1

Do you think that by modifying Newtonian dynamics to something like General Relativity one could explain the orbit of Neptune?

2

A very important result in dynamical systems is virial theorem. Can you reproduce it for Newtonian dynamics? (Show that the time average satisfies $2\langle T\rangle = -\langle V\rangle$ where T is the kinetic energy of a collection of particles, V the potential energy, $\frac{1}{\tau}\int_0^\tau dt... = \langle ...\rangle$ and we take the limit of large τ)

3

We know from the lectures that DM is almost collionless. Can you estimate a bound on the cross-section by assuming that the typical clusters do not interact when they collide? (assume the energy density of DM is $\sim \mathrm{GeV/cm^3}$ and recall that the typical size a cluster is $\sim$ few Mpc. Similarly, you can assume that the typical time between collisions should be larger than the crossing time of clusters. Assume this time to be $1\ Gyr$. You can leave the estimate in terms of the velocity in this second case).

4

You can become an cosmologist for one day. Go to the webpage `https://lambda.gsfc.nasa.gov/toolbox/tb_camb_form.cfm`. In the first page you can choose different values for $\Omega_b h^2$. Check how if one increases this value (compensate it by reducing the value of $\Omega_c h^2$, which is the value of the DM component such that $\Omega_b + \Omega_c$ is the same as before. If you click the 'Transfer Functions' box you also get the power spectrum. Plot the C_l (data from 'camb_xxxxx_scalcls.dat' shown as LinLog) and compare by eye with `https://wiki.cosmos.esa.int/planckpla2015/index.php/File:A15_TT.png`. If you have asked for 'Transfer functions' you can also loglog plot the power spectrum file 'camb_xxxxx_matterpower_z0.dat' and see how it changes.

5

Find the minimum value of dark matter mass allowed by quantum mechanics for bosonic and fermionic candidates. You need to fit the DM candidate to dwarf spheroidals ($r \sim$kpc, typical velocity $\sim 10^{-4}c$ and mean density $\sim 5\,\mathrm{GeV/cm^3}$)

6

What's H_0 in years? (there was a typo in the value I gave in the hand written notes, $H_0 \sim 0.7 km/s/Mpc$)

7

Compute the yield for a relativistic and non relativistic species. Estimate the yield that we need in order to reproduce the correct DM relic abundance $\Omega h^2 \approx 0.1$

8

Show that in terms of Y, the equation of evolution reads

$$\frac{dY}{dt} = -s\langle\sigma v\rangle \left(Y^2 - Y_{eq}^2\right). \tag{1}$$

9

DC Problem III: Using Boltzmann equation, expressed in terms of the yield $Y = n/s$, which reads

$$\frac{dY}{dx} = -\frac{\lambda\langle\sigma v\rangle}{x^2} \left(Y^2 - Y_{eq}^2\right), \tag{2}$$

define the quantity $\Delta Y = Y - Y_{eq}$ and show that, for non-relativistic particles, the solution can be approximated as

$$\begin{aligned}\Delta_Y &= -\frac{\frac{dY_{eq}}{dx}}{Y_{eq}}\frac{x^2}{2\lambda\langle\sigma v\rangle}, \quad 1 < x \ll x_f \\ \Delta_{Y_\infty} = Y_\infty &= \frac{x_f}{\lambda\left(a+\frac{b}{3x_f^2}\right)}, \quad x \gg x_f\end{aligned} \tag{3}$$

10

DC (David Cerdeño) Problem VII: See below

11

DC Problem VIII: See below.

12

What's the relic density of a species that is kept in equilibrium with SM particles through $3_{DM} \to 2_{SM}$ processes assuming it decouples at $T \sim m$?

13

Consider two massive bosonic fields coupled with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - m_1^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - m_2^2\phi_1^2 + g\phi_1^2\phi_2 \tag{4}$$

Assume that ϕ has a background value $\bar{\phi}_1$. Show that the fluctuations over this background satisfy (in Fourier space)

$$(\omega^2 - k^2 - m_1^2)\delta\phi_1 + g\bar{\phi}_1\delta\phi_2 = 0, \qquad (\omega^2 - k^2 - m_2^2)\delta\phi_2 + g\bar{\phi}_1\delta\phi_1 = 0. \tag{5}$$

If the system starts with initial conditions $\delta\phi_1 = \phi_0$ and $\delta\dot{\phi}_1 = \delta\dot{\phi}_2 = \delta\phi_2 = 0$, compute the value of $\delta\phi_2$ as the wave propagates in the limit where $m_1 = m_2$.

A David Cerdeño's problems

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Question 4 (Dark Matter relic density 1)

Consider a simple model in which the Dark Matter is a Dirac fermion, χ, which only couples to the Standard Model sector through the exchange of the a pseudoscalar particle A. The pseudoscalar A has couplings g_χ to the dark matter and g_b to b quarks as described by the Lagrangian

$$\mathcal{L} = i\left(g_\chi \bar{\chi}\gamma^5\chi + g_b \bar{b}\gamma^5 b\right) A$$

- Draw the Feynman diagram that corresponds to the pair-annihilation of two dark matter particles into $b\bar{b}$.
- Considering only Dark Matter annihilation into $b\bar{b}$, the annihilation cross section in the Early Universe can be expanded in plane waves as $\langle\sigma v\rangle \approx a_{b\bar{b}} + b_{b\bar{b}}\, x$, with (see e.g, Ref.[2])

  $$a_{b\bar{b}} \;\; = \;\; \frac{1}{m_\chi^2}\left(\frac{N_c}{32\pi}\left(1-\frac{4m_b^2}{s}\right)^{1/2}\frac{1}{2}\int_{-1}^{1} d\cos\theta_{CM}|\mathcal{M}_{\chi\chi\to bb}|^2\right)_{s=4m_\chi^2}$$

  Show that to leading order in velocity (i.e., $x=0$)

  $$\langle\sigma v\rangle \approx \frac{3}{2\pi}\frac{(g_\chi g_b)^2 m_\chi^2\sqrt{1-m_b^2/m_\chi^2}}{(4m_\chi^2-m_A^2)+m_A^2\Gamma_A^2}$$

  Remember to average over initial spins and sum over final ones. You will also need the following trace, $\mathrm{Tr}\left[(\not{p}_1 - m_\chi)\gamma^5(\not{p}_2+m_\chi)\gamma^5\right] = 4(-p_1\cdot p_2 - m_\chi^2)$.
- Show that if the mediator is a scalar particle instead of a pseudoscalar then $a_{b\bar{b}} = 0$.
- Given a dark matter mass $m_\chi = 100$ GeV and a pseudoscalar mass $m_A = 1000$ GeV, estimate the value of the coupling $g_\chi g_b$ for which the correct relic density is obtained. Neglect the pseudoscalar decay width, Γ_A and use that

  $$\Omega_\chi h^2 \approx \frac{3\times 10^{-10}\ \mathrm{GeV}^{-2}}{\langle\sigma v\rangle}$$

Question 5 (Dark Matter detection 1)

Consider now a scalar Dark Matter model, ϕ, which only couples to the Standard Model sector through the exchange of the Higgs boson. The coupling, $C_{\phi\phi H^0_{SM}}$ (which can be understood as coming from a quartic term $\phi\phi H^0_{SM} H^0_{SM}$) is fixed by imposing that the relic density is correct, $\Omega_\phi h^2 \approx 0.1$, obtaining $C_{\phi\phi H^0_{SM}} \approx 20 GeV$. Compute the prediction for the spin-independent scattering cross-section off protons, $\sigma^{SI}_{\phi-p}$, and compare it with current experimental constraints from LUX and SuperCDMS. Is this cadidate viable or is it excluded if it has a mass $m_\phi < 20$ GeV?

To do this,

- Write down the effective Lagrangian that describes the elastic scattering of ϕ with quarks and express the interaction strength, α_q, in terms of the fundamental coupling $C_{\phi\phi H^0_{SM}}$.
- Assume that the scattering off protons can be computed assuming that the contribution of s quarks is dominant.
- The expression for the scattering cross-section of scalar dark matter can be found, e.g., in Section 3.4 of Ref. [6].

$$\sigma^{SI}_{\phi-p} = \frac{f_p m_p^2}{4\pi(m_\phi + m_p)^2} \; , \tag{25}$$

where

$$\frac{f_p}{m_p} = \sum_{q_i=u,d,s} f^p_{T_{q_i}} \frac{\alpha_{q_i}}{m_{q_i}} + \frac{2}{27} f^p_{TG} \sum_{q_i=c,b,t} \frac{\alpha_{q_i}}{m_{q_i}} \; . \tag{26}$$

We can consider for simplicity that the s quark contribution dominates, and use $f_{T_{q_s}} = 0.229$.

Question 6 (Dark Matter detection 2)

In the previous question we noticed that the predictions for $\sigma^{SI}_{\chi-p}$ exceed the current experimental limits from the direct detection experiments LUX and SuperCDMS. Is there any way in which we can "fix" this model?

- Think about why the annihilation cross section and the scattering cross section are related in the example above. How can we break this relation?
- Consider enlarging the "exotic" sector by including more particles.

Question 7 (Neutrino decoupling 1)

In the Early Universe, neutrinos remain in equilibrium through the process $e^+ + e^- \longleftrightarrow \nu_e + \bar{\nu}_e$. Using that both the electron-positron and neutrino populations are relativistic and therefore their number density scales as $n \sim T^3$, the decoupling temperature of neutrinos can be roughly estimated by equating the annihilation rate $\Gamma = n\langle\sigma v\rangle$ and the Hubble expansion rate $H = \sqrt{8\pi G\rho/3}$. The energy density of the Universe scales as $\rho \sim T^4$. Show that neutrinos decouple at approximately $T \sim 1$ MeV.

Neutrinos keep in thermal equilibrium through interactions with electrons through the processes $e^- + e^+ \longleftrightarrow \nu_e + \bar{\nu}_e$ and $e^- + \nu_e \longleftrightarrow e^- + \nu_e$

Using dimensional arguments, the cross section of these processes at a temperature T (which defines the c.o.m. energy) is approximately $\sigma = G_F^2 T^2$, where $G_F = 1.17 \times 10^{-5}$ GeV^{-2}. Given that both neutrinos and electrons are fermions, their number density can be written as

$$n_{e,\,\nu_e} = \frac{g_{eff}}{\pi^2}\zeta(3)T^3 \approx 0.1 g_{e,\,\nu_e} T^3 \tag{33}$$

where $g_e = 2$ and $g_\nu = 1$.

The interaction rate of these processes therefore reads

$$\Gamma = n_e\langle\sigma v\rangle \approx 0.1(g_e + g_{\nu_e})T^3 G_F^2 T^2 \approx 0.3\, G_F^2\, T^5 \;, \tag{34}$$

where we have considered that both species are relativistic and therefore $v \sim c = 1$.

For a radiation dominated Universe the Hubble parameter reads

$$H = \frac{\pi}{\sqrt{90}} g_*^{1/2} \frac{T^2}{M_P} \tag{35}$$

In order to see when neutrinos decouple, we need to compare their annihilation rate with the Hubble parameter

$$\frac{\Gamma}{H} = M_P \frac{0.3\, G_F^2\, T^5}{0.3 g_*^{1/2} T^2} = \frac{M_P\; G_F^2 T^3}{g_*^{1/2}} \sim \left(\frac{T}{2\text{ MeV}}\right)^3 \tag{36}$$

In the last expression we have used that the number of relativistic degrees of freedom when neutrinos decouple is $g_* = 10.75$.

This approximation suggests that neutrinos decouple from the thermal bath at $T \sim 2$ MeV . A full numerical solution of Boltzmann equation yields $T \sim 1$ MeV, so this is still a good approximation.

Question 8 (Neutrino decoupling 2)

From the question above, we know that when neutrinos decouple, they are still relativistic. The other relativistic species in the thermal bath are electrons, positrons, photons and the three neutrinos and antineutrinos. With this information the relic density of neutrinos in the Universe today can be estimated as a function of the neutrino mass.

To do that, remember that for relativistic species the Yield at equilibrium can be written as $Y_{eq} = \frac{45}{2\pi^4}\zeta(3)\frac{g_{eff}}{g_{*s}} \approx 0.278\frac{g_{eff}}{g_{*s}}$.