Quantum Field Theory

Christoph Englert[1]

These notes are a write-up of lectures given at the RAL school for High Energy Physicists, which took place at Warwick in 2014. The aim is to introduce the canonical quantisation approach to QFT, and derive the Feynman rules for a scalar field.

1 Introduction

Quantum Field Theory is a highly important cornerstone of modern physics. It underlies, for example, the description of elementary particles i.e. the Standard Model of particle physics is a QFT. There is currently no observational evidence to suggest that QFT is insufficient in describing particle behaviour, and indeed many theories for beyond the Standard Model physics (e.g. supersymmetry, extra dimensions) are QFTs. There are some theoretical reasons, however, for believing that QFT will not work at energies above the Planck scale, at which gravity becomes important. Aside from particle physics, QFT is also widely used in the description of condensed matter systems, and there has been a fruitful interplay between the fields of condensed matter and high energy physics.

We will see that the need for QFT arises when one tries to unify special relativity and quantum mechanics, which explains why theories of use in high energy particle physics are quantum field theories. Historically, Quantum Electrodynamics (QED) emerged as the prototype of modern QFT's. It was developed in the late 1940s and early 1950s chiefly by Feynman, Schwinger and Tomonaga, and has the distinction of being the most accurately verified theory of all time: the anomalous magnetic dipole moment of the electron predicted by QED agrees with experiment with a stunning accuracy of one part in 10^{10}! Since then, QED has been understood as forming part of a larger theory, the Standard Model of particle physics, which also describes the weak and strong nuclear forces. As you will learn at this school, electromagnetism and the weak interaction can be unified into a single "electroweak" theory, and the theory of the strong force is described by Quantum Chromodynamics (QCD). QCD has been verified in a wide range of contexts, albeit not as accurately as QED (due to the fact that the QED force is much weaker, allowing more accurate calculations to be carried out).

As is clear from the above discussion, QFT is a type of theory, rather than a particular theory. In this course, our aim is to introduce what a QFT is, and how to derive scattering amplitudes in perturbation theory (in the form of Feynman rules). For this purpose, it is sufficient to consider the simple example of a single, real scalar field. More physically relevant examples will be dealt with

[1]SUPA, School of Physics and Astronomy, University of Glasgow.
Email: christoph.englert@glasgow.ac.uk

in the other courses. Throughout, we will follow the so-called canonical quantisation approach to QFT, rather than the path integral approach. Although the latter approach is more elegant, it is less easily presented in such a short course.

The structure of these notes is as follows. In the rest of the introduction, we review those aspects of classical and quantum mechanics which are relevant in discussing QFT. In particular, we go over the Lagrangian formalism in point particle mechanics, and see how this can also be used to describe classical fields. We then look at the quantum mechanics of non-relativistic point particles, and recall the properties of the quantum harmonic oscillator, which will be useful in what follows. We then briefly show how attempts to construct a relativistic analogue of the Schödinger equation lead to inconsistencies. Next, we discuss classical field theory, deriving the equations of motion that a relativistic scalar field theory has to satisfy, and examining the relationship between symmetries and conservation laws. We then discuss the quantum theory of free fields, and interpret the resulting theory in terms of particles, before showing how to describe interactions via the S-matrix and its relation to Green's functions. Finally, we describe how to obtain explicit results for scattering amplitudes using perturbation theory, which leads (via Wick's theorem) to Feynman diagrams.

1.1 Classical Mechanics

Let us begin this little review by considering the simplest possible system in classical mechanics, a single point particle of mass m in one dimension, whose coordinate and velocity are functions of time, $x(t)$ and $\dot{x}(t) = dx(t)/dt$, respectively. Let the particle be exposed to a time-independent potential $V(x)$. It's motion is then governed by Newton's law

$$m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x} = F(x), \tag{1}$$

where $F(x)$ is the force exerted on the particle. Solving this equation of motion involves two integrations, and hence two arbitrary integration constants to be fixed by initial conditions. Specifying, e.g., the position $x(t_0)$ and velocity $\dot{x}(t_0)$ of the particle at some initial time t_0 completely determines its motion: knowing the initial conditions and the equations of motion, we also know the evolution of the particle at all times (provided we can solve the equations of motion).

We can also derive the equation of motion using an entirely different approach, via the Lagrangian formalism. This is perhaps more abstract than Newton's force-based approach, but in fact is easier to generalise and technically more simple in complicated systems (such as field theory!), not least because it avoids us having to think about forces at all.
First, we introduce the *Lagrangian*

$$L(x, \dot{x}) = T - V = \frac{1}{2}m\dot{x}^2 - V(x), \tag{2}$$

which is a function of coordinates and velocities, and given by the difference between the kinetic and potential energies of the particle. Next, we define the *action*

$$S = \int_{t_0}^{t_1} dt\, L(x, \dot{x}). \tag{3}$$

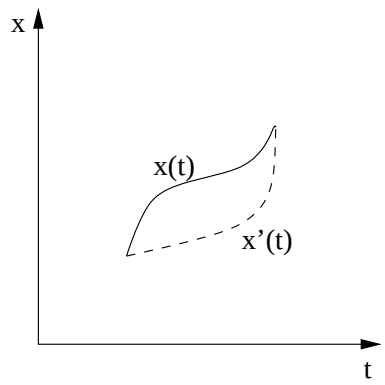


Figure 1: *Variation of particle trajectory with identified initial and end points.*

The equations of motion are then given by the *principle of least action*, which says that the trajectory $x(t)$ followed by the particle is precisely that such that S is extremised [2]. To verify this in the present case, let us rederive Newton's Second Law.

First let us suppose that $x(t)$ is indeed the trajectory that extremises the action, and then introduce a small perturbation

$$x(t) \to x(t) + \delta x(t), \tag{4}$$

such that the end points are fixed:

$$\left.\begin{array}{l} x'(t_1) = x(t_1) \\ x'(t_2) = x(t_2) \end{array}\right\} \Rightarrow \delta x(t_1) = \delta x(t_2) = 0. \tag{5}$$

This sends S to some $S + \delta S$, where $\delta S = 0$ if S is extremised. One may Taylor expand to give

$$\begin{aligned} S + \delta S &= \int_{t_1}^{t_2} L(x + \delta x, \dot{x} + \delta\dot{x})\, dt, \qquad \delta\dot{x} = \frac{d}{dt}\delta x \\ &= \int_{t_1}^{t_2} \left\{ L(x, \dot{x}) + \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \ldots \right\} dt \\ &= S + \frac{\partial L}{\partial \dot{x}}\delta x\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right\} \delta x\, dt, \end{aligned} \tag{6}$$

where we performed an integration by parts on the last term in the second line. The second and third term in the last line are the variation of the action, δS, under variations of the trajectory, δx. The second term vanishes because of the boundary conditions for the variation, and we are left with the third. Now the Principal of Least Action demands $\delta S = 0$. For the remaining integral to vanish for arbitrary δx is only possible if the integrand vanishes, leaving us with the Ēuler-Lagrange equation:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0. \tag{7}$$

[2] The name of the principle comes from the fact that, in most cases, S is indeed minimised.

If we insert the Lagrangian of our point particle, Eq. (2), into the Euler-Lagrange equation we obtain

$$\begin{aligned} \frac{\partial L}{\partial x} &= -\frac{\partial V(x)}{\partial x} = F \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= \frac{d}{dt} m\dot{x} = m\ddot{x} \\ &\Rightarrow \quad m\ddot{x} = F = -\frac{\partial V}{\partial x} \quad \text{(Newton's law).} \end{aligned} \tag{8}$$

Hence, we have derived the equation of motion (the Euler-Lagrange equation) using the Principal of Least Action and found it to be equivalent to Newton's Second Law. The benefit of the former is that it can be easily generalised to other systems in any number of dimensions, multi-particle systems, or systems with an infinite number of degrees of freedom, where the latter are needed for field theory.

For example, a general system of point particles has a set $\{q_i\}$ of *generalised coordinates*, which may not be simple positions but also angles etc. The equations of motion are then given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i},$$

by analogy with the one-dimensional case. That is, each coordinate has its own Euler-Lagrange equation (which may nevertheless depend on the other coordinates, so that the equations of motion are coupled). Another advantage of the Lagrangian formalism is that the relationship between symmetries and conserved quantities is readily understood - more on this later.
First, let us note that there is yet another way to think about classical mechanics (that we will see again in quantum mechanics / field theory), namely via the Hamiltonian formalism. Given a Lagrangian depending on generalised coordinates $\{q_i\}$, we may define the *conjugate momenta*

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

e.g. in the simple one-dimensional example given above, there is a single momentum $p = m\dot{x}$ conjugate to x. We recognise as the familiar definition of momentum, but it is not always true that $p_i = m\dot{q}_i$.

We may now define the *Hamiltonian*

$$H(\{q_i\}, \{p_i\}) = \sum_i \dot{q}_i p_i - L(\{q_i\}, \{\dot{q}_i\}).$$

As an example, consider again

$$L = \frac{1}{2} m\dot{x}^2 - V(x).$$

It is easy to show from the above definition that

$$H = \frac{1}{2} m\dot{x}^2 + V(x),$$

which we recognise as the total energy of the system. From the definition of the Hamiltonian one may derive (problem 1.1)

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial H}{\partial p_i} = \dot{x}_i,$$

which constitute *Hamilton's equations*. These are useful in proving the relation between symmetries and conserved quantities. For example, one readily sees from the above equations that the momentum p_i is conserved if H does not depend explicitly on q_i. That is, conservation of momentum is related to invariance under spatial translations, if q_i can be interpreted as a simple position coordinate.

1.2 Quantum mechanics

Having set up some basic formalism for classical mechanics, let us now move on to quantum mechanics. In doing so we shall use *canonical quantisation*, which is historically what was used first and what we shall later use to quantise fields as well. We remark, however, that one can also quantise a theory using path integrals.
Canonical quantisation consists of two steps. Firstly, the dynamical variables of a system are replaced by operators, which we denote by a hat. Secondly, one imposes commutation relations on these operators,

$$[\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij} \tag{9}$$

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0. \tag{10}$$

The physical state of a quantum mechanical system is encoded in state vectors $|\psi\rangle$, which are elements of a Hilbert space $\mathcal{H}$. The hermitian conjugate state is $\langle\psi| = (|\psi\rangle)^\dagger$, and the modulus squared of the scalar product between two states gives the probability for the system to go from state 1 to state 2,

$$|\langle\psi_1|\psi_2\rangle|^2 = \text{probability for} \quad |\psi_1\rangle \to |\psi_2\rangle. \tag{11}$$

On the other hand physical observables O, i.e. measurable quantities, are given by the expectation values of hermitian operators, $\hat{O} = \hat{O}^\dagger$,

$$O = \langle\psi|\hat{O}|\psi\rangle, \quad O_{12} = \langle\psi_2|\hat{O}|\psi_1\rangle. \tag{12}$$

Hermiticity ensures that expectation values are real, as required for measurable quantities. Due to the probabilistic nature of quantum mechanics, expectation values correspond to statistical averages, or mean values, with a variance

$$(\Delta O)^2 = \langle\psi|(\hat{O} - O)^2|\psi\rangle = \langle\psi|\hat{O}^2|\psi\rangle - \langle\psi|\hat{O}|\psi\rangle^2. \tag{13}$$

An important concept in quantum mechanics is that of eigenstates of an operator, defined by

$$\hat{O}|\psi\rangle = O|\psi\rangle. \tag{14}$$

Evidently, between eigenstates we have $\Delta O = 0$. Examples are coordinate eigenstates, $\hat{\mathbf{x}}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle$, and momentum eigenstates, $\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$, describing a particle at position $\mathbf{x}$ or with momentum $\mathbf{p}$, respectively. However, a state vector cannot be simultaneous eigenstate of non-commuting

operators. This leads to the Heisenberg uncertainty relation for any two non-commuting operators $\hat{A}, \hat{B}$,

$$\Delta A \Delta B \geq \frac{1}{2}|\langle\psi|[\hat{A}, \hat{B}]|\psi\rangle|. \tag{15}$$

Finally, sets of eigenstates can be orthonormalized and we assume completeness, i.e. they span the entire Hilbert space,

$$\langle \mathbf{p}'|\mathbf{p}\rangle = \delta(\mathbf{p} - \mathbf{p}'), \quad 1 = \int d^3p \; |\mathbf{p}\rangle\langle\mathbf{p}|. \tag{16}$$

As a consequence, an arbitrary state vector can always be expanded in terms of a set of eigenstates. We may then define the *position space wavefunction*

$$\psi(\mathbf{x}) = \langle \mathbf{x}|\psi\rangle,$$

so that

$$\begin{aligned}\langle\psi_1|\psi_2\rangle &= \int d^3\mathbf{x}\langle\psi_1|\mathbf{x}\rangle\langle\mathbf{x}|\psi_2\rangle \\ &= \int d^3\mathbf{x}\, \psi_1^*(\mathbf{x})\psi_2(\mathbf{x}).\end{aligned} \tag{17}$$

Acting on the wavefunction, the explicit form of the position and momentum operators is

$$\hat{\mathbf{x}} = \mathbf{x}, \quad \hat{\mathbf{p}} = -i\hbar\nabla, \tag{18}$$

so that the Hamiltonian operator is

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{x}) = -\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{x}). \tag{19}$$

Having quantised our system, we now want to describe its time evolution. This can be done in different "pictures", depending on whether we consider the state vectors or the operators (or both) to depend explicitly on t, such that expectation values remain the same. Two extreme cases are those where the operators do not depend on time (the *Schrödinger picture*), and when the state vectors do not depend on time (the *Heisenberg picture*). We discuss these two choices in the following sections.

1.3 The Schrödinger picture

In this approach state vectors are functions of time, $|\psi(t)\rangle$, while operators are time independent, $\partial_t \hat{O} = 0$. The time evolution of a system is described by the Schrödinger equation[3],

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x}, t) = \hat{H}\psi(\mathbf{x}, t). \tag{20}$$

If at some initial time t_0 our system is in the state $\Psi(\mathbf{x}, t_0)$, then the time dependent state vector

$$\Psi(\mathbf{x}, t) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\Psi(\mathbf{x}, t_0) \tag{21}$$

[3]Note that the Hamiltonian could itself have some time dependence in general, even in the Schrödinger picture, if the potential of a system depends on time. Here we assume that this is not the case.

solves the Schrödinger equation for all later times t.
The expectation value of some hermitian operator $\hat{O}$ at a given time t is then defined as

$$\langle \hat{O} \rangle_t = \int d^3x \, \Psi^*(\mathbf{x}, t) \hat{O} \Psi(\mathbf{x}, t), \tag{22}$$

and the normalisation of the wavefunction is given by

$$\int d^3x \, \Psi^*(\mathbf{x}, t) \Psi(\mathbf{x}, t) = \langle 1 \rangle_t. \tag{23}$$

Since $\Psi^*\Psi$ is positive, it is natural to interpret it as the *probability density* for finding a particle at position $\mathbf{x}$. Furthermore one can derive a conserved current $\mathbf{j}$, as well as a continuity equation by considering

$$\Psi^* \times (\text{Schr.Eq.}) - \Psi \times (\text{Schr.Eq.})^*. \tag{24}$$

The continuity equation reads

$$\frac{\partial}{\partial t} \rho = -\nabla \cdot \mathbf{j} \tag{25}$$

where the density ρ and the current $\mathbf{j}$ are given by

$$\rho = \Psi^* \Psi \qquad \text{(positive)}, \tag{26}$$

$$\mathbf{j} = \frac{\hbar}{2im} \left(\Psi^* \nabla \Psi - (\nabla \Psi^*) \Psi \right) \quad \text{(real)}. \tag{27}$$

Now that we have derived the continuity equation let us discuss the probability interpretation of Quantum Mechanics in more detail. Consider a finite volume V with boundary S. The integrated continuity equation is

$$\begin{aligned} \int_V \frac{\partial \rho}{\partial t} \, d^3x &= -\int_V \nabla \cdot \mathbf{j} \, d^3x \\ &= -\int_S \mathbf{j} \cdot d^2\mathbf{o} \end{aligned} \tag{28}$$

where in the last line we have used Gauss's theorem. Using Eq. (23) the left-hand side can be rewritten and we obtain

$$\frac{\partial}{\partial t} \langle 1 \rangle_t = -\int_S \mathbf{j} \cdot d^2\mathbf{o} = 0. \tag{29}$$

In other words, provided that $\mathbf{j} = 0$ everywhere at the boundary S, we find that the time derivative of $\langle 1 \rangle_t$ vanishes. Since $\langle 1 \rangle_t$ represents the total probability for finding the particle anywhere inside the volume V, we conclude that this probability must be conserved: particles cannot be created or destroyed in our theory. Non-relativistic Quantum Mechanics thus provides a consistent formalism to describe a single particle. The quantity $\Psi(\mathbf{x}, t)$ is interpreted as a one-particle wave function.

1.4 The Heisenberg picture

Here the situation is the opposite to that in the Schrödinger picture, with the state vectors regarded as constant, $\partial_t |\Psi_H\rangle = 0$, and operators which carry the time dependence, $\hat{O}_H(t)$. This is the concept

which later generalises most readily to field theory. We make use of the solution Eq. (21) to the Schrödinger equation in order to *define* a Heisenberg state vector through

$$\Psi(x,t) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\Psi(x,t_0) \equiv e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\Psi_H(x), \tag{30}$$

i.e. $\Psi_H(\mathbf{x}) = \Psi(\mathbf{x}, t_0)$. In other words, the Schrödinger vector at some time t_0 is defined to be equivalent to the Heisenberg vector, and the solution to the Schrödinger equation provides the transformation law between the two for all times. This transformation of course leaves the physics, i.e. expectation values, invariant,

$$\langle\Psi(t)|\hat{O}|\Psi(t)\rangle = \langle\Psi(t_0)|e^{\frac{i}{\hbar}\hat{H}(t-t_0)}\hat{O}e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}|\Psi(t_0)\rangle = \langle\Psi_H|\hat{O}_H(t)|\Psi_H\rangle, \tag{31}$$

with

$$\hat{O}_H(t) = e^{\frac{i}{\hbar}\hat{H}(t-t_0)}\hat{O}e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}. \tag{32}$$

From this last equation it is now easy to derive the equivalent of the Schrödinger equation for the Heisenberg picture, the Heisenberg equation of motion for operators:

$$i\hbar\frac{d\hat{O}_H(t)}{dt} = [\hat{O}_H, \hat{H}]. \tag{33}$$

Note that all commutation relations, like Eq. (9), with time dependent operators are now intended to be valid for all times. Substituting $\hat{x}, \hat{p}$ for $\hat{O}$ into the Heisenberg equation readily leads to

$$\begin{aligned} \frac{d\hat{x}_i}{dt} &= \frac{\partial\hat{H}}{\partial\hat{p}_i}, \\ \frac{d\hat{p}_i}{dt} &= -\frac{\partial\hat{H}}{\partial\hat{x}_i}, \end{aligned} \tag{34}$$

the quantum mechanical equivalent of the Hamilton equations of classical mechanics.

1.5 The quantum harmonic oscillator

Because of similar structures later in quantum field theory, it is instructive to also briefly recall the harmonic oscillator in one dimension. Its Hamiltonian is given by

$$\hat{H}(\hat{x},\hat{p}) = \frac{1}{2}\left(\frac{\hat{p}^2}{m} + m\omega^2\hat{x}^2\right). \tag{35}$$

Employing the canonical formalism we have just set up, we easily identify the momentum operator to be $\hat{p}(t) = m\partial_t\hat{x}(t)$, and from the Hamilton equations we find the equation of motion to be $\partial_t^2\hat{x} = -\omega^2\hat{x}$, which has the well known plane wave solution $\hat{x} \sim \exp i\omega t$.
An alternative path useful for later field theory applications is to introduce new operators, expressed in terms of the old ones,

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} + i\sqrt{\frac{1}{m\omega\hbar}}\,\hat{p}\right), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} - i\sqrt{\frac{1}{m\omega\hbar}}\,\hat{p}\right). \tag{36}$$

Using the commutation relation for $\hat{x}, \hat{p}$, one readily derives (see the preschool problems)

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{H}, \hat{a}] = -\hbar\omega\hat{a}, \quad [\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger. \tag{37}$$

With the help of these the Hamiltonian can be rewritten in terms of the new operators:

$$\hat{H} = \frac{1}{2}\hbar\omega\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) = \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hbar\omega. \tag{38}$$

With this form of the Hamiltonian it is easy to construct a complete basis of energy eigenstates $|n\rangle$,

$$\hat{H}|n\rangle = E_n|n\rangle. \tag{39}$$

Using the above commutation relations, one finds

$$\hat{a}^\dagger\hat{H}|n\rangle = (\hat{H}\hat{a}^\dagger - \hbar\omega\hat{a}^\dagger)|n\rangle = E_n\hat{a}^\dagger|n\rangle, \tag{40}$$

and therefore

$$\hat{H}\hat{a}^\dagger|n\rangle = (E_n + \hbar\omega)\hat{a}^\dagger|n\rangle. \tag{41}$$

Thus, the state $\hat{a}^\dagger|n\rangle$ has energy $E_n + \hbar\omega$, so that $\hat{a}^\dagger$ may be regarded as a "creation operator" for a quantum with energy $\hbar\omega$. Along the same lines one finds that $\hat{a}|n\rangle$ has energy $E_n - \hbar\omega$, and $\hat{a}$ is an "annihilation operator".
Let us introduce a vacuum state $|0\rangle$ with no quanta excited, for which $\hat{a}|n\rangle = 0$, because there cannot be any negative energy states. Acting with the Hamiltonian on that state we find

$$\hat{H}|0\rangle = \hbar\omega/2, \tag{42}$$

i.e. the quantum mechanical vacuum has a non-zero energy, known as vacuum oscillation or *zero point energy*. Acting with a creation operator onto the vacuum state one easily finds the state with one quantum excited, and this can be repeated n times to get

$$\begin{aligned} |1\rangle = \hat{a}^\dagger|0\rangle \quad &, \quad E_1 = (1 + \frac{1}{2})\hbar\omega, \quad \dots \\ |n\rangle = \frac{\hat{a}^\dagger}{\sqrt{n}}|n-1\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle \quad &, \quad E_n = (n + \frac{1}{2})\hbar\omega. \end{aligned} \tag{43}$$

The root of the factorial is there to normalise all eigenstates to one. Finally, the *number operator* $\hat{N} = \hat{a}^\dagger\hat{a}$ returns the number of quanta in a given energy eigenstate,

$$\hat{N}|n\rangle = n|n\rangle. \tag{44}$$

1.6 Relativistic Quantum Mechanics

So far we have only considered non-relativistic particles. In this section, we see what happens when we try to formulate a relativistic analogue of the Schrödinger equation. First, note that we can derive the non-relativistic equation starting from the energy relation

$$E = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}) \tag{45}$$

and replacing variables by their appropriate operators acting on a position space wavefunction $\psi(\mathbf{x}, t)$

$$E \to i\hbar\frac{\partial}{\partial t}, \quad \mathbf{p} \to -i\hbar\nabla, \quad \mathbf{x} \to \mathbf{x} \tag{46}$$

to give

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right]\psi(\mathbf{x},t) = i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t}. \tag{47}$$

As we have already seen, there is a corresponding positive definite probability density

$$\rho = |\psi(\mathbf{x},t)|^2 \geq 0, \tag{48}$$

with corresponding current

$$\mathbf{j} = \frac{\hbar}{2im}\left(\psi^*\nabla\psi - (\nabla\psi^*)\psi\right). \tag{49}$$

Can we also make a relativistic equation? By analogy with the above, we may start with the relativistic energy relation

$$E^2 = c^2\mathbf{p}^2 + m^2c^4, \tag{50}$$

and making the appropriate operator replacements leads to the equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\phi(\mathbf{x},t) \tag{51}$$

for some wavefunction $\phi(\mathbf{x},t)$. This is the *Klein-Gordon* equation, and one may try to form a probability density and current, as in the non-relativistic case. Firstly, one notes that to satisfy relativistic invariance, the probability density should be the zeroth component of a 4-vector $j^\mu = (\rho, \mathbf{j})$ satisfying

$$\partial_\mu j^\mu = 0. \tag{52}$$

In fact, one finds

$$\rho = \frac{i\hbar}{2m}\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right), \tag{53}$$

with $\mathbf{j}$ given as before. This is not positive definite! That is, this may (and will) become negative in general, so we cannot interpret this as the probability density of a single particle.

There is another problem with the Klein-Gordon equation as it stands, that is perhaps less abstract to appreciate. The relativistic energy relation gives

$$E = \pm\sqrt{c^2\mathbf{p}^2 + m^2c^4}, \tag{54}$$

and thus one has positive and negative energy solutions. For a free particle, one could restrict to having positive energy states only. However, an interacting particle may exchange energy with its environment, and there is nothing to stop it cascading down to energy states of more and more negative energy, thus emitting infinite amounts of energy.

We conclude that the Klein-Gordon equation does not make sense as a consistent quantum theory of a single particle. We thus need a different approach in unifying special relativity and quantum mechanics. This, as we will see, is QFT, in which we will be able to reinterpret the Klein-Gordon function as a field $\phi(\mathbf{x},t)$ describing many particles.

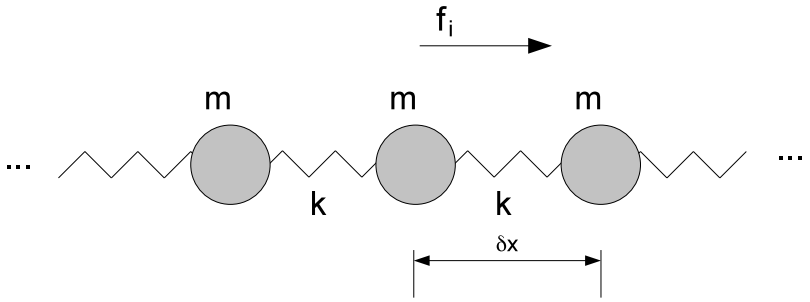


Figure 2: System of masses m joined by springs (of constant k), whose longitudinal displacements are $\{f_i\}$, and whose separation at rest is δx.

From now on, it will be extremely convenient to work in *natural units*, in which one sets $\hbar = c = 1$. The correct factors can always be reinstated by dimensional analysis. In these units, the Klein-Gordon equation becomes

$$(\Box + m^2)\phi(\mathbf{x}, t) = 0, \tag{55}$$

where

$$\Box = \partial^\mu \partial_\mu = \frac{\partial}{\partial t^2} - \nabla^2. \tag{56}$$

2 Classical Field Theory

In the previous section, we have seen how to describe point particles, both classically and quantum mechanically. In this section, we discuss classical field theory, as a precursor to considering quantum fields. A *field* associates a mathematical object (e.g. scalar, vector, tensor, spinor...) with every point in spacetime. Examples are the temperature distribution in a room (a scalar field), or the **E** and **B** fields in electromagnetism (vector fields). Just as point particles can be described by Lagrangians, so can fields, although it is more natural to think in terms of *Lagrangian densities*.

2.1 Example: Model of an Elastic Rod

Let us consider a particular example, namely a set of point masses connected together by springs, as shown in figure 2. Assume the masses m are equal, as also are the force constants of the springs k. Furthermore, we assume that the masses may move only longitudinally, where the i^{th} displacement is f_i, and that the separation of adjacent masses is δx when all f_i are zero. This system is an approximation to an elastic rod, with a displacement field $f(x, t)$. To see what this field theory looks like, we may first write the total kinetic and potential energies as

$$T = \sum_i \frac{1}{2} m \dot{f}_i^2, \quad V = \sum_i \frac{1}{2} k (f_{i+1} - f_i)^2 \tag{57}$$

respectively, where we have used Hooke's Law for the potential energy. Thus, the Lagrangian is

$$L = T - V = \sum_i \left[\frac{1}{2} m \dot{f}_i^2 - \frac{1}{2} k (f_{i+1} - f_i)^2 \right]. \tag{58}$$

Clearly this system becomes a better approximation to an elastic rod as the continuum limit is approached, in which the number of masses $N \to \infty$ and the separation $\delta x \to 0$. We can then rewrite the Lagrangian as

$$L = \sum_i \delta x \left[\frac{1}{2} \left(\frac{m}{\delta x} \right) \dot{f}_i^2 - \frac{1}{2} (k \delta x) \left(\frac{f_{i+1} - f_i}{\delta x} \right)^2 \right]. \tag{59}$$

We may recognise

$$\lim_{\delta x \to 0} m/\delta x = \rho \tag{60}$$

as the density of the rod, and also define the tension

$$\kappa = \lim_{\delta x \to 0} k \delta x. \tag{61}$$

Furthermore, the position index i gets replaced by the continuous variable x, and one has

$$\lim_{\delta x \to 0} \frac{f_{i+1} - f_i}{\delta x} = \frac{\partial f(x,t)}{\partial x}. \tag{62}$$

Finally, the sum over i becomes an integral so that the continuum Lagrangian is

$$L = \int dx \left[\frac{1}{2} \rho \dot{f}(x,t)^2 - \frac{1}{2} \kappa \left(\frac{\partial f}{\partial x} \right)^2 \right]. \tag{63}$$

This is the Lagrangian for the displacement field $f(x,t)$. It depends on a function of f and $\dot{f}$ which is integrated over all space coordinates (in this case there is only one, the position along the rod). We may therefore write the Lagrangian manifestly as

$$L = \int dx \mathcal{L}[f(x,t), \dot{f}(x,t)], \tag{64}$$

where $\mathcal{L}$ is the *Lagrangian density*

$$\mathcal{L}[f(x,t), \dot{f}(x,t)] = \frac{1}{2} \rho \dot{f}^2(x,t) - \frac{1}{2} \kappa \left(\frac{\partial f}{\partial x} \right)^2. \tag{65}$$

It is perhaps clear from the above example that for any field, there will always be an integration over all space dimensions, and thus it is more natural to think about the Lagrangian density rather than the Lagrangian itself. Indeed, we may construct the following dictionary between quantities in point particle mechanics, and corresponding field theory quantities (which may or may not be helpful to you in remembering the differences between particles and fields...!).

Classical Mechanics:		Classical Field Theory:	
$x(t)$	$\longrightarrow$	$\phi(x,t)$	(66)
$\dot{x}(t)$	$\longrightarrow$	$\dot{\phi}(x,t)$	
Index i	$\longrightarrow$	Coordinate x	(67)
$L(x, \dot{x})$	$\longrightarrow$	$\mathcal{L}[\phi, \dot{\phi}]$	(68)

Note that the action for the above field theory is given, as usual, by the time integral of the Lagrangian:

$$S = \int dt L = \int dt \int dx \, L[f, \dot{f}]. \tag{69}$$

2.2 Relativistic Fields

In the previous section we saw how fields can be described using Lagrangian densities, and illustrated this with a non-relativistic example. Rather than derive the field equations for this case, we do this explicitly here for relativistic theories, which we will be concerned with for the rest of the course (and, indeed, the school).
In special relativity, coordinates are combined into four-vectors, $x^\mu = (t, x_i)$ or $x = (t, \mathbf{x})$, whose length $x^2 = t^2 - \mathbf{x}^2$ is invariant under Lorentz transformations

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu. \tag{70}$$

A general function transforms as $f(x) \to f'(x')$, i.e. both the function and its argument transform. A Lorentz scalar is a function $\phi(x)$ which at any given point in space-time will have the same amplitude, regardless of which inertial frame it is observed in. Consider a space-time point given by x in the unprimed frame, and $x'(x)$ in the primed frame, where the function $x'(x)$ can be derived from eq. (70). Observers in both the primed and unprimed frames will see the same amplitude $\phi(x)$, although an observer in the primed frame will prefer to express this in terms of his or her own coordinate system x', hence will see

$$\phi(x) = \phi(x(x')) = \phi'(x'), \tag{71}$$

where the latter equality defines ϕ'.
Equation (71) defines the transformation law for a Lorentz scalar. A vector function transforms as

$$V'^\mu(x') = \Lambda^\mu{}_\nu \, V^\nu(x). \tag{72}$$

We will work in particular with $\partial_\mu \phi(x)$, where $x \equiv x^\mu$ denotes the 4-position. Note in particular that

$$(\partial_\mu \phi)(\partial^\mu \phi) = \left(\frac{\partial \phi}{\partial t}\right)^2 - \nabla\phi \cdot \nabla\phi$$
$$\partial_\mu \partial^\mu \phi = \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi.$$

In general, a relativistically invariant scalar field theory has action

$$S = \int d^4x \mathcal{L}[\phi, \partial_\mu \phi], \tag{73}$$

where

$$\int d^4x \equiv \int dt \, d^3\mathbf{x}, \tag{74}$$

and $\mathcal{L}$ is the appropriate Lagrangian density. We can find the equations of motion satisfied by the field ϕ using, as in point particle mechanics, the principle of least action. The field theory form of this is that the field $\phi(x)$ is such that the action of eq. (73) is extremised. Assuming $\phi(x)$ is indeed such a field, we may introduce a small perturbation

$$\phi(x) \to \phi(x) + \delta\phi(x), \tag{75}$$

which correspondingly perturbs the action according to

$$S \to S + \delta S = \int d^4x \left[\mathcal{L}(\phi, \partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right]. \tag{76}$$

Recognising the first term as the unperturbed action, one thus finds

$$\begin{aligned} \delta S &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right] \\ &= \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right]_{\text{boundary}} + \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi, \end{aligned}$$

where we have integrated by parts in the second line. Assuming the fields die away at infinity so that $\delta\phi = 0$ at the boundary of spacetime, the principle of least action $\delta S = 0$ implies

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}. \tag{77}$$

This is the *Euler-Lagrange field equation*. It tells us, given a particular Lagrangian density (which defines a particular field theory) the classical equation of motion which must be satisfied by the field ϕ. As a specific example, let us consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2, \tag{78}$$

from which one finds

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi, \tag{79}$$

so that the Euler-Lagrange equation gives

$$\partial_\mu \partial^\mu \phi + m^2 \phi = (\Box + m^2)\phi(x) = 0. \tag{80}$$

This is the Klein-Gordon equation! The above Lagrangian density thus corresponds to the classical field theory of a Klein-Gordon field. We see in particular that the coefficient of the quadratic term in the Lagrangian can be interpreted as the mass.

By analogy with point particle mechanics, one can define a *canonical momentum field* conjugate to ϕ:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}. \tag{81}$$

Then one can define the *Hamiltonian density*

$$\mathcal{H}[\phi, \pi] = \pi \dot{\phi} - \mathcal{L}, \tag{82}$$

such that

$$H = \int d^3x \, \mathcal{H}(\pi, \phi) \tag{83}$$

is the Hamiltonian (total energy carried by the field). For example, the Klein-Gordon field has conjugate momentum $\pi = \dot{\phi}$, and Hamiltonian density

$$\mathcal{H} = \frac{1}{2} \left[\pi^2(x) + (\nabla\phi)^2 + m^2 \phi^2 \right]. \tag{84}$$

2.3 Plane wave solutions to the Klein-Gordon equation

Let us consider real solutions to Eq. (80), characterised by $\phi^*(x) = \phi(x)$. To find them we try an ansatz of plane waves

$$\phi(x) \propto \mathrm{e}^{i(k^0 t - \mathbf{k}\cdot\mathbf{x})}. \tag{85}$$

The Klein-Gordon equation is satisfied if $(k^0)^2 - \mathbf{k}^2 = m^2$ so that

$$k^0 = \pm\sqrt{\mathbf{k}^2 + m^2}. \tag{86}$$

Defining the energy as

$$E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2} > 0, \tag{87}$$

we obtain two types of solution which read

$$\phi_+(x) \propto \mathrm{e}^{i(E(\mathbf{k})t - \mathbf{k}\cdot\mathbf{x})}, \quad \phi_-(x) \propto \mathrm{e}^{-i(E(\mathbf{k})t - \mathbf{k}\cdot\mathbf{x})}. \tag{88}$$

We may interpret these as positive and negative energy solutions, such that it does not matter which branch of the square root we take in eq. (87) (it is conventional, however, to define energy as a positive quantity). The general solution is a superposition of ϕ_+ and ϕ_-. Using

$$E(\mathbf{k})t - \mathbf{k}\cdot\mathbf{x} = k^\mu k_\mu = k_\mu k^\mu = k \cdot x \tag{89}$$

this solution reads

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\mathbf{k})} \left(\mathrm{e}^{ik\cdot x}\alpha^*(\mathbf{k}) + \mathrm{e}^{-ik\cdot x}\alpha(\mathbf{k})\right), \tag{90}$$

where $\alpha(\mathbf{k})$ is an arbitrary complex coefficient. Note that the coefficients of the positive and negative exponentials are related by complex conjugation. This ensures that the field $\phi(x)$ is real (as can be easily verified from eq. (90)), consistent with the Lagrangian we wrote down. Such a field has applications in e.g. the description of neutral mesons. We can also write down a Klein-Gordon Lagrangian for a complex field ϕ. This is really two independent fields (i.e. ϕ and ϕ^*), and thus can be used to describe a system of two particles (e.g. charged meson pairs). To simplify the discussion in this course, we will explicitly consider the real Klein-Gordon field. Note that the factors of 2 and π in eq. (90) are conventional, and the inverse power of the energy is such that the measure of integration is Lorentz invariant (problem 2.1), so that the whole solution is written in a manifestly Lorentz invariant way.

2.4 Symmetries and Conservation Laws

As was the case in point particle mechanics, one may relate symmetries of the Lagrangian density to conserved quantities in field theory. For example, consider the invariance of $\mathcal{L}$ under space-time translations

$$x^\mu \to x^\mu + \epsilon^\mu, \tag{91}$$

where ϵ^μ is constant. Under such a transformation one has

$$\mathcal{L}(x^\mu + \epsilon^\mu) = \mathcal{L}(x^\mu) + \epsilon^\mu \partial_\mu \mathcal{L}(x^\mu) + \ldots \tag{92}$$

$$\phi(x^\mu + \epsilon^\mu) = \phi(x^\mu) + \epsilon^\mu \partial_\mu \phi(x^\mu) + \ldots \tag{93}$$

$$\partial_\nu\phi(x^\mu + \epsilon^\mu) = \partial_\nu\phi(x^\mu) + \epsilon^\mu \partial_\mu\partial_\nu \phi(x^\mu) + \ldots, \tag{94}$$

$$\tag{95}$$

where we have used Taylor's theorem. But if $\mathcal{L}$ does not explicitly depend on x^μ (i.e. only through ϕ and $\partial_\mu\phi$) then one has

$$\begin{aligned}\mathcal{L}(x^\mu+\epsilon^\mu) &= \mathcal{L}[\phi(x^\mu+\epsilon^\mu),\partial_\nu\phi(x^\mu+\epsilon^\mu)] \\ &= \mathcal{L}+\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi+\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\delta(\partial_\nu\phi)+\ldots \qquad (96)\\ &= \mathcal{L}+\frac{\partial\mathcal{L}}{\partial\phi}\epsilon^\mu\partial_\mu\phi+\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\epsilon^\mu\partial_\mu\partial_\nu\phi+\ldots, \qquad (97)\end{aligned}$$

where we have used the fact that $\delta\phi=\epsilon^\mu\partial_\mu\phi$ in the third line, and all functions on the right-hand side are evaluated at x^μ. One may replace $\partial\mathcal{L}/\partial\phi$ by the LHS of the Euler-Lagrange equation to get

$$\begin{aligned}\mathcal{L}(x^\mu+\epsilon^\mu) &= \mathcal{L}+\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\epsilon^\mu\partial_\mu\phi+\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\epsilon^\mu\partial_\mu\partial_\nu\phi+\ldots \\ &= \mathcal{L}+\partial_\nu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\mu\phi\right]\epsilon^\mu, \qquad (98)\end{aligned}$$

and equating this with the alternative expression above, one finds

$$\partial_\nu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\mu\phi\right]\epsilon^\mu=\epsilon^\mu\partial_\mu\mathcal{L}. \qquad (99)$$

If this is true for all ϵ^μ, then one has

$$\partial^\nu\Theta_{\nu\mu}=0, \qquad (100)$$

where

$$\Theta_{\nu\mu}=\frac{\partial\mathcal{L}}{\partial(\partial^\nu\phi)}\partial_\mu\phi-g_{\mu\nu}\mathcal{L} \qquad (101)$$

is the *energy-momentum tensor*. We can see how this name arises by considering the components explicitly, for the case of the Klein Gordon field. One then finds

$$\Theta_{00}=\frac{\partial\mathcal{L}}{\partial\dot\phi}\dot\phi-g_{00}\mathcal{L}=\pi\dot\phi-\mathcal{L}=\mathcal{H}, \qquad (102)$$

$$\Theta_{0j}=\frac{\partial\mathcal{L}}{\partial\dot\phi}\partial_j\phi-g_{0j}\mathcal{L}=\pi\partial_j\phi\quad(j=1\ldots3). \qquad (103)$$

One then sees that Θ_{00} is the energy density carried by the field. Its conservation can then be shown by considering

$$\begin{aligned}\frac{\partial}{\partial t}\int_V d^3x\,\Theta_{00} &= \int_V d^3x\,\partial^0\Theta_{00} \\ &= \int_V d^3x\,\partial^j\Theta_{j0}=\int_S dS_j\cdot\Theta_{0j}=0, \qquad (104)\end{aligned}$$

where we have used Eq. (100) in the second line. The Hamiltonian density is a conserved quantity, provided that there is no energy flow through the surface S which encloses the volume V. In a

similar manner one can show that the 3-momentum p_j, which is related to Θ_{0j}, is conserved as well. It is then useful to define a conserved energy-momentum four-vector

$$P_\mu = \int d^3x \; \Theta_{0\mu}. \tag{105}$$

In analogy to point particle mechanics, we thus see that invariances of the Lagrangian density correspond to conservation laws. An entirely analogous procedure leads to conserved quantities like angular mometum and spin. Furthermore one can study so-called internal symmetries, i.e. ones which are not related to coordinate but other transformations. Examples are conservation of all kinds of charges, isospin, etc.
We have thus established the Lagrange-Hamilton formalism for classical field theory: we derived the equation of motion (Euler-Lagrange equation) from the Lagrangian and introduced the conjugate momentum. We then defined the Hamiltonian (density) and considered conservation laws by studying the energy-momentum tensor $\Theta_{\mu\nu}$.

3 Quantum Field Theory: Free Fields

3.1 Canonical Field Quantisation

In the previous sections we have reviewed the classical and quantum mechanics of point particles, and also classical field theory. We used the canonical quantisation procedure in discussing quantum mechanics, whereby classical variables are replaced by operators, which have non-trivial commutation relations. In this section, we see how to apply this procedure to fields, taking the explicit example of the Klein-Gordon field discussed previously. This is, as yet, a non-interacting field theory, and we will discuss how to deal with interactions later on in the course.

The Klein-Gordon Lagrangian density has the form

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2. \tag{106}$$

We have seen that in field theory the field $\phi(x)$ plays the role of the coordinates in ordinary point particle mechanics, and we defined a canonically conjugate momentum, $\pi(x) = \partial\mathcal{L}/\partial\dot{\phi} = \dot{\phi}(x)$. We then continue the analogy to point mechanics through the quantisation procedure, i.e. we now take our canonical variables to be operators,

$$\phi(x) \to \hat{\phi}(x), \quad \pi(x) \to \hat{\pi}(x). \tag{107}$$

Next we impose equal-time commutation relations on them,

$$\left[\hat{\phi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)\right] = i\delta^3(\mathbf{x}-\mathbf{y}), \tag{108}$$

$$\left[\hat{\phi}(\mathbf{x},t), \hat{\phi}(\mathbf{y},t)\right] = [\hat{\pi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)] = 0. \tag{109}$$

As in the case of quantum mechanics, the canonical variables commute among themselves, but not the canonical coordinate and momentum with each other. Note that the commutation relation is entirely analogous to the quantum mechanical case. There would be an $\hbar$, if it hadn't been set to

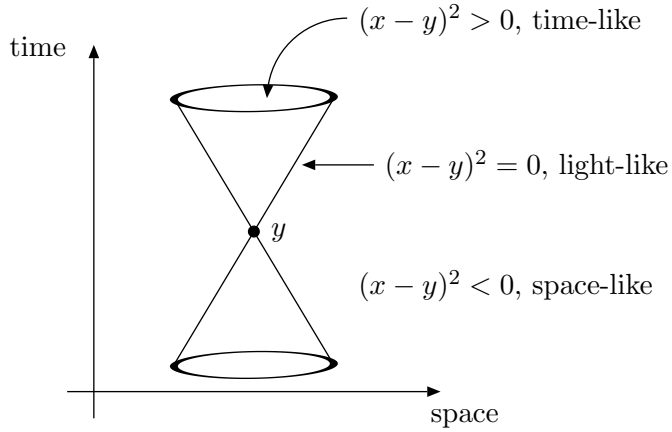


Figure 3: The light cone about y. Events occurring at points x and y are said to be time-like (space-like) if x is inside (outside) the light cone about y.

one earlier, and the delta-function accounts for the fact that we are dealing with fields. It is zero if the fields are evaluated at different space-time points.
After quantisation, our fields have turned into field operators. Note that within the relativistic formulation they depend on time, and hence they are Heisenberg operators.
In the previous paragraph we have formulated commutation relations for fields evaluated at equal time, which is clearly a special case when considering fields at general x, y. The reason has to do with maintaining causality in a relativistic theory. Let us recall the light cone about an event at y, as in Fig. 3. One important postulate of special relativity states that no signal and no interaction can travel faster than the speed of light. This has important consequences about the way in which different events can affect each other. For instance, two events which are characterised by space-time points x^μ and y^μ are said to be causal if the distance $(x-y)^2$ is time-like, i.e. $(x-y)^2 > 0$. By contrast, two events characterised by a space-like separation, i.e. $(x-y)^2 < 0$, cannot affect each other, since the point x is not contained inside the light cone about y.
In non-relativistic Quantum Mechanics the commutation relations among operators indicate whether precise and independent measurements of the corresponding observables can be made. If the commutator does not vanish, then a measurement of one observable affects that of the other. From the above it is then clear that the issue of causality must be incorporated into the commutation relations of the relativistic version of our quantum theory: whether or not independent and precise measurements of two observables can be made depends also on the separation of the 4-vectors characterising the points at which these measurements occur. Clearly, events with space-like separations cannot affect each other, and hence all fields must commute,

$$\left[\hat{\phi}(x), \hat{\phi}(y)\right] = [\hat{\pi}(x), \hat{\pi}(y)] = \left[\hat{\phi}(x), \hat{\pi}(y)\right] = 0 \quad \text{for} \quad (x-y)^2 < 0. \tag{110}$$

This condition is sometimes called micro-causality. Writing out the four-components of the time interval, we see that as long as $|t'-t| < |\mathbf{x}-\mathbf{y}|$, the commutator vanishes in a finite interval $|t'-t|$. It also vanishes for $t' = t$, as long as $\mathbf{x} \neq \mathbf{y}$. Only if the fields are evaluated at an equal space-time point can they affect each other, which leads to the equal-time commutation relations above. They can also affect each other everywhere within the light cone, i.e. for time-like intervals. It is not

hard to show that in this case (e.g. problem 3.1)

$$\left[\hat{\phi}(x), \hat{\phi}(y)\right] = [\hat{\pi}(x), \hat{\pi}(y)] = 0, \quad \text{for} \quad (x-y)^2 > 0 \tag{111}$$

$$\left[\hat{\phi}(x), \hat{\pi}(y)\right] = \frac{i}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(e^{ip\cdot(x-y)} + e^{-ip\cdot(x-y)}\right). \tag{112}$$

n.b. since the 4-vector dot product $p \cdot (x-y)$ depends on $p_0 = \sqrt{\mathbf{p}^2 + m^2}$, one cannot trivially carry out the integrals over $d^3\mathbf{p}$ here.

3.2 Creation and annihilation operators

After quantisation, the Klein-Gordon equation we derived earlier turns into an equation for operators. For its solution we simply promote the classical plane wave solution, Eq. (90), to operator status,

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\mathbf{k})} \left(\mathrm{e}^{ik\cdot x}\hat{a}^\dagger(\mathbf{k}) + \mathrm{e}^{-ik\cdot x}\hat{a}(\mathbf{k})\right). \tag{113}$$

Note that the complex conjugation of the Fourier coefficient turned into hermitian conjugation for an operator.
Let us now solve for the operator coefficients of the positive and negative energy solutions. In order to do so, we invert the Fourier integrals for the field and its time derivative,

$$\int d^3x\ \hat{\phi}(\mathbf{x},t)e^{ikx} = \frac{1}{2E}\left[\hat{a}(\mathbf{k}) + \hat{a}^\dagger(\mathbf{k})e^{2ik_0x_0}\right], \tag{114}$$

$$\int d^3x\ \dot{\hat{\phi}}(\mathbf{x},t)e^{ikx} = -\frac{i}{2}\left[\hat{a}(\mathbf{k}) - \hat{a}^\dagger(\mathbf{k})e^{2ik_0x_0}\right], \tag{115}$$

and then build the linear combination $iE(k)(114)-(115)$ to find

$$\int d^3x \left[iE(k)\hat{\phi}(\mathbf{x},t) - \dot{\hat{\phi}}(\mathbf{x},t)\right] e^{ikx} = i\hat{a}(\mathbf{k}), \tag{116}$$

Following a similar procedure for $\hat{a}^\dagger(k)$, and using $\hat{\pi}(x) = \dot{\hat{\phi}}(x)$ we find

$$\hat{a}(\mathbf{k}) = \int d^3x \left[E(k)\hat{\phi}(\mathbf{x},t) + i\hat{\pi}(\mathbf{x},t)\right] e^{ikx}, \tag{117}$$

$$\hat{a}^\dagger(\mathbf{k}) = \int d^3x \left[E(k)\hat{\phi}(\mathbf{x},t) - i\hat{\pi}(\mathbf{x},t)\right] e^{-ikx}. \tag{118}$$

Note that, as Fourier coefficients, these operators do not depend on time, even though the right hand side does contain time variables. Having expressions in terms of the canonical field variables $\hat{\phi}(x), \hat{\pi}(x)$, we can now evaluate the commutators for the Fourier coefficients. Expanding everything out and using the commutation relations Eq. (109), we find

$$\left[\hat{a}^\dagger(\mathbf{k}_1), \hat{a}^\dagger(\mathbf{k}_2)\right] = 0 \tag{119}$$

$$[\hat{a}(\mathbf{k}_1), \hat{a}(\mathbf{k}_2)] = 0 \tag{120}$$

$$\left[\hat{a}(\mathbf{k}_1), \hat{a}^\dagger(\mathbf{k}_2)\right] = (2\pi)^3\, 2E(\mathbf{k}_1)\delta^3(\mathbf{k}_1 - \mathbf{k}_2) \tag{121}$$

We easily recognise these for every $\mathbf{k}$ to correspond to the commutation relations for the harmonic oscillator, Eq. (37). This motivates us to also express the Hamiltonian and the energy momentum four-vector of our quantum field theory in terms of these operators. To do this, first note that the Hamiltonian is given by the integral of the Hamiltonian density (eq. (84)) over all space. One may then substitute eq. (113) to yield (see the problem sheet)

$$\hat{H} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})}\, E(\mathbf{k})\left(\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})\right), \tag{122}$$

$$\hat{\mathbf{P}} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})}\, \mathbf{k}\left(\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})\right). \tag{123}$$

We thus find that the Hamiltonian and the momentum operator are nothing but a continuous sum of excitation energies/momenta of one-dimensional harmonic oscillators! After a minute of thought this is not so surprising. We expanded the solution of the Klein-Gordon equation into a superposition of plane waves with momenta $\mathbf{k}$. But of course a plane wave solution with energy $E(\mathbf{k})$ is also the solution to a one-dimensional harmonic oscillator with the same energy. Hence, our free scalar field is simply a collection of infinitely many harmonic oscillators distributed over the whole energy/momentum range. These energies sum up to that of the entire system. We have thus reduced the problem of handling our field theory to oscillator algebra. From the harmonic oscillator we know already how to construct a complete basis of energy eigenstates, and thanks to the analogy of the previous section we can take this over to our quantum field theory.

3.3 Energy of the vacuum state and renormalisation

In complete analogy we begin again with the postulate of a vacuum state $|0\rangle$ with norm one, which is annihilated by the action of the operator a,

$$\langle 0|0\rangle = 1, \quad \hat{a}(\mathbf{k})|0\rangle = 0 \quad \text{for all} \quad \mathbf{k}. \tag{124}$$

Let us next evaluate the energy of this vacuum state, by taking the expectation value of the Hamiltonian,

$$E_0 = \langle 0|\hat{H}|0\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3\, 2E(\mathbf{k})}\, E(\mathbf{k})\left\{\langle 0|\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})|0\rangle + \langle 0|\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})|0\rangle\right\}. \tag{125}$$

The first term in curly brackets vanishes, since a annihilates the vacuum. The second can be rewritten as

$$\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})|0\rangle = \left\{\left[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k})\right] + \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})\right\}|0\rangle. \tag{126}$$

It is now the second term which vanishes, whereas the first can be replaced by the value of the commutator. Thus we obtain

$$E_0 = \langle 0|\hat{H}|0\rangle = \delta^3(0)\frac{1}{2}\int d^3k\, E(\mathbf{k}) = \delta^3(0)\frac{1}{2}\int d^3k\, \sqrt{\mathbf{k}^2 + m^2} = \infty, \tag{127}$$

which means that the energy of the ground state is infinite! This result seems rather paradoxical, but it can be understood again in terms of the harmonic oscillator. Recall that the simple quantum mechanical oscillator has a finite zero-point energy. As we have seen above, our field theory corresponds to an infinite collection of harmonic oscillators, i.e. the vacuum receives an infinite number of zero point contributions, and its energy thus diverges.

This is the first of frequent occurrences of infinities in quantum field theory. Fortunately, it is not too hard to work around this particular one. Firstly, we note that nowhere in nature can we observe absolute values of energy, all we can measure are energy differences relative to some reference scale, at best the one of the vacuum state, $|0\rangle$. In this case it does not really matter what the energy of the vacuum is. This then allows us to redefine the energy scale, by always subtracting the (infinite) vacuum energy from any energy we compute. This process is called "renormalisation".
We then *define* the renormalised vacuum energy to be zero, and take it to be the expectation value of a renormalised Hamiltonian,

$$E_0^R \equiv \langle 0|\hat{H}^R|0\rangle = 0. \tag{128}$$

According to this recipe, the renormalised Hamiltonian is our original one, minus the (unrenormalised) vacuum energy,

$$\begin{aligned}
\hat{H}^R &= \hat{H} - E_0 && (129)\\
&= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3\,2E(\mathbf{k})}\,E(\mathbf{k})\left\{\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k}) - \langle 0|\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})|0\rangle\right\}\\
&= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3\,2E(\mathbf{k})}\,E(\mathbf{k})\left\{2\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \left[\hat{a}(\mathbf{k}),\hat{a}^\dagger(\mathbf{k})\right] - \langle 0|\left[\hat{a}(\mathbf{k}),\hat{a}^\dagger(\mathbf{k})\right]|0\rangle\right\}. && (130)
\end{aligned}$$

Here the subtraction of the vacuum energy is shown explicitly, and we can rewrite is as

$$\begin{aligned}
\hat{H}^R &= \int \frac{d^3p}{(2\pi)^3\,2E(\mathbf{p})}\,E(\mathbf{p})\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p})\\
&\quad + \frac{1}{2}\int \frac{d^3p}{(2\pi)^3\,2E(\mathbf{p})}\,E(\mathbf{p})\left\{\left[\hat{a}(\mathbf{p}),\hat{a}^\dagger(\mathbf{p})\right] - \langle 0|\left[\hat{a}(\mathbf{p}),\hat{a}^\dagger(\mathbf{p})\right]|0\rangle\right\}.\\
&= \int \frac{d^3p}{(2\pi)^3\,2E(\mathbf{p})}\,E(\mathbf{p})\,\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) + \hat{H}^{\rm vac} && (131)
\end{aligned}$$

The operator $\hat{H}^{\rm vac}$ ensures that the vacuum energy is properly subtracted: if $|\psi\rangle$ and $|\psi'\rangle$ denote arbitrary N-particle states, then one can convince oneself that $\langle\psi'|\hat{H}^{\rm vac}|\psi\rangle = 0$. In particular we now find that

$$\langle 0|\hat{H}^R|0\rangle = 0, \tag{132}$$

as we wanted. A simple way to automatise the removal of the vacuum contribution is to introduce *normal ordering*. Normal ordering means that all annihilation operators appear to the right of any creation operator. The notation is

$$:\hat{a}\hat{a}^\dagger: = \hat{a}^\dagger\hat{a}, \tag{133}$$

i.e. the normal-ordered operators are enclosed within colons. For instance

$$:\tfrac{1}{2}\left(\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) + \hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{p})\right): \;=\; \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}). \tag{134}$$

It is important to keep in mind that $\hat{a}$ and $\hat{a}^\dagger$ *always* commute inside $:\cdots:$. This is true for an arbitrary string of $\hat{a}$ and $\hat{a}^\dagger$. With this definition we can write the normal-ordered Hamiltonian as

$$\begin{aligned}
:\hat{H}: &= :\frac{1}{2}\int \frac{d^3p}{(2\pi)^3\,2E(\mathbf{p})}\,E(\mathbf{p})\left(\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) + \hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{p})\right):\\
&= \int \frac{d^3p}{(2\pi)^3\,2E(\mathbf{p})}\,E(\mathbf{p})\,\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}), && (135)
\end{aligned}$$

and thus have the relation

$$\hat{H}^R =: \hat{H} : + \hat{H}^{\rm vac}. \tag{136}$$

Hence, we find that

$$\langle \psi' | : \hat{H} : |\psi\rangle = \langle \psi' | \hat{H}^R | \psi \rangle, \tag{137}$$

and, in particular, $\langle 0| : \hat{H} : |0\rangle = 0$. The normal ordered Hamiltonian thus produces a renormalised, sensible result for the vacuum energy.

3.4 Fock space and Particles

After this lengthy grappling with the vacuum state, we can continue to construct our basis of states in analogy to the harmonic oscillator, making use of the commutation relations for the operators $\hat{a}, \hat{a}^\dagger$. In particular, we define the state $|\mathbf{k}\rangle$ to be the one obtained by acting with the operator $a^\dagger(\mathbf{k})$ on the vacuum,

$$|\mathbf{k}\rangle = \hat{a}^\dagger(\mathbf{k})|0\rangle. \tag{138}$$

Using the commutator, its norm is found to be

$$\begin{aligned} \langle \mathbf{k}|\mathbf{k}'\rangle &= \langle 0|\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k}')|0\rangle = \langle 0|[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')]|0\rangle + \langle 0|\hat{a}^\dagger(\mathbf{k}')a(\mathbf{k})|0\rangle & (139)\\ &= (2\pi)^3 2E(\mathbf{k})\delta^3(\mathbf{k}-\mathbf{k}'), & (140) \end{aligned}$$

since the last term in the first line vanishes ($\hat{a}(\mathbf{k})$ acting on the vacuum). Next we compute the energy of this state, making use of the normal ordered Hamiltonian,

$$\begin{aligned} : \hat{H} : |\mathbf{k}\rangle &= \int \frac{d^3k'}{(2\pi)^3\, 2E(\mathbf{k}')} E(\mathbf{k}')\hat{a}^\dagger(\mathbf{k}')\hat{a}(\mathbf{k}')\hat{a}^\dagger(\mathbf{k})|0\rangle & (141)\\ &= \int \frac{d^3k'}{(2\pi)^3\, 2E(\mathbf{k}')} E(\mathbf{k}')(2\pi)^3 2E(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')\hat{a}^\dagger(\mathbf{k})|0\rangle & (142)\\ &= E(\mathbf{k})\hat{a}^\dagger(\mathbf{k})|0\rangle = E(\mathbf{k})|\mathbf{k}\rangle, & (143) \end{aligned}$$

and similarly one finds

$$: \hat{\mathbf{P}} : |\mathbf{k}\rangle = \mathbf{k}|\mathbf{k}\rangle. \tag{144}$$

Observing that the normal ordering did its job and we obtain renormalised, finite results, we may now interpret the state $|\mathbf{k}\rangle$. It is a one-particle state for a relativistic particle of mass m and momentum $\mathbf{k}$, since acting on it with the energy-momentum operator returns the relativistic one particle energy-momentum dispersion relation, $E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$. The $a^\dagger(\mathbf{k}), a(\mathbf{k})$ are creation and annihilation operators for particles of momentum $\mathbf{k}$.
In analogy to the harmonic oscillator, the procedure can be continued to higher states. One easily checks that (problem 3.4)

$$: \hat{P}^\mu : \hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle = (k_1^\mu + k_2^\mu)\hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle, \tag{145}$$

and so the state

$$|\mathbf{k}_2, \mathbf{k}_1\rangle = \frac{1}{\sqrt{2!}}\hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle \tag{146}$$

is a two-particle state (the factorial is there to have it normalised in the same way as the one-particle state), and so on for higher states. These are called *Fock states* in the textbooks (formally

speaking, a Fock space is a tensor product of Hilbert spaces, where the latter occur in ordinary Quantum Mechanics).
At long last we can now see how the field in our free quantum field theory is related to particles. A particle of momentum $\mathbf{k}$ corresponds to an excited Fourier mode of a field. Since the field is a superpositon of all possible Fourier modes, one field is enough to describe all possible configurations representing one or many particles of the same kind in any desired momentum state.
There are some rather profound ideas here about how nature works at fundamental scales. In classical physics we have matter particles, and forces which act on those particles. These forces can be represented by fields, such that fields and particles are distinct concepts. In non-relativistic quantum mechanics, one unifies the concept of waves and particles (particles can have wave-like characteristics), but fields are still distinct (e.g. one may quantise a particle in an electromagnetic field in QM, provided the latter is treated classically). Taking into account the effects of relativity for both particles and fields, one finds in QFT that all particles are excitation quanta of fields. That is, the concepts of *field* and *particle* are no longer distinct, but actually manifestations of the same thing, namely quantum fields. In this sense, QFT is more fundamental than either of its preceding theories. Each force field and each matter field have particles associated with it.
Returning to our theory for the free Klein-Gordon field, let us investigate what happens under interchange of the two particles. Since $[\hat{a}^\dagger(\mathbf{k}_1), \hat{a}^\dagger(\mathbf{k}_2)] = 0$ for all $\mathbf{k}_1, \mathbf{k}_2$, we see that

$$|\mathbf{k}_2, \mathbf{k}_1\rangle = |\mathbf{k}_1, \mathbf{k}_2\rangle, \tag{147}$$

and hence the state is symmetric under interchange of the two particles. Thus, the particles described by the scalar field are bosons.
Finally we complete the analogy to the harmonic oscillator by introducing a number operator

$$\hat{N}(\mathbf{k}) = \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}), \quad \hat{\mathcal{N}} = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})}\, \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}), \tag{148}$$

which gives us the number of bosons described by a particular Fock state,

$$\hat{\mathcal{N}}|0\rangle = 0, \quad \hat{\mathcal{N}}|\mathbf{k}\rangle = |\mathbf{k}\rangle, \quad \hat{\mathcal{N}}|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle = n|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle. \tag{149}$$

Of course the normal-ordered Hamiltonian can now simply be given in terms of this operator,

$$:\hat{H}: = \int \frac{d^3k}{(2\pi)^3\, 2E(\mathbf{k})} E(\mathbf{k})\hat{N}(\mathbf{k}), \tag{150}$$

i.e. when acting on a Fock state it simply sums up the energies of the individual particles to give

$$:\hat{H}: |\mathbf{k}_1 \ldots \mathbf{k}_n\rangle = (E(\mathbf{k}_1) + \ldots E(\mathbf{k}_n))\, |\mathbf{k}_1 \ldots \mathbf{k}_n\rangle. \tag{151}$$

This concludes the quantisation of our free scalar field theory. We have followed the canonical quantisation procedure familiar from quantum mechanics. Due to the infinite number of degrees of freedom, we encountered a divergent vacuum energy, which we had to renormalise. The renormalised Hamiltonian and the Fock states that we constructed describe free relativistic, uncharged spin zero particles of mass m, such as neutral pions, for example.

If we want to describe charged pions as well, we need to introduce complex scalar fields, the real and imaginary parts being necessary to describe opposite charges. For particles with spin we need

still more degrees of freedom and use vector or spinor fields, which have the appropriate rotation and Lorentz transformation properties. For fermion fields (which satisfy the Dirac equation rather than the Klein-Gordon equation), one finds that the condition of a positive-definite energy density requires that one impose anti-commutation relations rather than commutation relations. This in turn implies that multiparticle states are antisymmetric under interchange of identical fermions, which we recognise as the Pauli exclusion principle. Thus, not only does QFT provide a consistent theory of relativistic multiparticle systems; it also allows us to "derive" the Pauli principle, which is put in by hand in non-relativistic quantum mechanics.

More details on vector and spinor fields can be found in the other courses at this school. Here, we continue to restrict our attention to scalar fields, so as to more clearly illustrate what happens when interactions are present.

4 Quantum Field Theory: Interacting Fields

So far we have seen how to quantise the Klein-Gordon Lagrangian, and seen that this describes free scalar particles. For interesting physics, however, we need to know how to describe interactions, which lead to nontrivial scattering processes. This is the subject of this section.

From now on we shall always discuss quantised real scalar fields. It is then convenient to drop the "hats" on the operators that we have considered up to now. Interactions can be described by adding a term $\mathcal{L}_{\rm int}$ to the Lagrangian density, so that the full result $\mathcal{L}$ is given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int} \tag{152}$$

where

$$\mathcal{L}_0 = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 \tag{153}$$

is the free Lagrangian density discussed before. The Hamiltonian density of the interaction is related to $\mathcal{L}_{\rm int}$ simply by

$$\mathcal{H}_{\rm int} = \mathcal{H} - \mathcal{H}_0, \tag{154}$$

where $\mathcal{H}_0$ is the free Hamiltonian. If the interaction Lagrangian only depends on ϕ (we will consider such a case later in the course), one has

$$\mathcal{H}_{\rm int} = -\mathcal{L}_{\rm int}, \tag{155}$$

as can be easily shown from the definition above. We shall leave the details of $\mathcal{L}_{\rm int}$ unspecified for the moment. What we will be concerned with mostly are scattering processes, in which two initial particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ scatter, thereby producing a number of particles in the final state, characterised by momenta $\mathbf{k}_1, \ldots, \mathbf{k}_n$. This is schematically shown in Fig. 4. Our task is to find a description of such a scattering process in terms of the underlying quantum field theory.

4.1 The S-matrix

The timescales over which interactions happen are extremely short. The scattering (interaction) process takes place during a short interval around some particular time t with $-\infty \ll t \ll \infty$.

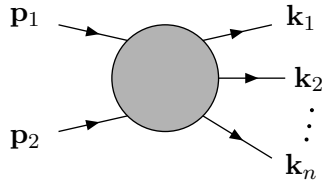


Figure 4: Scattering of two initial particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ into n particles with momenta $\mathbf{k}_1, \ldots, \mathbf{k}_n$ in the final state.

Long before t, the incoming particles evolve independently and freely. They are described by a field operator $\phi_{\rm in}$ defined through

$$\lim_{t\to-\infty} \phi(x) = \phi_{\rm in}(x), \tag{156}$$

which acts on a corresponding basis of $|{\rm in}\rangle$ states. Long after the collision the particles in the final state evolve again like in the free theory, and the corresponding operator is

$$\lim_{t\to+\infty} \phi(x) = \phi_{\rm out}(x), \tag{157}$$

acting on states $|{\rm out}\rangle$. The fields $\phi_{\rm in}$, $\phi_{\rm out}$ are the asymptotic limits of the Heisenberg operator ϕ. They both satisfy the free Klein-Gordon equation, i.e.

$$(\Box + m^2)\phi_{\rm in}(x) = 0, \qquad (\Box + m^2)\phi_{\rm out}(x) = 0. \tag{158}$$

Operators describing free fields can be expressed as a superposition of plane waves (see Eq. (113)). Thus, for $\phi_{\rm in}$ we have

$$\phi_{\rm in}(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\mathbf{k})} \left({\rm e}^{ik\cdot x} a^\dagger_{\rm in}(\mathbf{k}) + {\rm e}^{-ik\cdot x} a_{\rm in}(\mathbf{k}) \right), \tag{159}$$

with an entirely analogous expression for $\phi_{\rm out}(x)$. Note that the operators $a^\dagger$ and a also carry subscripts "in" and "out".

The above discussion assumes that the interaction is such that we can talk about free particles at asymptotic times $t \to \pm\infty$ i.e. that the interaction is only present at intermediate times. This is not always a reasonable assumption e.g. it does not encompass the phenomenon of bound states, in which incident particles form a composite object at late times, which no longer consists of free particles. Nevertheless, the assumption will indeed allow us to discuss scattering processes, which is the aim of this course. Note that we can only talk about well-defined particle states at $t \to \pm\infty$ (the states labelled by "in" and "out" above), as only at these times do we have a free theory, and thus know what the spectrum of states is (using the methods of section 3). At general times t, the interaction is present, and it is not possible in general to solve for the states of the quantum field theory. Remarkably, we will end up seeing that we can ignore all the complicated stuff at intermediate times, and solve for scattering probabilities purely using the properties of the asymptotic fields.

At the asymptotic times $t = \pm\infty$, we can use the creation operators $a^\dagger_{\rm in}$ and $a^\dagger_{\rm out}$ to build up Fock states from the vacuum. For instance

$$a^\dagger_{\rm in}(\mathbf{p}_1)\, a^\dagger_{\rm in}(\mathbf{p}_2)|0\rangle = |\mathbf{p}_1, \mathbf{p}_2; {\rm in}\rangle, \tag{160}$$

$$a^\dagger_{\rm out}(\mathbf{k}_1)\cdots a^\dagger_{\rm out}(\mathbf{k}_n)|0\rangle = |\mathbf{k}_1, \ldots, \mathbf{k}_n; {\rm out}\rangle. \tag{161}$$

We must now distinguish between Fock states generated by $a^\dagger_{\rm in}$ and $a^\dagger_{\rm out}$, and therefore we have labelled the Fock states accordingly. In eqs. (160) and (161) we have assumed that there is a stable and unique vacuum state of the free theory (the vacuum at general times t will be that of the full interacting theory, and thus differ from this in general):

$$|0\rangle = |0; {\rm in}\rangle = |0; {\rm out}\rangle. \tag{162}$$

Mathematically speaking, the $a^\dagger_{\rm in}$'s and $a^\dagger_{\rm out}$'s generate two different bases of the Fock space. Since the physics that we want to describe must be independent of the choice of basis, expectation values expressed in terms of "in" and "out" operators and states must satisfy

$$\langle {\rm in}|\, \phi_{\rm in}(x)\, |{\rm in}\rangle = \langle {\rm out}|\, \phi_{\rm out}(x)\, |{\rm out}\rangle\,. \tag{163}$$

Here $|{\rm in}\rangle$ and $|{\rm out}\rangle$ denote generic "in" and "out" states. We can relate the two bases by introducing a unitary operator S such that

$$\phi_{\rm in}(x) = S\, \phi_{\rm out}(x)\, S^\dagger \tag{164}$$

$$|{\rm in}\rangle = S\, |{\rm out}\rangle\,, \quad |{\rm out}\rangle = S^\dagger\, |{\rm in}\rangle\,, \quad S^\dagger S = 1. \tag{165}$$

S is called the S-matrix or S-operator. Note that the plane wave solutions of $\phi_{\rm in}$ and $\phi_{\rm out}$ also imply that

$$a^\dagger_{\rm in} = S\, a^\dagger_{\rm out}\, S^\dagger, \qquad \hat{a}_{\rm in} = S\, \hat{a}_{\rm out}\, S^\dagger. \tag{166}$$

By comparing "in" with "out" states one can extract information about the interaction – this is the very essence of detector experiments, where one tries to infer the nature of the interaction by studying the products of the scattering of particles that have been collided with known energies. As we will see below, this information is contained in the elements of the S-matrix.

By contrast, in the absence of any interaction, i.e. for $\mathcal{L}_{\rm int} = 0$ the distinction between $\phi_{\rm in}$ and $\phi_{\rm out}$ is not necessary. They can thus be identified, and then the relation between different bases of the Fock space becomes trivial, $S = 1$, as one would expect.

What we are ultimately interested in are transition amplitudes between an initial state i of, say, two particles of momenta $\mathbf{p}_1, \mathbf{p}_2$, and a final state f, for instance n particles of unequal momenta. The transition amplitude is then given by

$$\langle f,\, {\rm out}|\, i,\, {\rm in}\rangle = \langle f,\, {\rm out}|\, S\, |i,\, {\rm out}\rangle = \langle f,\, {\rm in}|\, S\, |i,\, {\rm in}\rangle \equiv S_{\rm fi}. \tag{167}$$

The S-matrix element $S_{\rm fi}$ therefore describes the transition amplitude for the scattering process in question. The scattering cross section, which is a measurable quantity, is then proportional to $|S_{\rm fi}|^2$. All information about the scattering is thus encoded in the S-matrix, which must therefore be closely related to the interaction Hamiltonian density $\mathcal{H}_{\rm int}$. However, before we try to derive the relation between S and $\mathcal{H}_{\rm int}$ we have to take a slight detour.

4.2 More on time evolution: Dirac picture

The operators $\phi(\mathbf{x},t)$ and $\pi(\mathbf{x},t)$ which we have encountered are Heisenberg fields and thus time-dependent. The state vectors are time-independent in the sense that they do not satisfy a non-trivial equation of motion. Nevertheless, state vectors in the Heisenberg picture can carry a time label. For instance, the "in"-states of the previous subsection are defined at $t=-\infty$. The relation of the Heisenberg operator $\phi_H(x)$ with its counterpart ϕ_S in the Schrödinger picture is given by

$$\phi_H(\mathbf{x},t) = \mathrm{e}^{iHt}\,\phi_S\,\mathrm{e}^{-iHt}, \qquad H = H_0 + H_{\rm int}, \tag{168}$$

Note that this relation involves the *full* Hamiltonian $H = H_0 + H_{\rm int}$ in the interacting theory. We have so far found solutions to the Klein-Gordon equation in the free theory, and so we know how to handle time evolution in this case. However, in the interacting case the Klein-Gordon equation has an extra term,

$$(\Box + m^2)\phi(x) + \frac{\partial V_{\rm int}(\phi)}{\partial\phi} = 0, \tag{169}$$

due to the potential of the interactions. Apart from very special cases of this potential, the equation cannot be solved anymore in closed form, and thus we no longer know the time evolution. It is therefore useful to introduce a new quantum picture for the interacting theory, in which the time dependence is governed by H_0 only. This is the so-called Dirac or Interaction picture. The relation between fields in the Interaction picture, ϕ_I, and in the Schrödinger picture, ϕ_S, is given by

$$\phi_I(\mathbf{x},t) = \mathrm{e}^{iH_0t}\,\phi_S\,\mathrm{e}^{-iH_0t}. \tag{170}$$

At $t=-\infty$ the interaction vanishes, i.e. $H_{\rm int}=0$, and hence the fields in the Interaction and Heisenberg pictures are identical, i.e. $\phi_H(\mathbf{x},t)=\phi_I(\mathbf{x},t)$ for $t\to-\infty$. The relation between ϕ_H and ϕ_I can be worked out easily:

$$\begin{aligned}
\phi_H(\mathbf{x},t) &= \mathrm{e}^{iHt}\,\phi_S\,\mathrm{e}^{-iHt} \\
&= \mathrm{e}^{iHt}\,\mathrm{e}^{-iH_0t}\underbrace{\mathrm{e}^{iH_0t}\phi_S\,\mathrm{e}^{-iH_0t}}_{\phi_I(\mathbf{x},t)}\,\mathrm{e}^{iH_0t}\,\mathrm{e}^{-iHt} \\
&= U^{-1}(t)\,\phi_I(\mathbf{x},t)\,U(t),
\end{aligned} \tag{171}$$

where we have introduced the unitary operator $U(t)$

$$U(t) = \mathrm{e}^{iH_0t}\,\mathrm{e}^{-iHt}, \qquad U^\dagger U = 1. \tag{172}$$

The field $\phi_H(\mathbf{x},t)$ contains the information about the interaction, since it evolves over time with the full Hamiltonian. In order to describe the "in" and "out" field operators, we can now make the following identifications:

$$t\to-\infty \;:\; \phi_{\rm in}(\mathbf{x},t) = \phi_I(\mathbf{x},t) = \phi_H(\mathbf{x},t), \tag{173}$$

$$t\to+\infty \;:\; \phi_{\rm out}(\mathbf{x},t) = \phi_H(\mathbf{x},t). \tag{174}$$

Furthermore, since the fields ϕ_I evolve over time with the free Hamiltonian H_0, they always act in the basis of "in" vectors, such that

$$\phi_{\rm in}(\mathbf{x},t) = \phi_I(\mathbf{x},t), \qquad -\infty < t < \infty. \tag{175}$$

The relation between ϕ_I and ϕ_H at any time t is given by

$$\phi_I(\mathbf{x},t) = U(t)\,\phi_H(\mathbf{x},t)\,U^{-1}(t). \tag{176}$$

As $t \to \infty$ the identifications of eqs. (174) and (175) yield

$$\phi_{\rm in} = U(\infty)\,\phi_{\rm out}\,U^\dagger(\infty). \tag{177}$$

From the definition of the S-matrix, Eq. (164) we then read off that

$$\lim_{t\to\infty} U(t) = S. \tag{178}$$

We have thus derived a formal expression for the S-matrix in terms of the operator $U(t)$, which tells us how operators and state vectors deviate from the free theory at time t, measured relative to $t_0 = -\infty$, i.e. long before the interaction process.
An important boundary condition for $U(t)$ is

$$\lim_{t\to-\infty} U(t) = 1. \tag{179}$$

What we mean here is the following: the operator U actually describes the evolution relative to some initial time t_0, which we will normally suppress, i.e. we write $U(t)$ instead of $U(t,t_0)$. We regard t_0 merely as a time label and fix it at $-\infty$, where the interaction vanishes. Equation (179) then simply states that U becomes unity as $t \to t_0$, which means that in this limit there is no distinction between Heisenberg and Dirac fields.
Using the definition of $U(t)$, Eq. (172), it is an easy exercise to derive the equation of motion for $U(t)$:

$$i\frac{d}{dt}\,U(t) = H_{\rm int}(t)\,U(t), \qquad H_{\rm int}(t) = \mathrm{e}^{iH_0t}\,H_{\rm int}\,\mathrm{e}^{-iH_0t}. \tag{180}$$

The time-dependent operator $H_{\rm int}(t)$ is defined in the interaction picture, and depends on the fields $\phi_{\rm in}$, $\pi_{\rm in}$ in the "in" basis. Let us now solve the equation of motion for $U(t)$ with the boundary condition $\lim_{t\to-\infty} U(t) = 1$. Integrating Eq. (180) gives

$$\begin{aligned}
\int_{-\infty}^{t} \frac{d}{dt_1}U(t_1)\,dt_1 &= -i\int_{-\infty}^{t} H_{\rm int}(t_1)\,U(t_1)\,dt_1 \\
U(t) - U(-\infty) &= -i\int_{-\infty}^{t} H_{\rm int}(t_1)\,U(t_1)\,dt_1 \\
\Rightarrow\; U(t) &= 1 - i\int_{-\infty}^{t} H_{\rm int}(t_1)\,U(t_1)\,dt_1.
\end{aligned} \tag{181}$$

The right-hand side still depends on U, but we can substitute our new expression for $U(t)$ into the integrand, which gives

$$\begin{aligned}
U(t) &= 1 - i\int_{-\infty}^{t} H_{\rm int}(t_1)\left\{1 - i\int_{-\infty}^{t_1} H_{\rm int}(t_2)\,U(t_2)\,dt_2\right\}dt_1 \\
&= 1 - i\int_{-\infty}^{t} H_{\rm int}(t_1)dt_1 - \int_{-\infty}^{t} dt_1\,H_{\rm int}(t_1)\int_{-\infty}^{t_1} dt_2\,H_{\rm int}(t_2)\,U(t_2),
\end{aligned} \tag{182}$$

where $t_2 < t_1 < t$. This procedure can be iterated further, so that the nth term in the sum is

$$(-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n \, H_{\rm int}(t_1)\, H_{\rm int}(t_2) \cdots H_{\rm int}(t_n). \tag{183}$$

This iterative solution could be written in much more compact form, were it not for the fact that the upper integration bounds were all different, and that the ordering $t_n < t_{n-1} < \ldots < t_1 < t$ had to be obeyed. Time ordering is an important issue, since one has to ensure that the interaction Hamiltonians act at the proper time, thereby ensuring the causality of the theory. By introducing the time-ordered product of operators, one can use a compact notation, such that the resulting expressions still obey causality. The time-ordered product of two fields $\phi(t_1)$ and $\phi(t_2)$ is defined as

$$\begin{aligned} T\left\{\phi(t_1)\,\phi(t_2)\right\} &= \begin{cases} \phi(t_1)\phi(t_2) & t_1 > t_2 \\ \phi(t_2)\phi(t_1) & t_1 < t_2 \end{cases} \\ &\equiv \theta(t_1 - t_2)\,\phi(t_1)\phi(t_2) + \theta(t_2 - t_1)\,\phi(t_2)\phi(t_1), \end{aligned} \tag{184}$$

where θ denotes the step function. The generalisation to products of n operators is obvious. Using time ordering for the nth term of Eq. (183) we obtain

$$\frac{(-i)^n}{n!} \prod_{i=1}^{n} \int_{-\infty}^{t} dt_i \; T\left\{H_{\rm int}(t_1)\, H_{\rm int}(t_2) \cdots H_{\rm int}(t_n)\right\}. \tag{185}$$

Here we have replaced each upper limit of integration with t. Each specific ordering in the time-ordered product gives a term identical to eq. (183), where applying the T operator corresponds to setting the upper limit of integration to the relevant t_i in each integral. However, we have overcounted by a factor $n!$, corresponding to the number of ways of ordering the fields in the time ordered product. Thus one must divide by $n!$ as shown. We may recognise eq. (185) as the nth term in the series expansion of an exponential, and thus can finally rewrite the solution for $U(t)$ in compact form as

$$U(t) = T \, \exp\left\{-i \int_{-\infty}^{t} H_{\rm int}(t')\, dt'\right\}, \tag{186}$$

where the "T" in front ensures the correct time ordering.

4.3 S-matrix and Green's functions

The S-matrix, which relates the "in" and "out" fields before and after the scattering process, can be written as

$$S = 1 + iT, \tag{187}$$

where T is commonly called the T-matrix. The fact that S contains the unit operator means that also the case where none of the particles scatter is encoded in S. On the other hand, the non-trivial case is described by the T-matrix, and this is what we are interested in. So far we have derived an expression for the S-matrix in terms of the interaction Hamiltonian, and we could use this in principle to calculate scattering processes. However, there is a slight complication owing to the fact that the vacuum of the free theory is not the same as the true vacuum of the full, interacting

theory. Instead, we will follow the approach of Lehmann, Symanzik and Zimmerman, which relates the S-matrix to *n-point Green's functions*

$$G_n(x_1, \dots x_n) = \langle 0|T(\phi(x_1) \dots \phi(x_n))|0\rangle \tag{188}$$

i.e. vacuum expectation values of Heisenberg fields. We will see later how to calculate these in terms of vacuum expectation values of "in" fields (i.e. in the free theory).
In order to relate S-matrix elements to Green's functions, we have to express the "in/out"-states in terms of creation operators $a^\dagger_{\rm in/out}$ and the vacuum, then express the creation operators by the fields $\phi_{\rm in/out}$, and finally use the time evolution to connect those with the fields ϕ in our Lagrangian.
Let us consider again the scattering process depicted in Fig. 4. The S-matrix element in this case is

$$\begin{aligned} S_{\rm fi} &= \Big\langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n; \text{out} \Big| \mathbf{p}_1, \mathbf{p}_2; \text{in} \Big\rangle \\ &= \Big\langle \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n; \text{out} \Big| a^\dagger_{\rm in}(\mathbf{p}_1) \Big| \mathbf{p}_2; \text{in} \Big\rangle, \end{aligned} \tag{189}$$

where $a^\dagger_{\rm in}$ is the creation operator pertaining to the "in" field $\phi_{\rm in}$. Our task is now to express $a^\dagger_{\rm in}$ in terms of $\phi_{\rm in}$, and repeat this procedure for all other momenta labelling our Fock states.
The following identities will prove useful

$$\begin{aligned} a^\dagger(\mathbf{p}) &= i \int d^3x \left\{ \left(\partial_0 \, \mathrm{e}^{-iq\cdot x}\right) \phi(x) - \mathrm{e}^{-iq\cdot x} \, \left(\partial_0 \phi(x)\right) \right\} \\ &\equiv -i \int d^3x \, \mathrm{e}^{-iq\cdot x} \overset{\longleftrightarrow}{\partial_0} \phi(x), \end{aligned} \tag{190}$$

$$\begin{aligned} \hat{a}(\mathbf{p}) &= -i \int d^3x \left\{ \left(\partial_0 \, \mathrm{e}^{iq\cdot x}\right) \phi(x) - \mathrm{e}^{iq\cdot x} \, \left(\partial_0 \phi(x)\right) \right\} \\ &\equiv i \int d^3x \, \mathrm{e}^{iq\cdot x} \overset{\longleftrightarrow}{\partial_0} \phi(x). \end{aligned} \tag{191}$$

The S-matrix element can then be rewritten as

$$\begin{aligned} S_{\rm fi} &= -i \int d^3x_1 \, \mathrm{e}^{-ip_1\cdot x_1} \overset{\longleftrightarrow}{\partial_0} \Big\langle \mathbf{k}_1, \dots, \mathbf{k}_n; \text{out} \Big| \phi_{\rm in}(x_1) \Big| \mathbf{p}_2; \text{in} \Big\rangle \\ &= -i \lim_{t_1 \to -\infty} \int d^3x_1 \, \mathrm{e}^{-ip_1\cdot x_1} \overset{\longleftrightarrow}{\partial_0} \Big\langle \mathbf{k}_1, \dots, \mathbf{k}_n; \text{out} \Big| \phi(x_1) \Big| \mathbf{p}_2; \text{in} \Big\rangle, \end{aligned} \tag{192}$$

where in the last line we have used Eq. (156) to replace $\phi_{\rm in}$ by ϕ. We can now rewrite $\lim_{t_1\to-\infty}$ using the following identity, which holds for an arbitrary, differentiable function $f(t)$, whose limit $t \to \pm\infty$ exists:

$$\lim_{t\to-\infty} f(t) = \lim_{t\to+\infty} f(t) - \int_{-\infty}^{+\infty} \frac{df}{dt} \, dt. \tag{193}$$

The S-matrix element then reads

$$\begin{aligned} S_{\rm fi} &= -i \lim_{t_1 \to +\infty} \int d^3x_1 \, \mathrm{e}^{-ip_1\cdot x_1} \overset{\longleftrightarrow}{\partial_0} \Big\langle \mathbf{k}_1, \dots, \mathbf{k}_n; \text{out} \Big| \phi(x_1) \Big| \mathbf{p}_2; \text{in} \Big\rangle \\ &\quad + i \int_{-\infty}^{+\infty} dt_1 \, \frac{\partial}{\partial t_1} \left\{ \int d^3x_1 \, \mathrm{e}^{-ip_1\cdot x_1} \overset{\longleftrightarrow}{\partial_0} \Big\langle \mathbf{k}_1, \dots, \mathbf{k}_n; \text{out} \Big| \phi(x_1) \Big| \mathbf{p}_2; \text{in} \Big\rangle \right\}. \end{aligned} \tag{194}$$

The first term in this expression involves $\lim_{t_1\to+\infty}\phi=\phi_{\rm out}$, which gives rise to a contribution

$$\propto\Big\langle\mathbf{k}_1,\ldots,\mathbf{k}_n;\mathrm{out}\Big|a^\dagger_{\rm out}(\mathbf{p}_1)\Big|\mathbf{p}_2;\mathrm{in}\Big\rangle. \tag{195}$$

This is non-zero only if $\mathbf{p}_1$ is equal to one of $\mathbf{k}_1,\ldots,\mathbf{k}_n$. This, however, means that the particle with momentum $\mathbf{p}_1$ does not scatter, and hence the first term does not contribute to the T-matrix of Eq. (187). We are then left with the following expression for $S_{\rm fi}$:

$$S_{\rm fi}=-i\int d^4x_1\,\Big\langle\mathbf{k}_1,\ldots,\mathbf{k}_n;\mathrm{out}\Big|\partial_0\left\{\left(\partial_0\mathrm{e}^{-ip_1\cdot x_1}\right)\phi(x_1)-\mathrm{e}^{-ip_1\cdot x_1}\left(\partial_0\phi(x_1)\right)\right\}\Big|\mathbf{p}_2;\mathrm{in}\Big\rangle. \tag{196}$$

The time derivatives in the integrand can be worked out:

$$\begin{aligned}&\partial_0\left\{\left(\partial_0\mathrm{e}^{-ip_1\cdot x_1}\right)\phi(x_1)-\mathrm{e}^{-ip_1\cdot x_1}\left(\partial_0\phi(x_1)\right)\right\}\\&=-\left[E(\mathbf{p}_1)\right]^2\,\mathrm{e}^{-ip_1\cdot x_1}\,\phi(x_1)-\mathrm{e}^{-ip_1\cdot x_1}\,\partial_0^2\phi(x_1)\\&=-\left\{\left(\left(-\nabla^2+m^2\right)\,\mathrm{e}^{-ip_1\cdot x_1}\right)\phi(x_1)+\mathrm{e}^{-ip_1\cdot x_1}\,\partial_0^2\,\phi(x_1)\right\},\end{aligned} \tag{197}$$

where we have used that $-\nabla^2\mathrm{e}^{-ip_1\cdot x_1}=\mathbf{p}_1^2\,\mathrm{e}^{-ip_1\cdot x_1}$. For the S-matrix element one obtains

$$\begin{aligned}S_{\rm fi}&=i\int d^4x_1\,\mathrm{e}^{-ip_1\cdot x_1}\Big\langle\mathbf{k}_1,\ldots,\mathbf{k}_n;\mathrm{out}\Big|\left(\partial_0^2-\nabla^2+m^2\right)\phi(x_1)\Big|\mathbf{p}_2;\mathrm{in}\Big\rangle\\&=i\int d^4x_1\,\mathrm{e}^{-ip_1\cdot x_1}\,\left(\Box_{x_1}+m^2\right)\Big\langle\mathbf{k}_1,\ldots,\mathbf{k}_n;\mathrm{out}\Big|\phi(x_1)\Big|\mathbf{p}_2;\mathrm{in}\Big\rangle,\end{aligned} \tag{198}$$

where we have used integration by parts twice so that ∇^2 acts on $\phi(x_1)$ rather than on $e^{-ip_1\cdot x_1}$. What we have obtained after this rather lengthy step of algebra is an expression in which the (Heisenberg) field operator is sandwiched between Fock states, one of which has been reduced to a one-particle state. We can now successively eliminate all momentum variables from the Fock states, by repeating the procedure for the momentum $\mathbf{p}_2$, as well as the n momenta of the "out" state. The final expression for $S_{\rm fi}$ is

$$\begin{aligned}S_{\rm fi}&=(i)^{n+2}\int d^4x_1\int d^4x_2\int d^4y_1\cdots\int d^4y_n\,\mathrm{e}^{(-ip_1\cdot x_1-ip_2\cdot x_2+ik_1\cdot y_1+\cdots+ik_n\cdot y_n)}\\&\times\left(\Box_{x_1}+m^2\right)\left(\Box_{x_2}+m^2\right)\left(\Box_{y_1}+m^2\right)\cdots\left(\Box_{y_n}+m^2\right)\\&\times\Big\langle0;\mathrm{out}\Big|T\{\phi(y_1)\cdots\phi(y_n)\phi(x_1)\phi(x_2)\}\Big|0;\mathrm{in}\Big\rangle,\end{aligned} \tag{199}$$

where the time-ordering inside the vacuum expectation value (VEV) ensures that causality is obeyed. The above expression is known as the *Lehmann-Symanzik-Zimmermann* (LSZ) *reduction formula*. It relates the formal definition of the scattering amplitude to a vacuum expectation value of time-ordered fields. Since the vacuum is uniquely the same for "in/out", the VEV in the LSZ formula for the scattering of two initial particles into n particles in the final state is recognised as the $(n+2)$-point Green's function:

$$G_{n+2}(y_1,y_2,\ldots,y_n,x_1,x_2)=\Big\langle0\Big|T\{\phi(y_1)\cdots\phi(y_n)\phi(x_1)\phi(x_2)\}\Big|0\Big\rangle. \tag{200}$$

You will note that we still have not calculated or evaluated anything, but merely rewritten the expression for the scattering matrix elements. Nevertheless, the LSZ formula is of tremendous

importance and a central piece of QFT. It provides the link between fields in the Lagrangian and the scattering amplitude $S_{\rm fi}^2$, which yields the cross section, measurable in an experiment. Up to here no assumptions or approximations have been made, so this connection between physics and formalism is rather tight. It also illustrates a profound phenomenon of QFT and particle physics: the scattering properties of particles, in other words their interactions, are encoded in the vacuum structure, i.e. the vacuum is non-trivial!

4.4 How to compute Green's functions

Of course, in order to calculate cross sections, we need to compute the Green's functions. Alas, for any physically interesting and interacting theory this cannot be done exactly, contrary to the free theory discussed earlier. Instead, approximation methods have to be used in order to simplify the calculation, while hopefully still giving reliable results. Or one reformulates the entire QFT as a lattice field theory, which in principle allows to compute Green's functions without any approximations (in practice this still turns out to be a difficult task for physically relevant systems). This is what many theorists do for a living. But the formalism stands, and if there are discrepancies between theory and experiments, one "only" needs to check the accuracy with which the Green's functions have been calculated or measured, before approving or discarding a particular Lagrangian.
In the next section we shall discuss how to compute the Green's function of scalar field theory in perturbation theory. Before we can tackle the actual computation, we must take a further step. Let us consider the n-point Green's function

$$G_n(x_1,\ldots,x_n) = \langle 0\,|T\{\phi(x_1)\cdots\phi(x_n)\}|\,0\rangle\,. \tag{201}$$

The fields ϕ which appear in this expression are Heisenberg fields, whose time evolution is governed by the full Hamiltonian $H_0 + H_{\rm int}$. In particular, the ϕ's are *not* the $\phi_{\rm in}$'s. We know how to handle the latter, because they correspond to a free field theory, but not the former, whose time evolution is governed by the interacting theory, whose solutions we do not know. Let us thus start to isolate the dependence of the fields on the interaction Hamiltonian. Recall the relation between the Heisenberg fields $\phi(t)$ and the "in"-fields[4]

$$\phi(t) = U^{-1}(t)\,\phi_{\rm in}(t)\,U(t). \tag{202}$$

We now assume that the fields are properly time-ordered, i.e. $t_1 > t_2 > \ldots > t_n$, so that we can forget about writing $T(\cdots)$ everywhere. After inserting Eq. (202) into the definition of G_n one obtains

$$\begin{aligned} G_n = \big\langle 0\big|U^{-1}(t_1)\phi_{\rm in}(t_1)U(t_1)\,U^{-1}(t_2)\phi_{\rm in}(t_2)U(t_2)\cdots \\ \times U^{-1}(t_n)\phi_{\rm in}(t_n)U(t_n)\big|0\big\rangle. \end{aligned} \tag{203}$$

Now we introduce another time label t such that $t \gg t_1$ and $-t \ll t_1$. For the n-point function we now obtain

$$\begin{aligned} G_n = \Big\langle 0\Big|U^{-1}(t)\Big\{U(t)U^{-1}(t_1)\phi_{\rm in}(t_1)U(t_1)\,U^{-1}(t_2)\phi_{\rm in}(t_2)U(t_2)\cdots \\ \times U^{-1}(t_n)\phi_{\rm in}(t_n)U(t_n)U^{-1}(-t)\Big\}U(-t)\Big|0\Big\rangle. \end{aligned} \tag{204}$$

[4]Here and in the following we suppress the spatial argument of the fields for the sake of brevity.

The expression in curly braces is now time-ordered by construction. An important observation at this point is that it involves pairs of U and its inverse, for instance

$$U(t)U^{-1}(t_1) \equiv U(t, t_1). \tag{205}$$

One can easily convince oneself that $U(t, t_1)$ provides the net time evolution from t_1 to t. We can now write G_n as

$$G_n = \Big\langle 0 \Big| U^{-1}(t)\, T\Big\{ \phi_{\rm in}(t_1) \cdots \phi_{\rm in}(t_n) \underbrace{U(t, t_1)\, U(t_1, t_2) \cdots U(t_n, -t)}_{U(t,-t)} \Big\} U(-t) \Big| 0 \Big\rangle, \tag{206}$$

where we have used the fact that we may commute the U operators within the time-ordered product. Let us now take $t \to \infty$. The relation between $U(t)$ and the S-matrix Eq. (178), as well as the boundary condition Eq. (179) tell us that

$$\lim_{t\to\infty} U(-t) = 1, \qquad \lim_{t\to\infty} U(t, -t) = S, \tag{207}$$

which can be inserted into the above expression. We still have to work out the meaning of $\langle 0|U^{-1}(\infty)$ in the expression for G_n. In a paper by Gell-Mann and Low it was argued that the time evolution operator must leave the vacuum invariant (up to a phase), which justifies the ansatz

$$\langle 0|U^{-1}(\infty) = K\langle 0|, \tag{208}$$

with K being the phase[5]. Multiplying this relation with $|0\rangle$ from the right gives

$$\langle 0|U^{-1}(\infty)|0\rangle = K\langle 0|0\rangle = K. \tag{209}$$

Furthermore, Gell-Mann and Low showed that

$$\langle 0|U^{-1}(\infty)|0\rangle = \frac{1}{\langle 0|U(\infty)|0\rangle}, \tag{210}$$

which implies

$$K = \frac{1}{\langle 0|S|0\rangle}. \tag{211}$$

After inserting all these relations into the expression for G_n we obtain

$$G_n(x_1, \ldots, x_n) = \frac{\langle 0|\, T\left\{\phi_{\rm in}(x_1) \cdots \phi_{\rm in}(x_n)\, S\right\} |0\rangle}{\langle 0|S|0\rangle}. \tag{212}$$

The S-matrix is given by

$$S = T\, \exp\left\{ -i \int_{-\infty}^{+\infty} H_{\rm int}(t)\, dt \right\}, \quad H_{\rm int} = H_{\rm int}(\phi_{\rm in}, \pi_{\rm in}), \tag{213}$$

and thus we have finally succeeded in expressing the n-point Green's function exclusively in terms of the "in"-fields. This completes the derivation of a relation between the general definition of the scattering amplitude $S_{\rm fi}$ and the VEV of time-ordered "in"-fields. This has been a long and

[5] As hinted at earlier, K relates the vacuum of the free theory to the true vacuum of the interacting theory.

technical discussion, but the main points are the following:

Scattering probabilities are related to S-matrix elements. To calculate S-matrix elements for an n particle scattering process, one must first calculate the n particle Green's function (eq. (212)). Then one plugs this into the LSZ formula (eq. (199)).

In fact, the Green's functions cannot be calculated exactly using eq. (212). Instead, one can only obtain answers in the limit in which the interaction strength λ is small. This is the subject of the following sections.

5 Perturbation Theory

In this section we are going to calculate the Green's functions of scalar quantum field theory explicitly. We will specify the interaction Lagrangian in detail and use an approximation known as perturbation theory. At the end we will derive a set of rules, which represent a systematic prescription for the calculation of Green's functions, and can be easily generalised to apply to other, more complicated field theories. These are the famous *Feynman rules*.
We start by making a definite choice for the interaction Lagrangian $\mathcal{L}_{\text{int}}$. Although one may think of many different expressions for $\mathcal{L}_{\text{int}}$, one has to obey some basic principles: firstly, $\mathcal{L}_{\text{int}}$ must be chosen such that the potential it generates is bounded from below – otherwise the system has no ground state. Secondly, our interacting theory should be *renormalisable*. Despite being of great importance, the second issue will not be addressed in these lectures. The requirement of renormalisability arises because the non-trivial vacuum, much like a medium, interacts with particles to modify their properties. Moreover, if one computes quantities like the energy or charge of a particle, one typically obtains a divergent result[6]. There are classes of quantum field theories, called renormalisable, in which these divergences can be removed by suitable redefinitions of the fields and the parameters (masses and coupling constants).
For our theory of a real scalar field in four space-time dimensions, it turns out that the only interaction term which leads to a renormalisable theory must be quartic in the fields. Thus we choose

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4(x), \tag{214}$$

where the coupling constant λ describes the strength of the interaction between the scalar fields, much like, say, the electric charge describing the strength of the interaction between photons and electrons. The factor 4! is for later convenience. The full Lagrangian of the theory then reads

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{215}$$

and the explicit expressions for the interaction Hamiltonian and the S-matrix are

$$\begin{aligned} &\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}, \quad H_{\text{int}} = \frac{\lambda}{4!}\int d^3x\, \phi^4_{\text{in}}(\mathbf{x},t) \\ &S = T\,\exp\left\{-i\frac{\lambda}{4!}\int d^4x\, \phi^4_{\text{in}}(x)\right\}. \end{aligned} \tag{216}$$

[6]This is despite the subtraction of the vacuum energy discussed earlier.

The n-point Green's function is

$$G_n(x_1,\ldots,x_n) = \frac{\displaystyle\sum_{r=0}^{\infty}\left(-\frac{i\lambda}{4!}\right)^r\frac{1}{r!}\left\langle 0\left|T\left\{\phi_{\rm in}(x_1)\cdots\phi_{\rm in}(x_n)\left(\int d^4y\,\phi_{\rm in}^4(y)\right)^r\right\}\right|0\right\rangle}{\displaystyle\sum_{r=0}^{\infty}\left(-\frac{i\lambda}{4!}\right)^r\frac{1}{r!}\left\langle 0\left|T\left(\int d^4y\,\phi_{\rm in}^4(y)\right)^r\right|0\right\rangle}. \tag{217}$$

This expression cannot be dealt with as it stands. In order to evaluate it we must expand G_n in powers of the coupling λ and truncate the series after a finite number of terms. This only makes sense if λ is sufficiently small. In other words, the interaction Lagrangian must act as a small perturbation on the system. As a consequence, the procedure of expanding Green's functions in powers of the coupling is referred to as *perturbation theory*. We will see that there is a natural diagrammatic representation of this expansion (Feynman diagrams). First, we need to know how to calculate the vacuum expectation values of time ordered products. This is the subject of the next section.

5.1 Wick's Theorem

The n-point Green's function in Eq. (217) involves the time-ordered product over at least n fields. There is a method to express VEV's of n fields, i.e. $\langle 0|T\left\{\phi_{\rm in}(x_1)\cdots\phi_{\rm in}(x_n)\right\}|0\rangle$ in terms of VEV's involving two fields only. This is known as Wick's theorem.
Let us for the moment ignore the subscript "in" and return to the definition of normal-ordered fields. The normal-ordered product $:\phi(x_1)\phi(x_2):$ differs from $\phi(x_1)\phi(x_2)$ by the vacuum expectation value, i.e.

$$\phi(x_1)\phi(x_2) = :\phi(x_1)\phi(x_2): + \langle 0|\phi(x_1)\phi(x_2)|0\rangle. \tag{218}$$

We are now going to combine normal-ordered products with time ordering. The time-ordered product $T\{\phi(x_1)\phi(x_2)\}$ is given by

$$\begin{aligned} T\{\phi(x_1)\phi(x_2)\} &= \phi(x_1)\phi(x_2)\theta(t_1-t_2)+\phi(x_2)\phi(x_1)\theta(t_2-t_1) \\ &= :\phi(x_1)\phi(x_2): \Big(\theta(t_1-t_2)+\theta(t_2-t_1)\Big) \\ &\quad +\langle 0|\phi(x_1)\phi(x_2)\theta(t_1-t_2)+\phi(x_2)\phi(x_1)\theta(t_2-t_1)|0\rangle. \end{aligned} \tag{219}$$

Here we have used the important observation that

$$:\phi(x_1)\phi(x_2): = :\phi(x_2)\phi(x_1):, \tag{220}$$

which means that normal-ordered products of fields are automatically time-ordered.[7] Equation (219) is Wick's theorem for the case of two fields:

$$T\{\phi(x_1)\phi(x_2)\} = :\phi(x_1)\phi(x_2): + \langle 0|T\left\{\phi(x_1)\phi(x_2)\right\}|0\rangle. \tag{221}$$

For the case of three fields, Wick's theorem yields

$$\begin{aligned} T\{\phi(x_1)\phi(x_2)\phi(x_3)\} &= :\phi(x_1)\phi(x_2)\phi(x_3): + :\phi(x_1):\langle 0|T\{\phi(x_2)\phi(x_3)\}|0\rangle \\ &\quad + :\phi(x_2):\langle 0|T\{\phi(x_1)\phi(x_3)\}|0\rangle + :\phi(x_3):\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle \end{aligned} \tag{222}$$

[7] The reverse is, however, not true!

At this point the general pattern becomes clear: any time-ordered product of fields is equal to its normal-ordered version plus terms in which pairs of fields are removed from the normal-ordered product and sandwiched between the vacuum to form 2-point functions. Then one sums over all permutations. Without proof we give the expression for the general case of n fields (n even):

$$\begin{aligned}
T\{\phi(x_1)\cdots\phi(x_n)\} = & \\
& :\phi(x_1)\cdots\phi(x_n): \\
& +:\phi(x_1)\cdots\widehat{\phi(x_i)}\cdots\widehat{\phi(x_j)}\cdots\phi(x_n):\langle 0|T\{\phi(x_i)\phi(x_j)\}|0\rangle + \text{perms.} \\
& +:\phi(x_1)\cdots\widehat{\phi(x_i)}\cdots\widehat{\phi(x_j)}\cdots\widehat{\phi(x_k)}\cdots\widehat{\phi(x_l)}\cdots\phi(x_n): \\
& \quad\times\langle 0|T\{\phi(x_i)\phi(x_j)\}|0\rangle\langle 0|T\{\phi(x_k)\phi(x_l)\}|0\rangle + \text{perms.} \\
& +\ldots+ \\
& +\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle\langle 0|T\{\phi(x_3)\phi(x_4)\}|0\rangle\cdots\langle 0|T\{\phi(x_{n-1})\phi(x_n)\}|0\rangle \\
& \quad + \text{perms.}.
\end{aligned} \tag{223}$$

The symbol $\widehat{\phi(x_i)}$ indicates that $\phi(x_i)$ has been removed from the normal-ordered product.
Let us now go back to $\langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle$. If we insert Wick's theorem, then we find that only the contribution in the last line of Eq. (223) survives: by definition the VEV of a normal-ordered product of fields vanishes, and it is precisely the last line of Wick's theorem in which no normal-ordered products are left. The only surviving contribution is that in which all fields have been paired or "contracted". Sometimes a contraction is represented by the notation:

$$\phi\,\underbrace{(x_i)\phi}(x_j) \equiv \langle 0|T\{\phi(x_i)\phi(x_j)\}|0\rangle, \tag{224}$$

i.e. the pair of fields which is contracted is joined by the braces. Wick's theorem can now be rephrased as

$$\langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle \;=\; \text{sum of all possible contractions of } n \text{ fields.} \tag{225}$$

An example of this result is the 4-point function

$$\begin{aligned}
\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle = & \;\phi\,\underbrace{(x_1)\phi}(x_2)\phi\,\underbrace{(x_3)\phi}(x_4) \\
& +\phi\,\underbrace{(x_1)\phi\,\overbrace{(x_2)\phi}(x_3)\phi}(x_4) + \underbrace{\phi(x_1)\phi\,\overbrace{(x_2)\phi}(x_3)\phi}(x_4)
\end{aligned} \tag{226}$$

5.2 The Feynman propagator

Using Wick's Theorem one can relate any n-point Green's functions to an expression involving only 2-point functions. Let us have a closer look at

$$G_2(x,y) = \langle 0|T\{\phi_{\rm in}(x)\phi_{\rm in}(y)\}|0\rangle. \tag{227}$$

We can now insert the solution for ϕ in terms of $\hat{a}$ and $\hat{a}^\dagger$. If we assume $t_x > t_y$ then $G_2(x,y)$ can be written as

$$\begin{aligned} G_2(x,y) &= \int \frac{d^3p\, d^3q}{(2\pi)^6\, 4E(\mathbf{p})E(\mathbf{q})} \\ &\quad \times \left\langle 0 \left| \left(\hat{a}^\dagger(\mathbf{p})\, \mathrm{e}^{ip\cdot x} + \hat{a}(\mathbf{p})\, \mathrm{e}^{-ip\cdot x}\right) \left(\hat{a}^\dagger(\mathbf{q})\, \mathrm{e}^{iq\cdot y} + \hat{a}(\mathbf{q})\, \mathrm{e}^{-iq\cdot y}\right) \right| 0 \right\rangle \\ &= \int \frac{d^3p\, d^3q}{(2\pi)^6\, 4E(\mathbf{p})E(\mathbf{q})}\, \mathrm{e}^{-ip\cdot x+iq\cdot y} \left\langle 0 \left| \hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{q}) \right| 0 \right\rangle . \end{aligned} \tag{228}$$

This shows that G_2 can be interpreted as the amplitude for a meson which is created at y and destroyed again at point x. We can now replace $\hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{q})$ by its commutator:

$$\begin{aligned} G_2(x,y) &= \int \frac{d^3p\, d^3q}{(2\pi)^6\, 4E(\mathbf{p})E(\mathbf{q})}\, \mathrm{e}^{-ip\cdot x+iq\cdot y} \left\langle 0 \left| \left[\hat{a}(\mathbf{p}), \hat{a}^\dagger(\mathbf{q})\right] \right| 0 \right\rangle \\ &= \int \frac{d^3p}{(2\pi)^3\, 2E(\mathbf{p})}\, \mathrm{e}^{-ip\cdot(x-y)}, \end{aligned} \tag{229}$$

and the general result, after restoring time-ordering, reads

$$G_2(x,y) = \int \frac{d^3p}{(2\pi)^3\, 2E(\mathbf{p})} \left(\mathrm{e}^{-ip\cdot(x-y)}\theta(t_x - t_y) + \mathrm{e}^{ip\cdot(x-y)}\theta(t_y - t_x)\right). \tag{230}$$

Furthermore, using contour integration one can show that this expression can be rewritten as a 4-dimensional integral

$$G_2(x,y) = i \int \frac{d^4p}{(2\pi)^4}\, \frac{\mathrm{e}^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}, \tag{231}$$

where ϵ is a small parameter which ensures that G_2 does not develop a pole. This calculation has established that $G_2(x,y)$ actually depends only on the difference $(x-y)$. Equation (231) is called the *Feynman propagator* $G_F(x-y)$:

$$G_F(x-y) \equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = i \int \frac{d^4p}{(2\pi)^4}\, \frac{\mathrm{e}^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}. \tag{232}$$

The Feynman propagator is a Green's function of the Klein-Gordon equation, i.e. it satisfies

$$\left(\Box_x + m^2\right)\, G_F(x-y) = -i\delta^4(x-y), \tag{233}$$

and describes the propagation of a meson between the space-time points x and y.

5.3 Two-particle scattering to O(λ)

Let us now consider a scattering process in which two incoming particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ scatter into two outgoing ones with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$, as shown in Fig. 5. The S-matrix element in this case is

$$\begin{aligned} S_{\mathrm{fi}} &= \langle \mathbf{k}_1, \mathbf{k}_2; \mathrm{out} | \mathbf{p}_1, \mathbf{p}_2; \mathrm{in} \rangle \\ &= \langle \mathbf{k}_1, \mathbf{k}_2; \mathrm{in} | S | \mathbf{p}_1, \mathbf{p}_2; \mathrm{in} \rangle, \end{aligned} \tag{234}$$

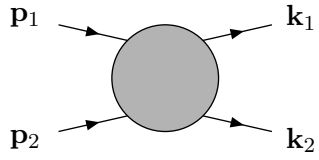


Figure 5: Scattering of two initial particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ into 2 particles with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$.

and $S = 1 + iT$. The LSZ formula Eq. (199) tells us that we must compute G_4 in order to obtain $S_{\rm fi}$. Let us work out G_4 in powers of λ using Wick's theorem.

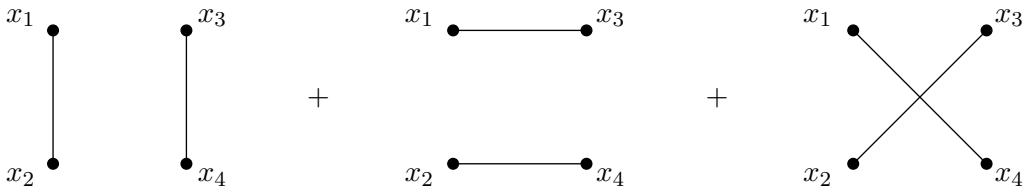
Suppressing the subscripts "in" from now on, the expression we have to evaluate order by order in λ is

$$G_4(x_1,\ldots,x_4) \tag{235}$$
$$= \frac{\displaystyle\sum_{r=0}^{\infty}\left(-\frac{i\lambda}{4!}\right)^r\frac{1}{r!}\left\langle 0\left|T\left\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\left(\int d^4y\,\phi^4(y)\right)^r\right\}\right|0\right\rangle}{\displaystyle\sum_{r=0}^{\infty}\left(-\frac{i\lambda}{4!}\right)^r\frac{1}{r!}\left\langle 0\left|T\left(\int d^4y\,\phi^4(y)\right)^r\right|0\right\rangle}.$$

At $\mathcal{O}(\lambda^0)$, the denominator is 1, and the numerator gives

$$\begin{aligned}\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle &= G_F(x_1-x_2)\,G_F(x_3-x_4)+G_F(x_1-x_3)\,G_F(x_2-x_4)\\ &\quad + G_F(x_1-x_4)\,G_F(x_2-x_3),\end{aligned} \tag{236}$$

where we have used Wick's theorem. We may represent this graphically as follows:

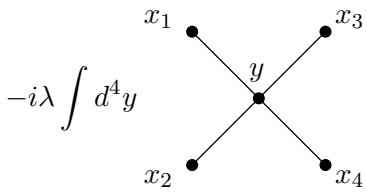


But this is the same answer as if we had set $\lambda = 0$, so $\mathcal{O}(\lambda^0)$ does not describe scattering and hence is not a contribution to the T-matrix.

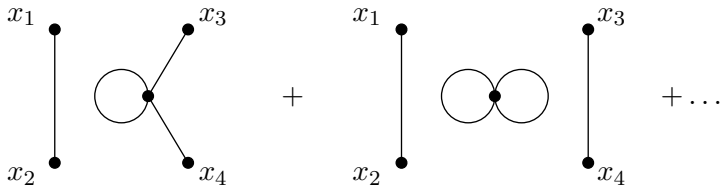
The first non-trivial scattering happens at $\mathcal{O}(\lambda)$. For example, the expansion of the above formula includes the contribution (from the numerator)

$$\begin{aligned}-\frac{i\lambda}{4!}\langle 0|T[\phi(x_1)\ldots\phi(x_4)\int d^4y\phi^4(y)|0\rangle &= -\frac{i\lambda}{4!}\int d^4y\,4!G_F(x_1-y)G_F(x_2-y)G_F(x_3-y)\\ &\quad\times G_F(x_4-y),\end{aligned} \tag{237}$$

where the 4! inside the integral arises from all possible contractions in Wick's theorem. This has the graphical representation



where each line corresponds to a propagator, and we have assinged a vertex to each space-time point. Also at this order, we have the graphs



We will see later on that neither of these graphs contributes to the S-matrix element (after substituting the Green's function into the LSZ formula of eq. (199)), as they are not fully connected. By this we mean that not all external particle vertices are connected to the same graph. At yet higher orders, we may have graphs wich involve fully connected pieces, dressed by additional "vacuum bubbles" (such as that which is sitting in the middle of the right-most figure above). These vacuum bubbles are cancelled by the denominator in eq. (212) which, given that it contains no external fields, generates all possible vacuum graphs. The presence of these vacuum graphs explains why the vacuum of the interacting theory is different to that of the free theory, as mentioned earlier.

To summarise, the final answer for the scattering amplitude to $\mathrm{O}(\lambda)$ is given by Eq. (237).

5.4 Graphical representation of the Wick expansion: Feynman rules

We have already encountered the graphical representation of the expansion of Green's functions in perturbation theory after applying Wick's theorem. It is possible to formulate a simple set of rules which allow us to draw the graphs directly without using Wick's theorem and to write down the corresponding algebraic expressions.
We again consider a neutral scalar field whose Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{238}$$

Suppose now that we want to compute the $\mathrm{O}(\lambda^m)$ contribution to the n-point Green's function $G_n(x_1, \ldots, x_n)$. This is achieved by going through the following steps:

(1) Draw all *distinct* diagrams with n external lines and m 4-fold vertices:

- Draw n dots and label them $x_1, \ldots, x_n$ (external points)
- Draw m dots and label them $y_1, \ldots, y_m$ (vertices)

- Join the dots according to the following rules:
  - only one line emanates from each x_i
  - exactly four lines run into each y_j
  - the resulting diagram must be connected, i.e. there must be a continuous path between any two points.

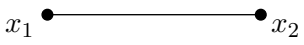
(2) Assign a factor $-\dfrac{i\lambda}{4!}\displaystyle\int d^4y_i$ to the vertex at y_i

(3) Assign a factor $G_F(x_i - y_j)$ to the line joining x_i and y_j

(4) Multiply by the number of contractions $\mathcal{C}$ from the Wick expansion which lead to the same diagram.

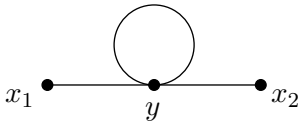
These are the Feynman rules for scalar field theory in *position space*.
Let us look at an example, namely the 2-point function. According to the Feynman rules the contributions up to order λ^2 are as follows:

$\mathcal{O}(1)$:



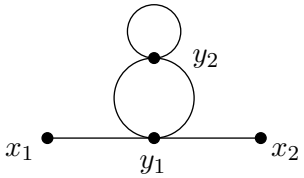
$= G_F(x_1 - x_2)$

$\mathcal{O}(\lambda)$:



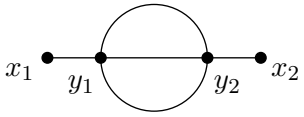
$= \frac{-i\lambda}{2}\int d^4y G_F(x_1 - y)G_F(x_2 - y)G_F(0)$

$\mathcal{O}(\lambda^2)$:



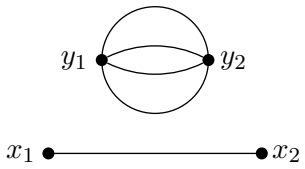
$= -\frac{\lambda^2}{4}\int d^4y \int d^4z G_F(x_1 - y)G_F(x_2 - y)$
$\times G_F^2(y - z)G_F(0)$

$\mathcal{O}(\lambda^2)$:



$$= \mathcal{C}\left(-\frac{i\lambda}{4!}\right)^2 \int d^4y_1 d^4y_2\; G_F(x_1 - y_1)\left[G_F(y_1 - y_2)\right]^3 G_F(y_2 - x_2)$$

The combinatorial factor for this contribution is worked out as $\mathcal{C} = 4 \cdot 4!$. Note that the same graph, but with the positions of y_1 and y_2 interchanged is topologically distinct. Numerically it has the same value as the above graph, and so the corresponding expression has to be multiplied by a factor 2.
Another contribution at order λ^2 is

$O(\lambda^2)$: 

vacuum contribution;
not connected

This contribution must be discarded, since not all of the points are connected via a continuous line.

5.5 Feynman rules in momentum space

It is often simpler to work in momentum space, and hence we will discuss the derivation of Feynman rules in this case. This also reflects what is typically done in scattering experiments (i.e. incoming and outgoing particles have definite momentum). If one works in momentum space, the Green's functions are related to those in position space by a Fourier transform

$$\tilde{G}_n(p_1,\ldots,p_n) = \int d^4x_1 \cdots \int d^4x_n \, \mathrm{e}^{ip_1\cdot x_1+\ldots+ip_n\cdot x_n} \, G_n(x_1,\ldots,x_n). \tag{239}$$

The Feynman rules then serve to compute the Green's function $\widetilde{G}_n(p_1,\ldots,p_n)$ order by order in the coupling.

Let us see how this works for the $2\to 2$ scattering example we considered above. At $\mathcal{O}(\lambda)$ this was given in eq. (237), which we may simplify slightly to

$$-i\lambda \int d^4y \, G_F(x_1-y)G_F(x_2-y)G_F(x_3-y)G_F(x_4-y). \tag{240}$$

We may now substitute in the momentum space form of each propagator (eq. (232)) to give

$$\begin{aligned} &-i\lambda \int d^4y \left(\prod_{i=1}^4 \int \frac{d^4p_i}{(2\pi)^4}\frac{i}{p_i^2-m^2+i\epsilon}\right) e^{-i\sum_i p_i\cdot(x_i-y)} \\ &= -i\lambda(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\left(\prod_{i=1}^4 \int \frac{d^4p_i}{(2\pi)^4}\frac{i}{p_i^2-m^2+i\epsilon}\right) e^{-i\sum_i p_i\cdot x_i}, \end{aligned}$$

where we have carried out the y integration in the second line. Substituting this into eq. (239) and carrying out the integrals over each x_i, one finds

$$\begin{aligned} \tilde{G}_4(p_1,\ldots,p_n) &= -i\lambda(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\left(\prod_i^4 \int \frac{d^4p_i}{(2\pi)^4}\frac{i}{p_i^2-m^2+i\epsilon}(2\pi)^4\delta(p_i)\right) \\ &= -i\lambda(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\prod_i \frac{i}{p_i^2-m^2+i\epsilon} \end{aligned}$$

We will not repeat the above derivation for a general Green's function. Rather, we now state the Feynman rules in momentum space, and the reader may easily verify that the above example is a special case.

Feynman rules (momentum space)

(1) Draw all *distinct* diagrams with n external lines and m 4-fold vertices:

- Assign momenta $p_1, \ldots, p_n$ to the external lines
- Assign momenta k_j to the internal lines

(2) Assign to each external line a factor

$$\frac{i}{p_k^2 - m^2 + i\epsilon}$$

(3) Assign to each internal line a factor

$$\int \frac{d^4k_j}{(2\pi)^4} \frac{i}{k_j^2 - m^2 + i\epsilon}$$

(4) Each vertex contributes a factor

$$-\frac{i\lambda}{4!}(2\pi)^4\delta^4\left(\sum \text{momenta}\right),$$

(the delta function ensures that momentum is conserved at each vertex).

(5) Multiply by the combinatorial factor $\mathcal{C}$, which is the number of contractions leading to the same momentum space diagram (note that $\mathcal{C}$ may be different from the combinatorial factor for the same diagram considered in position space!)

Alternatively, one may rephrase (4) and (5) as follows:

(4*) Each vertex carries a factor

$$-i\lambda(2\pi)^4\delta^4\left(\sum \text{momenta}\right),$$

(5*) Divide by the *symmetry factor* i.e. the dimension of the group of symmetry transformations that leaves the diagram invariant.

5.6 S-matrix and truncated Green's functions

The final topic in these lectures is the derivation of a simple relation between the S-matrix element and a particular momentum space Green's function, which has its external legs amputated: the so-called truncated Green's function. This further simplifies the calculation of scattering amplitudes using Feynman rules.

Let us return to the LSZ formalism and consider the scattering of m initial particles (momenta $\mathbf{p}_1, \ldots, \mathbf{p}_m$) into n final particles with momenta $\mathbf{k}_1, \ldots, \mathbf{k}_n$. The LSZ formula (eq. (199)) tells us that the S-matrix element is given by

$$\begin{aligned}
&\left\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \middle| \mathbf{p}_1, \ldots, \mathbf{p}_m; \text{in} \right\rangle \\
&\quad = (i)^{n+m} \int \prod_{i=1}^{m} d^4x_i \int \prod_{j=1}^{n} d^4y_j \exp\left\{ -i\sum_{i=1}^{m} p_i \cdot x_i + i\sum_{j=1}^{n} k_j \cdot y_j \right\} \\
&\qquad \times \prod_{i=1}^{m} (\Box_{x_i} + m^2) \prod_{j=1}^{n} (\Box_{y_j} + m^2)\; G_{n+m}(x_1, \ldots, x_m, y_1, \ldots, y_n).
\end{aligned} \tag{241}$$

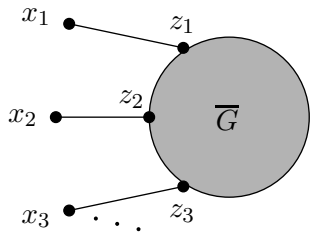


Figure 6: The construction of the truncated Green's function in position space.

Let us have a closer look at $G_{n+m}(x_1,\ldots,x_m,y_1,\ldots,y_n)$. As shown in Fig. 6 it can be split into Feynman propagators, which connect the external points to the vertices at $z_1,\ldots,z_{n+m}$, and a remaining Green's function $\overline{G}_{n+m}$, according to

$$G_{n+m} = \int d^4z_1\cdots d^4z_{n+m}\, G_F(x_1-z_1)\cdots G_F(y_n-z_{n+m})\,\overline{G}_{n+m}(z_1,\ldots,z_{n+m}), \tag{242}$$

where, perhaps for obvious reasons, $\overline{G}_{n+m}$ is called the *truncated* Green's function.
Putting Eq. (242) back into the LSZ expression for the S-matrix element, and using that

$$\left(\Box_{x_i}+m^2\right)\, G_F(x_i-z_i) = -i\delta^4(x_i-z_i) \tag{243}$$

one obtains

$$\begin{aligned}
&\Big\langle \mathbf{k}_1,\ldots,\mathbf{k}_n;\mathrm{out}\Big|\mathbf{p}_1,\ldots,\mathbf{p}_m;\mathrm{in}\Big\rangle \\
&\quad = \quad (i)^{n+m}\int\prod_{i=1}^m d^4x_i\int\prod_{j=1}^n d^4y_j\exp\left\{-i\sum_{i=1}^m p_i\cdot x_i + i\sum_{j=1}^n k_j\cdot y_j\right\} \\
&\qquad\quad \times(-i)^{n+m}\int d^4z_1\cdots d^4z_{n+m}\;\delta^4(x_1-z_1)\cdots\delta^4(y_n-z_{n+m})\,\overline{G}_{n+m}(z_1,\ldots,z_{n+m}).
\end{aligned} \tag{244}$$

After performing all the integrations over the z_k's, the final relation becomes

$$\begin{aligned}
&\Big\langle \mathbf{k}_1,\ldots,\mathbf{k}_n;\mathrm{out}\Big|\mathbf{p}_1,\ldots,\mathbf{p}_m;\mathrm{in}\Big\rangle \\
&\quad = \quad \int\prod_{i=1}^m d^4x_i\prod_{j=1}^n d^4y_j\exp\left\{-i\sum_{i=1}^m p_i\cdot x_i + i\sum_{j=1}^n k_j\cdot y_j\right\} \\
&\qquad \times\,\overline{G}_{n+m}(x_1,\ldots,x_m,y_1,\ldots,y_n) \\
&\quad \equiv \quad \overline{\mathcal{G}}_{n+m}(p_1,\ldots,p_m,k_1,\ldots,k_n),
\end{aligned} \tag{245}$$

where $\overline{\mathcal{G}}_{n+m}$ is the truncated $n+m$-point function in momentum space. This result shows that the scattering matrix element is directly given by the truncated Green's function in momentum space. In other words, calculating the S-matrix is much the same as calculating the Green's function, but without the free propagators associated with the external legs. Note that this renders zero any graph which is not fully connected - any diagram in which not all external points are connected to the same graph vanishes upon multiplication by the $(p_i^2+m^2)$ factors. This is what allowed us to neglect such graphs in the previous section.

6 Summary

That completes this introductory look at quantum field theory. Although we did not get as far as some of the more relevant physical applications of QFT, we have looked in detail at what a QFT is, and how the description of scattering amplitudes leads to Feynman diagrams. To recap how we did this:

1. We reviewed the Lagrangian formalism for classical field theory, and also the canonical quantisation approach to quantum mechanics.

2. We constructed the Lagrangian for a relativistic field theory (the free Klein-Gordon field), and applied the techniques of canonical quantisation to this field theory.

3. States in this theory were found to represent particle excitations, such that a particle of momentum $\mathbf{p}$ was found to be a quantum of excitation in the relevant Fourier mode of the field.

4. We then studied the interacting theory, arguing that at initial and final times (when the interaction dies away) we can work with free fields. These were related by an operator S, whose matrix elements represented the transition probability to go from a given initial to a given final state.

5. Using the interaction picture for time evolution, we found an expression for the S matrix in terms of an evolution operator U, describing how the fields at general time t deviate from the initial free fields.

6. We also found a formula which related S matrix elements to n-particle Green's functions (vacuum expectation values of time-ordered fields). This was the LSZ formula of eq. (199).

7. We related the Green's functions involving Heisenberg fields to those involving the "in" fields at time $t \to -\infty$ (eq. (212)).

8. We then found how to compute these Green's functions in perturbation theory, valid when the strength of the interaction is weak. This involved having to calculate vacuum expectation values of time-ordered products, for which we could use Wick's theorem.

9. We developed a graphical representation of Wick's theorem, which led to simple rules (Feynman rules) for the calculation of Green's functions in position or momentum space.

10. These can easily be converted to S matrix elements by truncating the free propagators associated with the external lines.

Needless to say, there are many things we did not have time to talk about. Some of these will be explored by the other courses at this school:

- Here we calculated S-matrix elements without explaining how to turn these into decay rates or cross-sections, which are the measurable quantities. This is dealt with in the QED / QCD course.

- The Klein-Gordon field involves particles of spin zero, which are bosons. One may also construct field theories for fermions of spin $\frac{1}{2}$, and vector bosons (spin 1). Physical examples include QED and QCD.

- Fields may have internal symmetries (e.g. local gauge invariance). Again, see the QED / QCD and Standard Model courses.

- Diagrams involving loops are divergent, ultimately leading to infinite renormalisation of the couplings and masses. The renormalisation procedure can only be carried out in certain theories. The Standard Model is one example, but other well-known physical theories (e.g. general relativity) fail this criterion.

- There is an alternative formulation of QFT in terms of path integrals (i.e sums over all possible configurations of fields). This alternative formulation involves some extra conceptual overhead, but allows a much more straightforward derivation of the Feynman rules. More than this, the path integral approach makes many aspects of field theory manifest i.e. is central to our understanding of what a quantum field theory is. This will not be covered at all in this school, but the interested student will find many excellent textbooks on the subject.

There are other areas which are not covered at this school, but nonetheless are indicative of the fact that field theory is still very much an active research area, with many exciting new developments:

- Calculating Feynman diagrams at higher orders is itself a highly complicated subject, and there are a variety of interesting mathematical ideas (e.g. from number theory and complex analysis) involved in current research.

- Sometimes perturbation theory is not well-behaved, in that there are large coefficients at each order of the expansion in the coupling constant. Often the physics of these large contributions can be understood, and summed up to all orders in the coupling. This is known as *resummation*, and is crucial to obtaining sensible results for many cross-sections, especially in QCD.

- Here we have "solved" for scattering probabilities using a perturbation expansion. It is sometimes possible to numerically solve the theory fully non-perturbatively. Such approaches are known as lattice field theory, due to the fact that one discretizes space and time into a lattice of points. It is then possible (with enough supercomputing power!) to calculate things like hadron masses, which are completely incalculable in perturbation theory.

- Here we set up QFT in Minkowski (flat space). If one attempts to do the same thing in curved space (i.e. a strong gravitational field), many weird things happen that give us tantalising hints of what a quantum field of gravity should look like.

- There are some very interesting recent correspondences between certain limits of certain string theories, and a particular quantum field theory in the strong coupling limit. This has allowed us to gain new insights into nonperturbative field theory from an analytic point of view, and there have been applications in heavy ion physics and even condensed matter systems.

I could go on of course, and many of the more formal developments of current QFT research are perhaps not so interesting to a student in experimental particle physics. However, at the present

time some of the more remarkable and novel extensions to the Standard Model (SUSY, extra dimensions) are not only testable, but are actively being looked for. Thus QFT, despite its age, is very much at the forefront of current research efforts and may yet surprise us!

Acknowledgments

I am very grateful to Chris White and Mrinal Dasgupta for providing a previous set of lecture notes, on which these notes are heavily based.

A Books on QFT

There are numerous textbooks already and a surprisingly high number of new books are appearing all the time. As with anything in theoretical physics, exploring a multitude of approaches to a certain field is encouraged.
In the following list, [1] is said to be a good introductory text and a lot of my colleagues use this one for their introduction to QFT classes. Mark has also put a "try-before-buy" version on his webpage, which is an early version of the entire textbook. You can judge yourself if it's worth the investment.
My first encounter with QFT was [2]. It's a very good book that heavily makes use of the Path Integral Formalism (not discussed in these lectures), it also includes topics which are normally not featured in general purpose QFT books (e.g. SUSY, topological aspects). A modern classic is [3], which many use as a standard text. It covers a lot of ground and develops an intuitive approach to QFT (but you aren't spared the hard bits!). It also touches other areas where QFT finds application (e.g. Statistical Physics). In my opinion, it isn't very good to look things up because Peskin's pedagogical approach forces logically-connected topics to be scattered across the text. Unless you are very familar with the book, it can take ages to find certain things again. My personal favorite by far is [4], mostly owing to the authors' focus on particle theory applications of QFT. But you'll probably need a bit of exposure to one of the introductory texts to fully appreciate the depth and technical details that the authors have put into it. Yes, it's expensive (like most of the Graduate-level textbooks), but having an advanced QFT book by a bunch of German authors on your shelf will not go unnoticed by your colleagues. Another good text is [5]. Finally, those who are not faint of heart and who like their field theory from the horse's mouth may like to consult Weinberg's monumental three volume set [6].

References

[1] M. Srednicki, *Quantum Field Theory*, CUP 2007.

[2] L. Ryder, *Quantum Field Theory*, CUP 1985.

[3] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison Wesley 1995.

[4] M. Böhm, A. Denner, H. Joos, *Gauge Theories*, Teubner 2001.

[5] T.-P. Cheng, L.-F. Li, *Gauge Theories of Elementary Particle Physics*, Clarendon 1982.

[6] S. Weinberg, *The Quantum Theory of Fields*, CUP 1995.

B Notation and conventions

4-vectors:

$$
\begin{aligned}
&x^\mu = (x^0, \mathbf{x}) = (t, \mathbf{x}) \\
&x_\mu = g_{\mu\nu}\, x^\nu = (x^0, -\mathbf{x}) = (t, -\mathbf{x}) \\
&\text{Metric tensor:} \quad g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned}
$$

Scalar product:

$$
\begin{aligned}
x^\mu x_\mu &= x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3 \\
&= t^2 - \mathbf{x}^2
\end{aligned}
$$

Gradient operators:

$$
\begin{aligned}
&\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla\right) \\
&\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right) \\
&\text{d'Alembertian:} \quad \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box
\end{aligned}
$$

Momentum operator:

$$
\hat{p}^\mu = i\hbar\partial^\mu = \left(i\hbar\frac{\partial}{\partial t},\, -i\hbar\nabla\right) = \left(\hat{E},\, \hat{\mathbf{p}}\right) \quad \text{(as it should be)}
$$

δ-functions:

$$
\begin{aligned}
&\int d^3p\, f(\mathbf{p})\, \delta^3(\mathbf{p} - \mathbf{q}) = f(\mathbf{q}) \\
&\int d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^3 \delta^3(\mathbf{p}) \\
&\int \frac{d^3p}{(2\pi)^3}\, e^{-i\mathbf{p}\cdot\mathbf{x}} = \delta^3(\mathbf{x})
\end{aligned}
$$

(similarly in four dimensions)

Note:

$$
\begin{aligned}
\delta(x^2 - x_0^2) &= \delta\{(x - x_0)(x + x_0)\} \\
&= \frac{1}{2x}\left\{\delta(x - x_0) + \delta(x + x_0)\right\}
\end{aligned}
$$