Quantum Field Theory - Tuesday Problems

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1 Classical Physics

1.1 Consider a transformed Lagrangian L', which is related to another Lagrangian L as follows:

$$L'(\dot{q}, q, t) = L(\dot{q}, q, t) + \frac{dF(q,t)}{dt} \ . \tag{1}$$

Here, F is an arbitrary function of q and t but is not a function of $\dot{q}$.
Show that the Euler-Lagrange equations are invariant under this transformation.
What does this imply about the uniqueness of the Lagrangian for a given physical system (e.g. the Lagrangian for the Simple Harmonic Oscillator)?

1.2 Show that if the Hamiltonian does not depend on time explicitly (i.e. $\partial H/\partial t = 0$), then H is a constant of motion.
In many cases when H is a constant of the motion, it is identified with a well known quantity. Which quantity?

1.3 Verify that

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E(\mathbf{k})}} \left\{ e^{ik\cdot x} a(\mathbf{k}) + e^{-ik\cdot x} b(\mathbf{k}) \right\}$$

is a solution of the Klein-Gordon equation if $E(\mathbf{k})^2 = \mathbf{k}^2 + m^2$.
Show that a real scalar field $\phi^*(x) = \phi(x)$ requires the condition $b(\mathbf{k}) = a^*(\mathbf{k})$.

1.4 The Lagrangian density for classical 'ϕ^4-theory' is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4 \ .$$

Use the Euler-Lagrange equations to find the field equation that ϕ satisfies.

1.5 Derive the components P_0, $\mathbf{P}$ of the energy-momentum four-vector P^μ for classical ϕ^4-theory.

1.6 Calculate the Hamiltonian density $\mathcal{H}$ for ϕ^4-theory. Is this Hamiltonian density Lorentz invariant?