Quantum Field Theory - Wednesday Problems

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1 Free quantum fields: working with commutators

1.1 Consider the Heisenberg equation of motion for the momentum operator $\hat{p}$ of the harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{1}{2}\left(\frac{\hat{p}^2}{m} + m\omega^2\hat{x}^2\right),$$

and show that it is equivalent to Newton's law for the position operator $\hat{x}$.

1.2 Starting from the expression

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E(\mathbf{k})}}\left(e^{ik\cdot x}\hat{a}^\dagger(\mathbf{k}) + e^{-ik\cdot x}\hat{a}(\mathbf{k})\right) ,$$

find the corresponding expression for the canonical momentum operator $\hat{\pi}(x) = \partial_0\hat{\phi}(x)$.

1.3 Show that the equal time commutation relations

$$\left[\hat{\phi}(\mathbf{x},t),\hat{\pi}(\mathbf{x}',t)\right] = i\delta^3(\mathbf{x}-\mathbf{x}') \quad , \quad \left[\hat{\phi}(\mathbf{x},t),\hat{\phi}(\mathbf{x}',t)\right] = \left[\hat{\pi}(\mathbf{x},t),\hat{\pi}(\mathbf{x}',t)\right] = 0 ,$$

imply that

$$[\hat{a}(\mathbf{k}),\hat{a}(\mathbf{p})] = 0 .$$

(A similar method would also show that $\left[\hat{a}^\dagger(\mathbf{k}),\hat{a}^\dagger(\mathbf{p})\right] = 0$.)

1.4 In this problem, we want to show that the scalar field operator $\hat{\phi}(\mathbf{x},t)$ satisfies the Klein-Gordon equation:

$$\partial_\mu\partial^\mu\hat{\phi}(\mathbf{x},t) + m^2\hat{\phi}(\mathbf{x},t) = 0 .$$

We know already that

$$\hat{\pi}(\mathbf{x},t) = \partial_t\hat{\phi}(\mathbf{x},t) .$$

We now need to find an equation for $\partial_t\hat{\pi}(\mathbf{x},t)$. This can be done with the Heisenberg equation of motion, which for a general field operator $\hat{O}$ is

$$\frac{\partial}{\partial t}\hat{O} = i[\hat{H},\hat{O}] ,$$

where

$$\hat{H} = \frac{1}{2}\int d^3x \left\{\hat{\pi}^2 + (\nabla\hat{\phi})^2 + m^2\hat{\phi}^2\right\} .$$

Assuming the equal time commutation relations

$$\left[\hat{\phi}(\mathbf{x},t),\hat{\pi}(\mathbf{x}',t)\right] = i\delta^3(\mathbf{x}-\mathbf{x}') \quad , \quad \left[\hat{\phi}(\mathbf{x},t),\hat{\phi}(\mathbf{x}',t)\right] = \left[\hat{\pi}(\mathbf{x},t),\hat{\pi}(\mathbf{x}',t)\right] = 0 ,$$

evaluate $[\hat{H},\hat{\pi}(\mathbf{x},t)]$ to show that

$$\partial_t\hat{\pi}(\mathbf{x},t) = \nabla^2\hat{\phi}(\mathbf{x},t) - m^2\hat{\phi}(\mathbf{x},t) .$$

[Hint: You will need to integrate terms involving $(\nabla\hat{\phi})^2$ by parts and assume that the fields vanish at the boundary of space (spatial infinity).]