle of mass M with velocity $\boldsymbol{v}$ decays into two massless particles with energies E_1 and E_2. Determine an expression for the angle θ between the two final particles. Show that the minimum possible value of this angle obeys the relation

$$\sin(\theta_{\min}/2) = \sqrt{1-\boldsymbol{v}^2}. \tag{139}$$

13. Using eqs. (114) and (115), show that

$$s + t + u = \sum_{i=1}^{4} m_i^2 \ . \tag{140}$$

14. Show that the total energy in the center-of-mass frame, $E_1^{\text{c.m.}} + E_2^{\text{c.m.}}$, is a relativistic invariant of the process, and equals $\sqrt{s}$.

15. Show that in general we must have

$$\sqrt{s} \geq m_3 + m_4 \ . \tag{141}$$

16. Suppose that a particle with mass m and kinetic energy T strikes a particle of equal mass m at rest. Show that the kinetic energy T' of the particle scattered elastically at an angle θ is given by

$$T' = \frac{T\cos^2\theta}{1 + T\sin^2\theta/(2m)} \ . \tag{142}$$

[Hint: Evaluate the invariant t in two different ways in the rest frame mentioned above, and use energy-momentum conservation. Use also the fact that $T = E - m$.]

5.4 Electromagnetism

17. Show that $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are unchanged under the gauge transformation given in eq. (120).

18. Check using eq. (124) that F^{i0} give the $\boldsymbol{E}$ components and that F^{ij} (for $i \neq j$) give the $\boldsymbol{B}$ components.

19. Show that the pair of inhomogeneous Maxwell equations in terms of the potentials take the form

$$\begin{aligned}\left(\partial_t^2 - \boldsymbol{\nabla}^2\right)\phi - \partial_t\left(\partial_t\phi + \boldsymbol{\nabla}\cdot\boldsymbol{A}\right) &= \rho \quad , \\ \left(\partial_t^2 - \boldsymbol{\nabla}^2\right)\boldsymbol{A} + \boldsymbol{\nabla}\left(\partial_t\phi + \boldsymbol{\nabla}\cdot\boldsymbol{A}\right) &= \boldsymbol{J} \quad . \end{aligned} \tag{143}$$

20. Verify that the equations of motion eq. (143) can be rewritten in relativistically covariant form as

$$\Box A^\mu - \partial^\mu\partial_\nu A^\nu = j^\mu \, , \tag{144}$$

where $\partial^\mu = (\partial_t, -\boldsymbol{\nabla})$ and $\Box \equiv \partial_\mu\partial^\mu$, and show this is equivalent to eq. (125).

21. The Lorentz gauge-fixing condition is defined as

$$\partial_\nu A^\nu = 0 \ . \tag{145}$$

Eq. (145) only fixes the gauge up to residual gauge transformations of the form eq. 123 with

$$\Box\lambda = 0 \ . \tag{146}$$

In the Lorentz gauge the equations of motion eq. (144) take the form

$$\Box A^\mu = j^\mu \ . \tag{147}$$

By doing a Fourier transformation on eq. (147), show that

$$\widetilde{A}^\mu = \frac{-g^{\mu\nu}}{q^2}\,\widetilde{j}_\nu = -\frac{\widetilde{j}^\mu}{q^2} \tag{148}$$

where the Fourier transforms are

$$\widetilde{A}^\mu = \int d^4x\; e^{-iqx}A^\mu \quad , \quad \widetilde{j}^\mu = \int d^4x\; e^{-iqx}j^\mu \ . \tag{149}$$