SUDAKOVS

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JAMES STIRLING MEMORIAL CONFERENCE,
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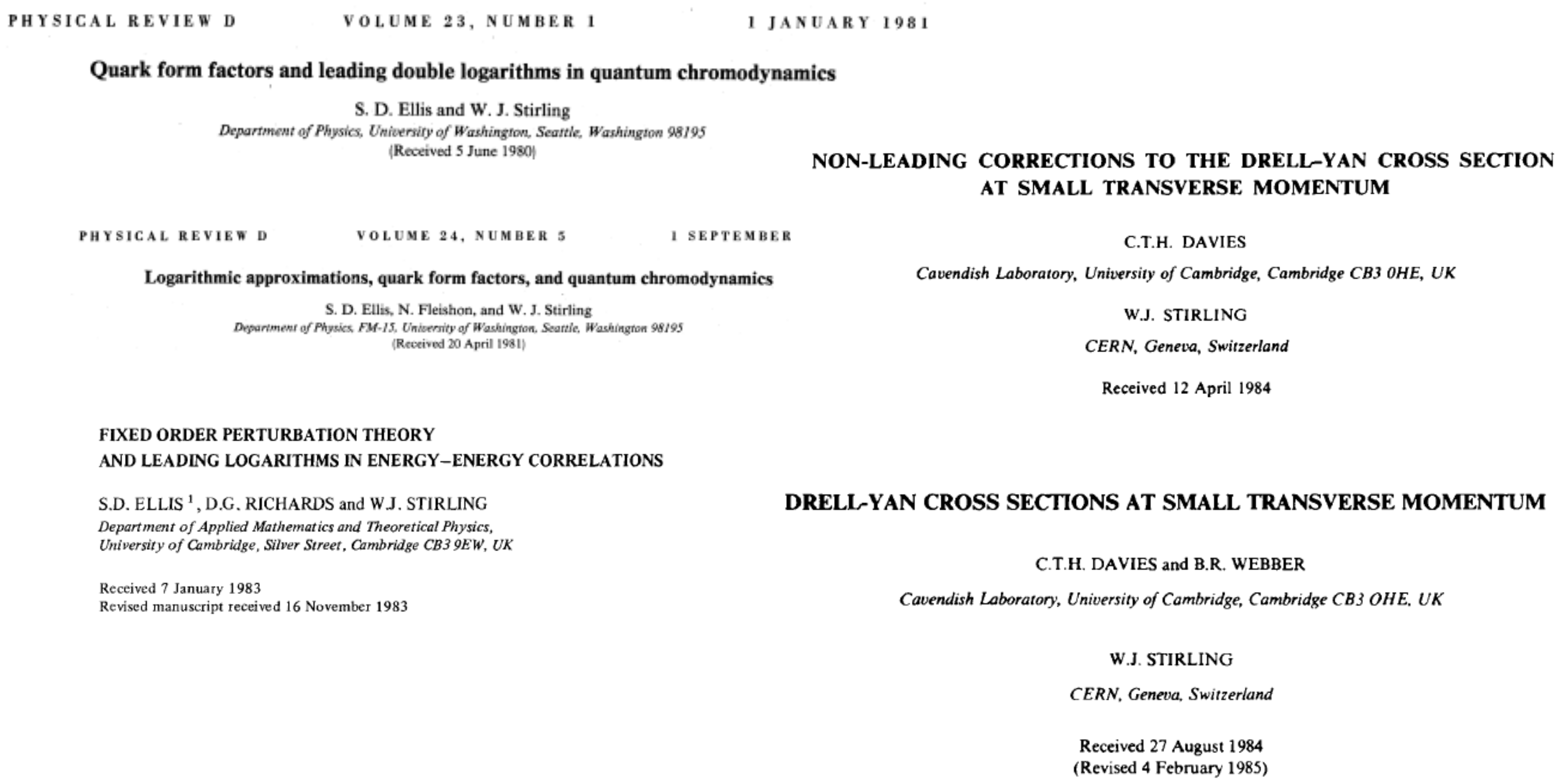
JAMES AND SOFT QCD RADIATION



PHYSICAL REVIEW D VOLUME 23, NUMBER 1 1 JANUARY 1981

Quark form factors and leading double logarithms in quantum chromodynamics

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(Received 5 June 1980)

PHYSICAL REVIEW D VOLUME 24, NUMBER 5 1 SEPTEMBER

Logarithmic approximations, quark form factors, and quantum chromodynamics

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(Received 20 April 1981)

FIXED ORDER PERTURBATION THEORY
AND LEADING LOGARITHMS IN ENERGY–ENERGY CORRELATIONS

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NON-LEADING CORRECTIONS TO THE DRELL–YAN CROSS SECTION AT SMALL TRANSVERSE MOMENTUM

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Received 12 April 1984

DRELL–YAN CROSS SECTIONS AT SMALL TRANSVERSE MOMENTUM

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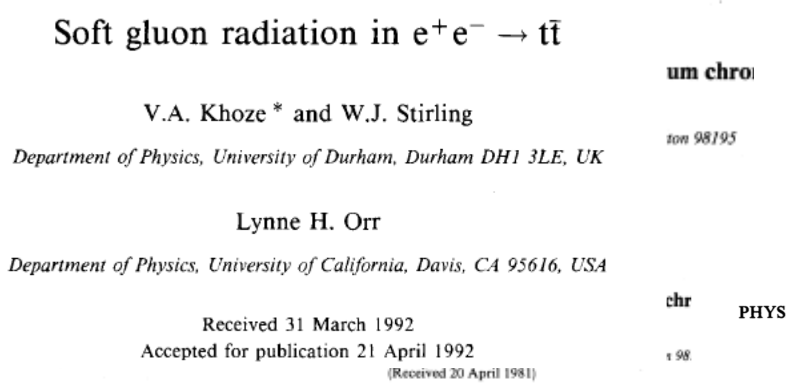
(Revised 4 February 1985)

JAMES AND SOFT QCD RADIATION



Soft gluon radiation in $e^+e^- \to t\bar{t}$

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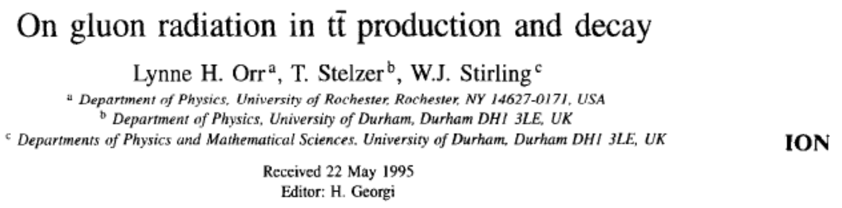

On gluon radiation in $t\bar{t}$ production and decay

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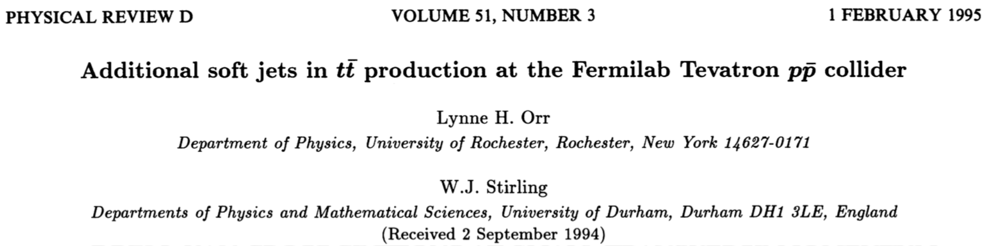

PHYSICAL REVIEW D VOLUME 51, NUMBER 3 1 FEBRUARY 1995

Additional soft jets in $t\bar{t}$ production at the Fermilab Tevatron $p\bar{p}$ collider

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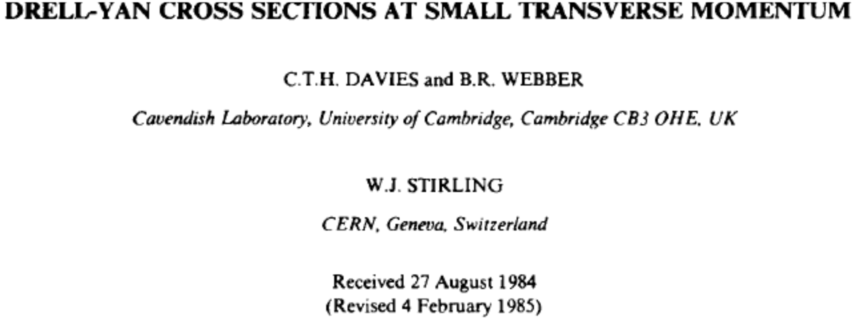

DRELL-YAN CROSS SECTIONS AT SMALL TRANSVERSE MOMENTUM

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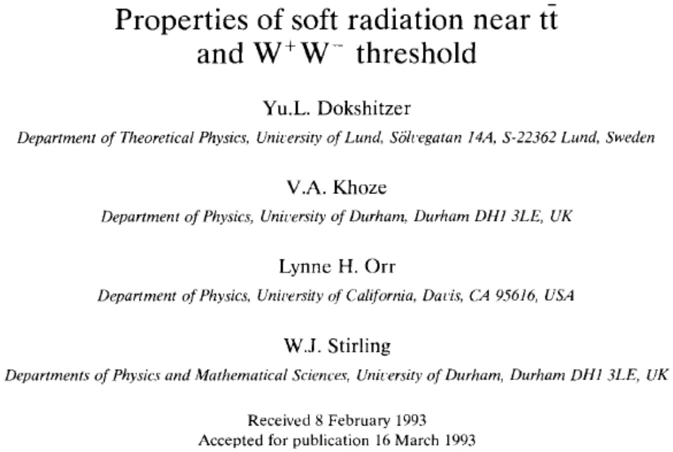

Properties of soft radiation near $t\bar{t}$ and W^+W^- threshold

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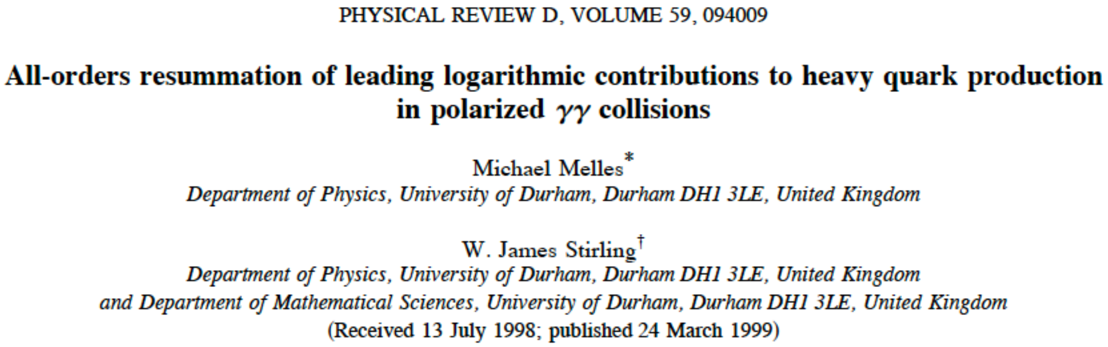

PHYSICAL REVIEW D, VOLUME 59, 094009

All-orders resummation of leading logarithmic contributions to heavy quark production in polarized $\gamma\gamma$ collisions

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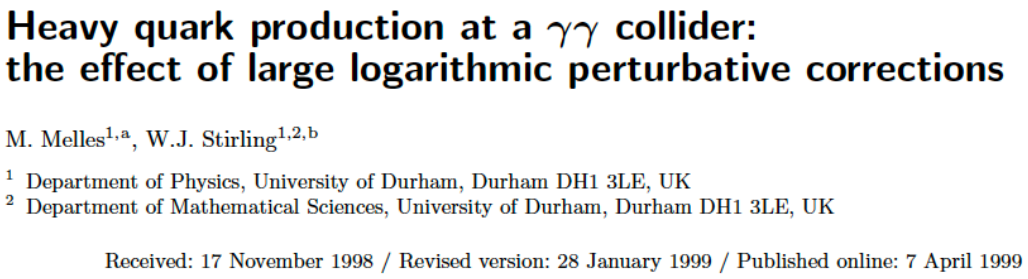

Heavy quark production at a $\gamma\gamma$ collider: the effect of large logarithmic perturbative corrections

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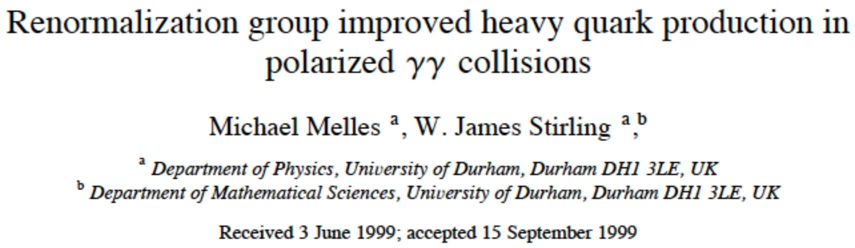

Renormalization group improved heavy quark production in polarized $\gamma\gamma$ collisions

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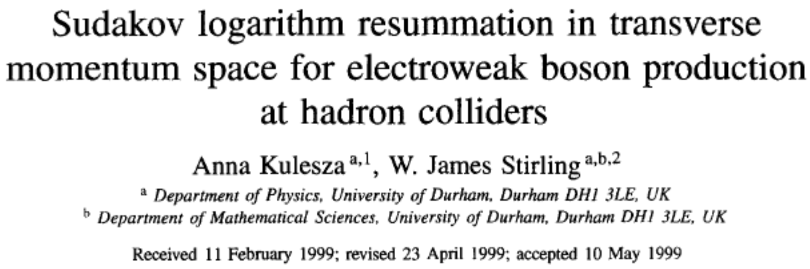
JAMES AND SOFT QCD RADIATION



Sudakov logarithm resummation in transverse momentum space for electroweak boson production at hadron colliders

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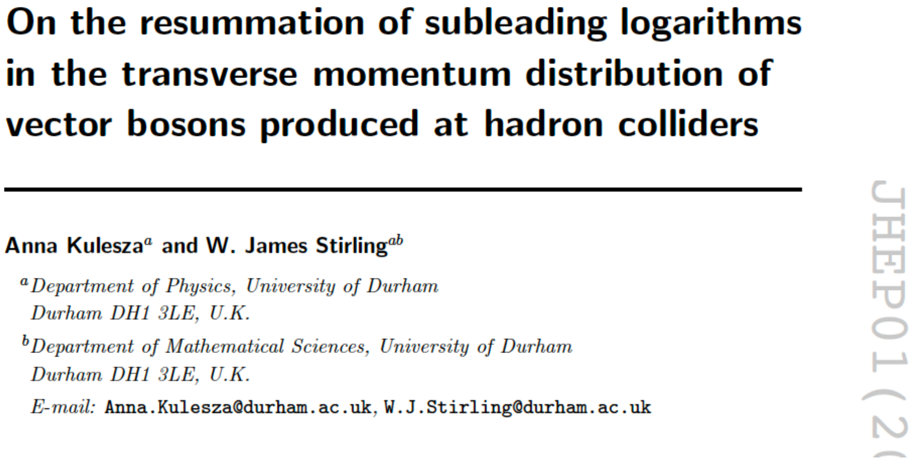

On the resummation of subleading logarithms in the transverse momentum distribution of vector bosons produced at hadron colliders

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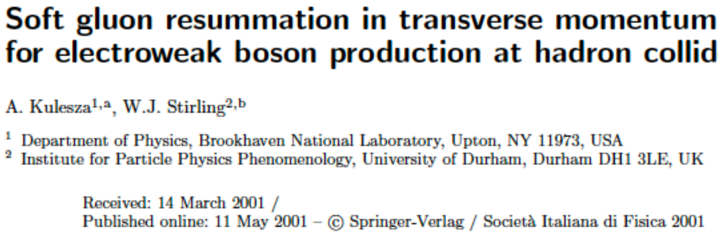

Soft gluon resummation in transverse momentum space for electroweak boson production at hadron colliders

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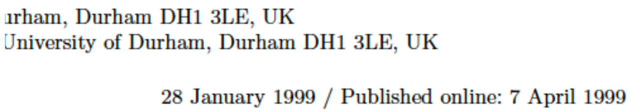

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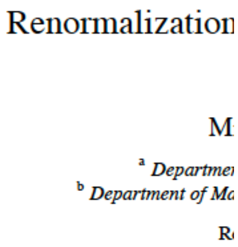
28 January 1999 / Published online: 7 April 1999



Renormalization

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[b] Department of Ma

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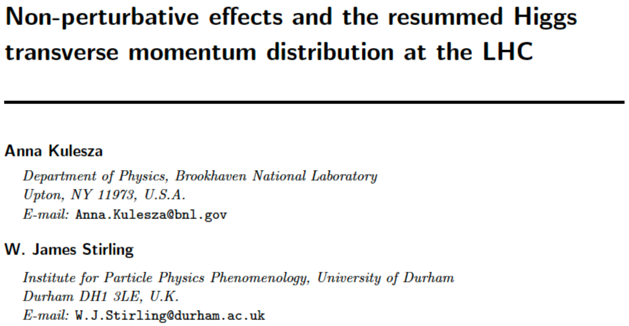

Non-perturbative effects and the resummed Higgs transverse momentum distribution at the LHC

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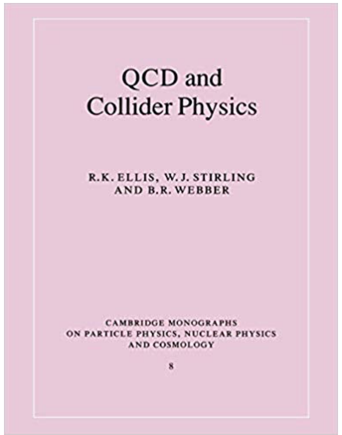
My PhD Topic

- Studies of electroweak vector (W, Z) boson production at small transverse momentum (p_T)in hadron collisions
- p_T distribution extensively investigated: a way to discrimate between the naive parton model and QCD
- Bulk of data at small $p_{T,}$ but fixed-order theory predictions diverge

  in the limit $p_T \rightarrow 0$



QCD and
Collider Physics

R.K. ELLIS, W.J. STIRLING
AND B.R. WEBBER

CAMBRIDGE MONOGRAPHS
ON PARTICLE PHYSICS, NUCLEAR PHYSICS
AND COSMOLOGY

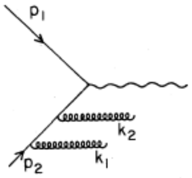
8

$$\frac{d\sigma^{R+V}}{dp_T^2} = \alpha_S \left(A \left[\frac{\ln(M^2/p_T^2)}{p_T^2} \right]_+ + B \left[\frac{1}{p_T^2} \right]_+ + \overline{C}(p_T^2) \right) , \qquad (9.55)$$

..and that happens to all orders

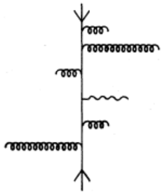
SUDAKOV FACTORS I

- These dominant contributions can be taken into account to all orders in perturbation theory by means of summing them (*resummation*) → Sudakov factors
- Double Leading Log Approximation (DLLA) *[Dokshitzer, Dyakonov, Troyan'80][Soper'80][S.Ellis, Stirling'80]*



$$\frac{1}{\sigma_0}\frac{d\sigma}{dq_T^2} = \frac{\alpha_s A}{2\pi q_T^2}\ln\left(\frac{Q^2}{q_T^2}\right)\exp\left(\frac{-\alpha_s A}{4\pi}\ln^2\left(\frac{Q^2}{q_T^2}\right)\right)$$

  - derived under assumption of strong ordering: $k_{T,1}^2 \ll k_{T,2}^2 \ll \ldots \ll k_{T,n}^2 \sim q_T^2 \ll Q^2$
  - the Sudakov factor leads to dampening of the cross section as $q_T \to 0$
- Transverse momentum conservation can be accounted for properly in Fourier space *[Parisi, Petronzio'79]*



$$\delta^{(2)}\left(\sum_{i=1}^{N}\boldsymbol{k}_{T_i} - \boldsymbol{q}_T\right) = \int d^2b\,\frac{1}{4\pi^2}e^{-i\boldsymbol{b}\boldsymbol{q}_T}\prod_{i=1}^{N}e^{i\boldsymbol{b}\boldsymbol{k}_{T_i}}$$

SUDAKOV FACTORS II

- Resummation of all terms at least as singular as $1/q_T^2$ *[Collins, Soper'81-'82][Kodaira, Trentadue'82-'83][Collins, Soper, Sterman'85]*

$$\frac{d\sigma}{dq_T^2} = \frac{\sigma_0}{2}\int_0^\infty bdb\, J_0(q_T b) e^{S(b,Q^2)}$$

$$S(b,Q^2) = -\int_{\frac{b_0^2}{b^2}}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[\ln\left(\frac{Q^2}{\bar{\mu}^2}\right) A(\alpha_S(\bar{\mu}^2)) + B(\alpha_S(\bar{\mu}^2))\right]$$

$$A(\alpha_S) = \sum_{i=1}^{\infty}\left(\frac{\alpha_S}{2\pi}\right)^i A^{(i)}\ , \quad B(\alpha_S) = \sum_{i=1}^{\infty}\left(\frac{\alpha_S}{2\pi}\right)^i B^{(i)}$$

$B_q^{(2)}$ coefficient calculated by James and C. Davies in '84, enabling more precise comparisons with data *[Davies, Stirling, Webber'85]*

- Further work by *[Catani, d'Emilio, Trentadue'88]*, `ResBos` collaboration *[Balazs, Qiu, Yuan, Nadolsky, Berger, Cao, Chen,.. ,'94-...]* and *[Catani, Grazzini, de Florian, Bozzi, Ferrera, Cieri, Sargsyan,Tommasini'01-...]* `(Hqt, HRES)` as well as in the SCET framework *[Becher, Neubert'10][Chiu, Jain, Neill, Rothstein Echevarria, Scimemi, Idilbi, Ebert, Tackmann, Stewart,..][Li,Zhu'16][Gehrmann, Lübert, Yang'18][Chen et al.'18]*

WORK ON SUDAKOVS

- New work by Keith Ellis and collaborators *[K. Ellis, Ross, Veseli'97] [K. Ellis, Veseli'97]*
  - In practice, resummation in b-space has some drawbacks, e.g. needs prescriptions on how to deal with the non-perturbative region of large b and for matching with fixed-order results, which would be overcome by resummation in q_T space *[K. Ellis, Veseli'97]*
  - Extension of the DLLA formula down to and including $\alpha_s^n \log\left(\frac{Q^2}{q_T^2}\right)^{2n-3}$ terms, closed analytical form

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  - Extension of the DLLA formula down to and including $\alpha_s^n \log\left(\frac{Q^2}{q_T^2}\right)^{2n-3}$ terms, closed analytical form
- James' work with S. Elllis in Seattle
  - Contributions from all soft-collinear but one hard-collinear exponentiate, shown directly in p_T space *[S. Ellis, Stirling'80]*
  - Logarithms of „kinematical"origin (i.e. from transverse momentum conservation) fill the dip at $q_T \to 0$ *[S. Ellis, Fleishon,Stirling'80]*
  - They enter at $\alpha_s^n \log\left(\frac{Q^2}{q_T^2}\right)^{2n-4}$ level and below
- Same discussion applies to energy-energy correlations in e^+e^- collisions

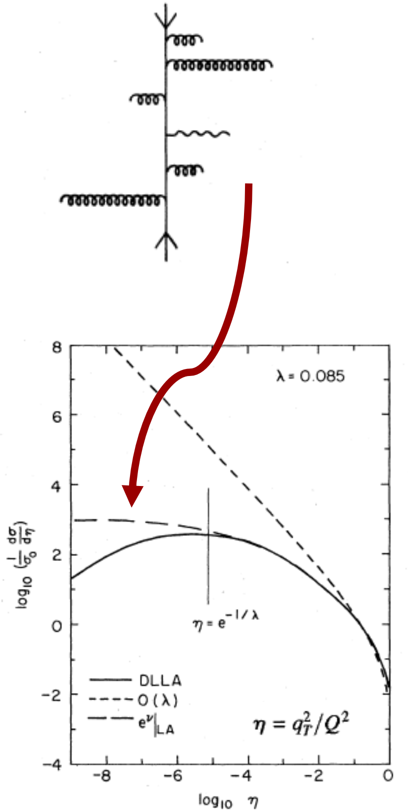
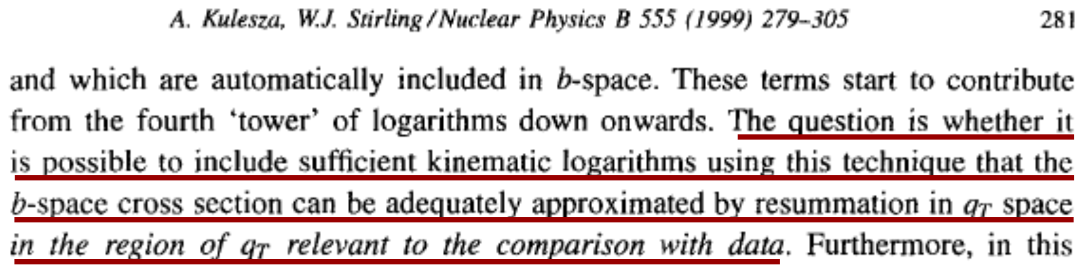


FIG. 4. Theoretical approximations to the cross section defined in the text. The long-dashed line is the soft logarithmic approximation [LA, (1), (2), (3)]. The solid line is the DLLA Eq. (2.12). The dashed line is the corresponding one-gluon contribution.

WORK ON SUDAKOVS II



A. Kulesza, W.J. Stirling / Nuclear Physics B 555 (1999) 279–305 281

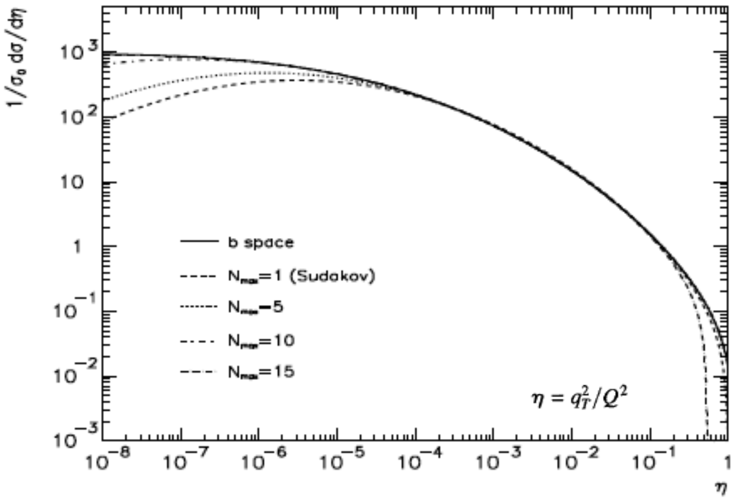
and which are automatically included in *b*-space. These terms start to contribute from the fourth 'tower' of logarithms down onwards. The question is whether it is possible to include sufficient kinematic logarithms using this technique that the *b*-space cross section can be adequately approximated by resummation in q_T space *in the region of q_T relevant to the comparison with data*. Furthermore, in this

- Toy model with subleading logarithms only due to kinematics, derived from *b*-space expression

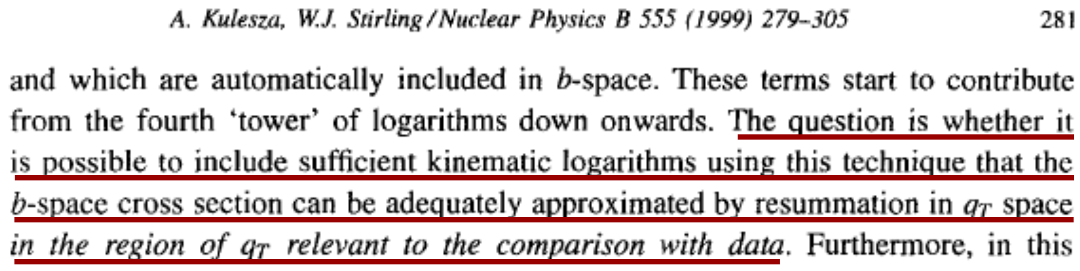
$$\frac{1}{\sigma_0}\frac{d\sigma}{dq_T^2} = \frac{\lambda}{q_T^2} e^{\frac{-\lambda}{2}L^2} \sum_{N=1}^{\infty} \frac{(-2\lambda)^{(N-1)}}{(N-1)!} \sum_{m=0}^{N-1} \left(\begin{array}{c} N-1 \\ m \end{array} \right) L^{N-1-m}\left[2\tau_{N+m} + L\tau_{N+m-1}\right]$$

$$L = \ln(Q^2/q_T^2),\ \lambda = \alpha_S C_F/\pi$$

$$\tau_m \equiv \int_0^\infty dy J_1(y) \ln^m\left(\frac{y}{b_0}\right)$$



WORK ON SUDAKOVS II



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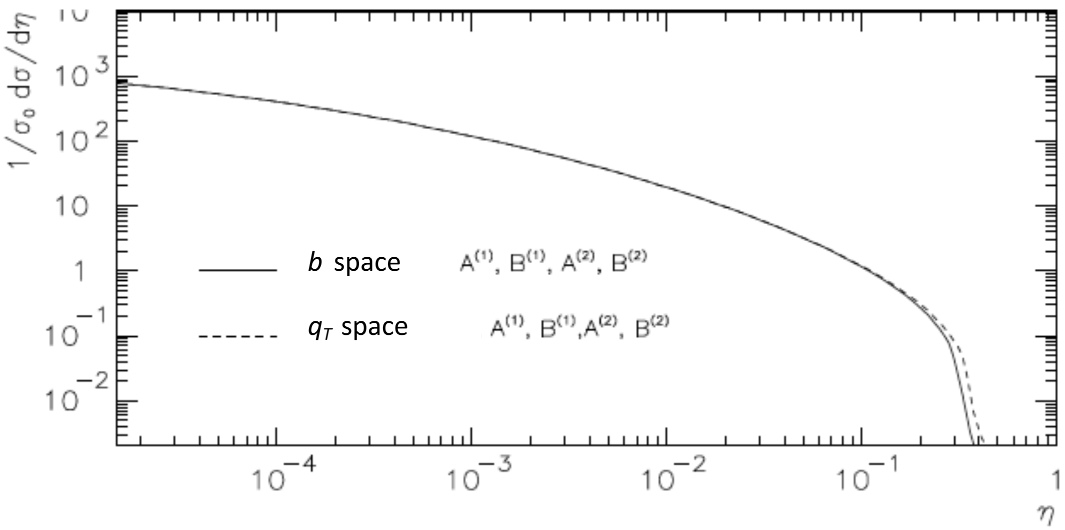
and which are automatically included in *b*-space. These terms start to contribute from the fourth 'tower' of logarithms down onwards. The question is whether it is possible to include sufficient kinematic logarithms using this technique that the *b*-space cross section can be adequately approximated by resummation in q_T space *in the region of* q_T *relevant to the comparison with data*. Furthermore, in this

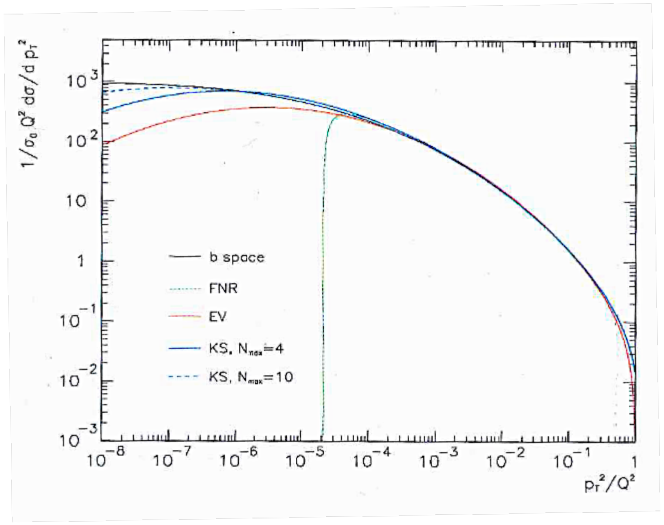
- More realistic scenario, with subleading logarithms from the matrix element



$\eta = q_T^2/Q^2$

WORK ON SUDAKOVS III

- Another approach to resummation of transverse spectra directly in q_T space *[Frixione, Nason Ridolfi'98]* (FNR)
- We focused on investigating differences between our approach and that of FNR, especially regarding kinematical logarithms



WORK ON SUDAKOVS III

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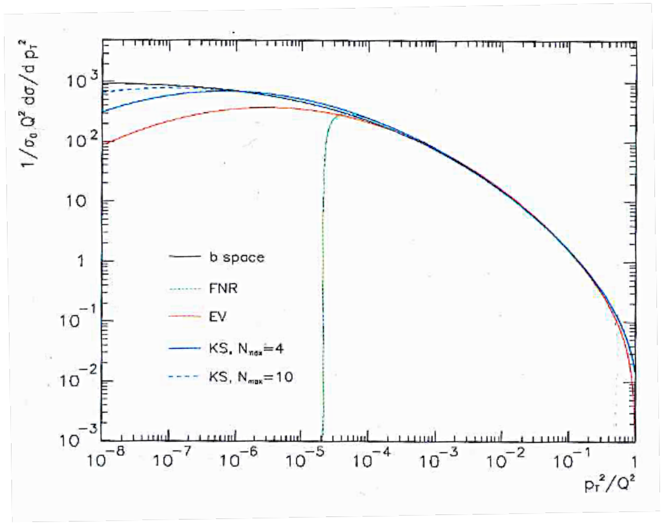
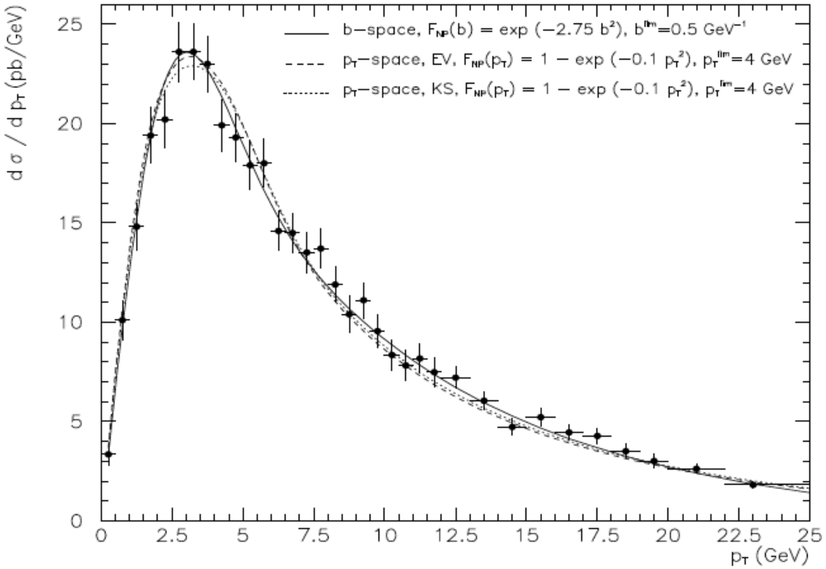


- Recent renewed interest in resummation in momentum space: *[Monni, Re, Torrielli'16][Bizon et al.'17]*
  - up to N^3LL +NNLO accuracy

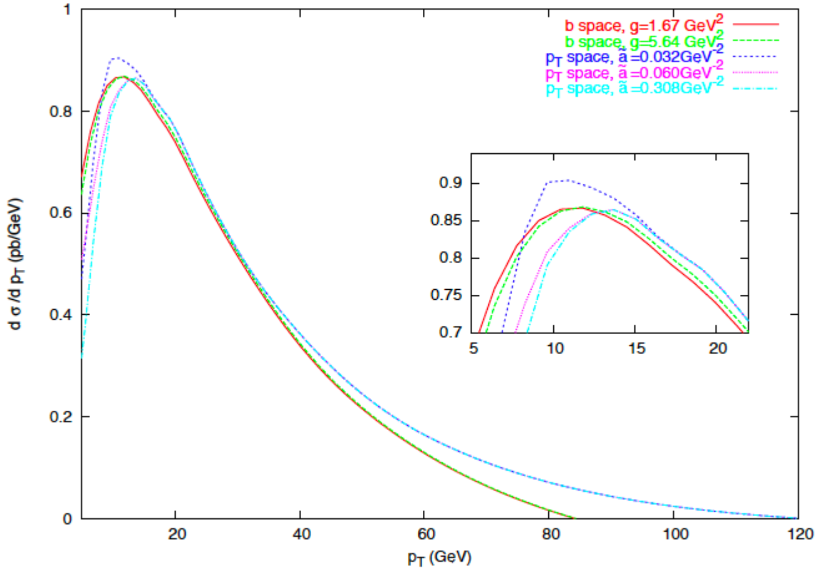
APPLICATIONS

[AK, Stirling'01]



Comparison between CDF data on Z production and theoretical predictions

[AK, Stirling'03]



Higgs boson p_T distributions at the LHC, as predicted by the b space and p_T space

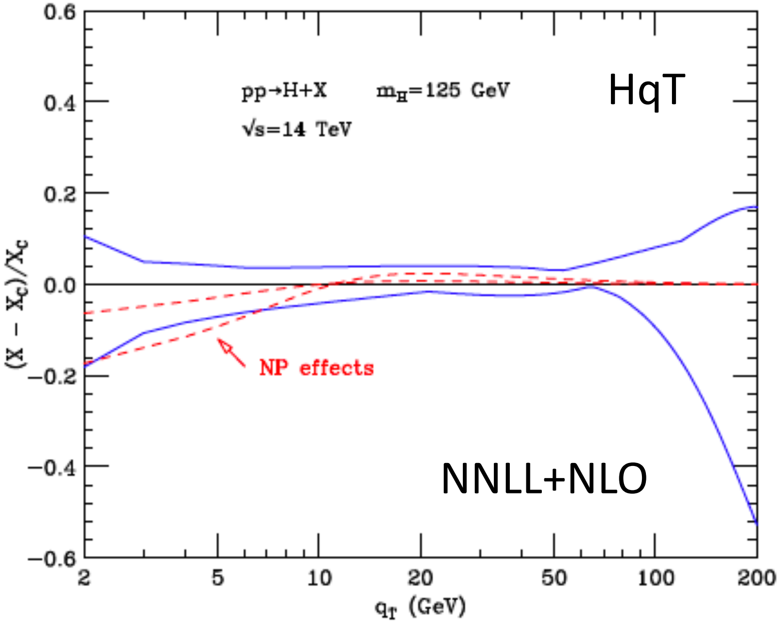
$$F^{NP} = e^{-gb^2}$$

$$\tilde{F}^{NP} = 1 - \exp\left[-\tilde{a}\, p_T^2\right]$$

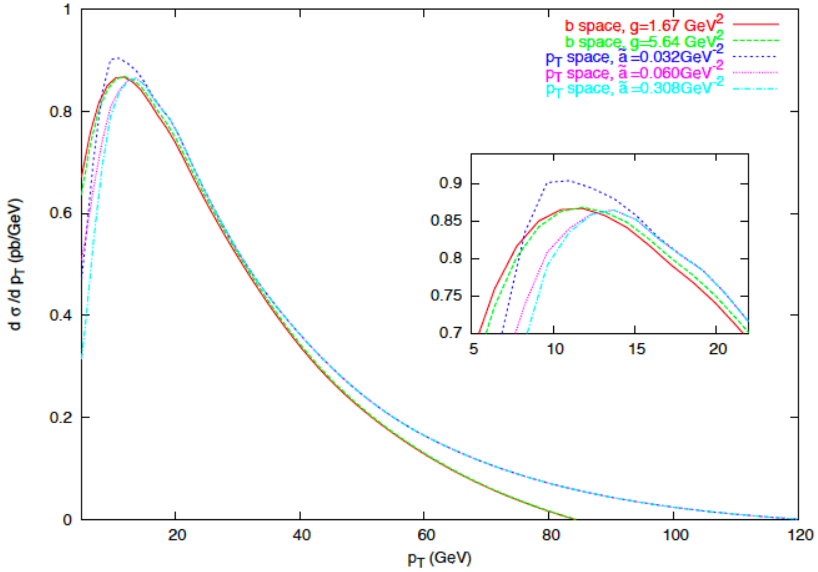
-> Data on Υ production used to determine the NP contribution

APPLICATIONS

[de Florian, Ferrera, Grazzini, Tommasini'11]



[AK, Stirling'03]



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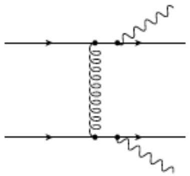
-> Data on Υ production used to determine the NP contribution

DOUBLE PARTON SCATTERING

- Substantial part of my collaborative work with James
- Started after a workshop at CERN which we both attended → talk by D. Treleani on DPS background to Higgs production at the LHC
- James had (of course!) also worked on the DPS in the past *[Halzen, Hoyer, Stirling'87]*
- James' idea was to consider production of same sign W's as a probe of DPS *[AK, Stirling'00]*

Single Parton Scattering

$\mathcal{O}(\alpha_S^2\alpha_W^2)$



vs

Double Parton Scattering

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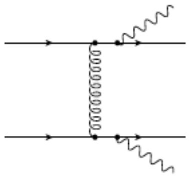
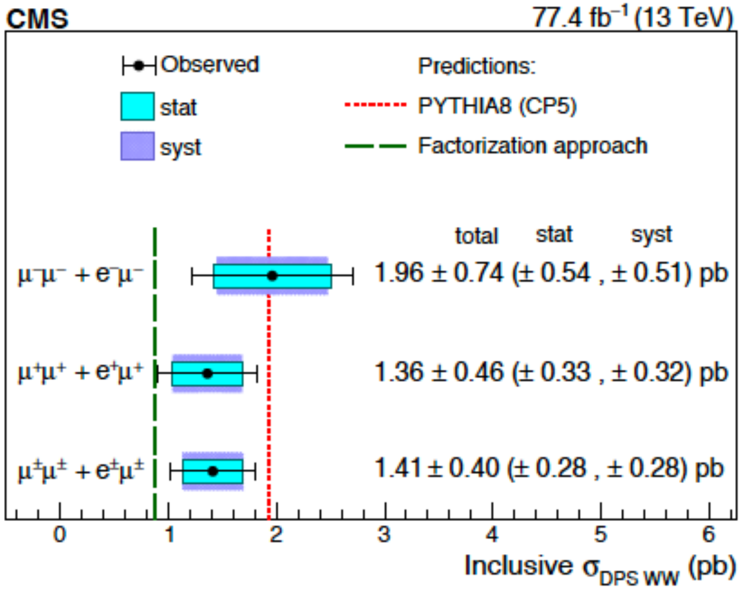
Single Parton Scattering

$\mathcal{O}(\alpha_S^2\alpha_W^2)$

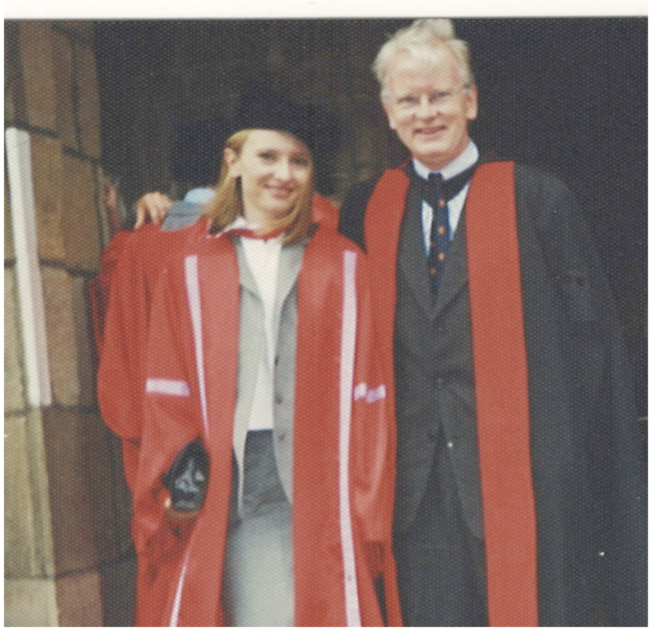


vs

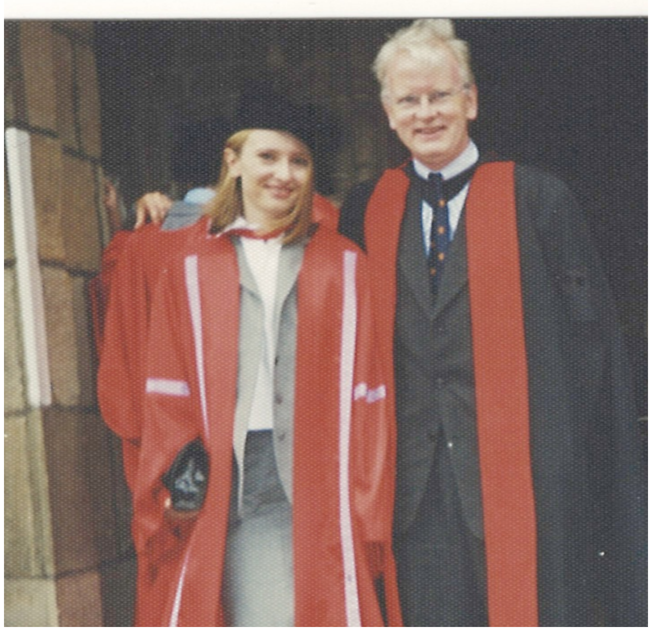


- Now, CMS measures it! (Moriond 2019)
- More work followed *[Gaunt, Kom, AK, Stirling‘11] [Kom, AK, Stirling‘11]*→ see Jo‘s talk

JAMES AS SUPERVISOR

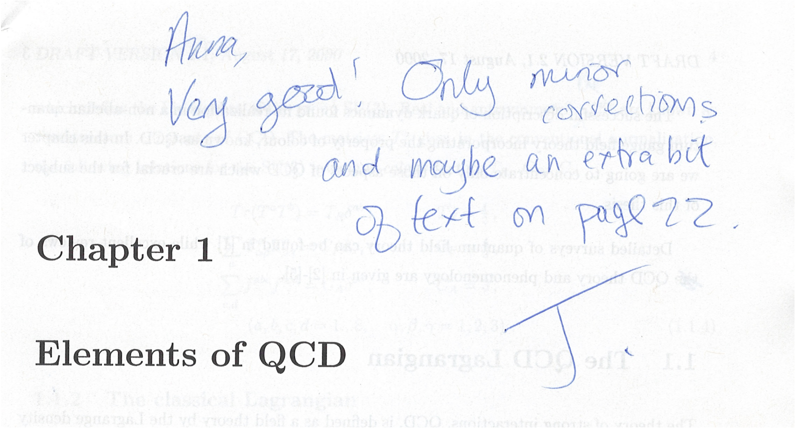


JAMES AS SUPERVISOR



- It was an extreme privilege (and luck) to be a PhD student, and later a collaborator of James. I am indebted to him for teaching me particle physics and showing the brilliance of research work
- James was the most kind and supportive, all the way through my PhD and later on in my career

JAMES AS SUPERVISOR

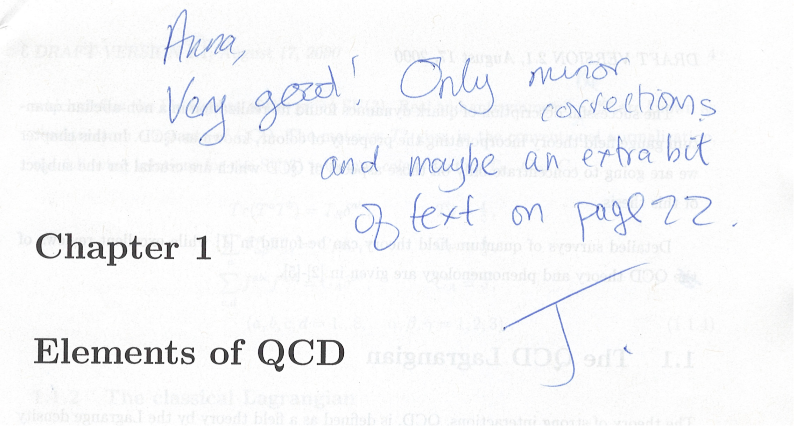


Anna,
Very good! Only minor corrections and maybe an extra bit of text on page 22.
J

Chapter 1

Elements of QCD

JAMES AS SUPERVISOR



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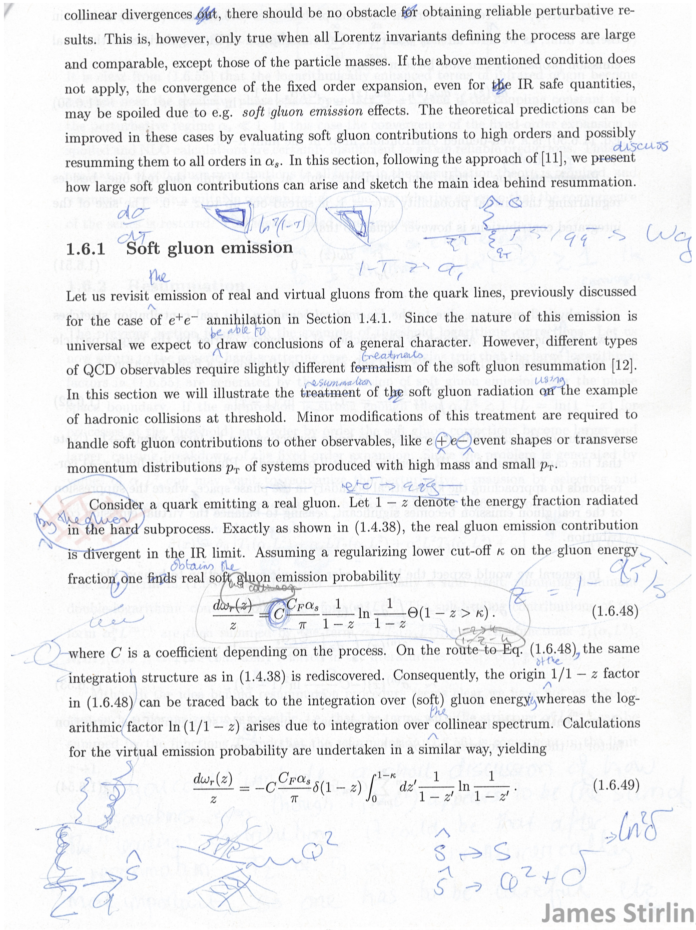
Chapter 1

Elements of QCD



collinear divergences ~~out~~, there should be no obstacle ~~for~~ obtaining reliable perturbative results. This is, however, only true when all Lorentz invariants defining the process are large and comparable, except those of the particle masses. If the above mentioned condition does not apply, the convergence of the fixed order expansion, even for ~~the~~ IR safe quantities, may be spoiled due to e.g. *soft gluon emission* effects. The theoretical predictions can be improved in these cases by evaluating soft gluon contributions to high orders and possibly resumming them to all orders in α_s. In this section, following the approach of [11], we ~~present~~ discuss how large soft gluon contributions can arise and sketch the main idea behind resummation.

1.6.1 Soft gluon emission

Let us revisit the emission of real and virtual gluons from the quark lines, previously discussed for the case of e^+e^- annihilation in Section 1.4.1. Since the nature of this emission is universal we expect to be able to draw conclusions of a general character. However, different types of QCD observables require slightly different ~~formalism~~ treatments of the soft gluon resummation [12]. In this section we will illustrate the ~~treatment~~ resummation of ~~the~~ soft gluon radiation ~~on~~ using the example of hadronic collisions at threshold. Minor modifications of this treatment are required to handle soft gluon contributions to other observables, like $e+e-$ event shapes or transverse momentum distributions p_T of systems produced with high mass and small p_T.

Consider a quark emitting a real gluon. Let $1-z$ denote the energy fraction radiated by the gluon ~~in the hard subprocess~~. Exactly as shown in (1.4.38), the real gluon emission contribution is divergent in the IR limit. Assuming a regularizing lower cut-off κ on the gluon energy fraction, one ~~finds~~ obtains the real soft gluon emission probability

$$\frac{d\omega_r(z)}{z} = C\frac{C_F\alpha_s}{\pi}\frac{1}{1-z}\ln\frac{1}{1-z}\Theta(1-z>\kappa)\,. \tag{1.6.48}$$

where C is a coefficient depending on the process. On the route to Eq. (1.6.48), the same integration structure as in (1.4.38) is rediscovered. Consequently, the origin of the $1/1-z$ factor in (1.6.48) can be traced back to the integration over (soft) gluon energy, whereas the logarithmic factor $\ln(1/1-z)$ arises due to integration over the collinear spectrum. Calculations for the virtual emission probability are undertaken in a similar way, yielding

$$\frac{d\omega_r(z)}{z} = -C\frac{C_F\alpha_s}{\pi}\delta(1-z)\int_0^{1-\kappa}dz'\frac{1}{1-z'}\ln\frac{1}{1-z'}\,. \tag{1.6.49}$$

$\frac{d\sigma}{d\tau}$

$\ln^2(1-\tau)$

$1-\tau \to q_T$

$e^+e^- \to q\bar{q}g$

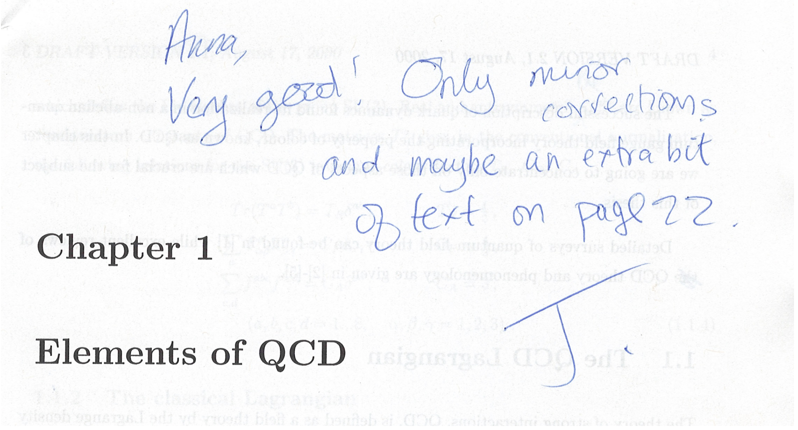
$z = 1 - \frac{Q^2}{\hat{s}}$

$\hat{s} \to s$

$\hat{s} \to Q^2 + \int$

$\to \ln\delta$

JAMES AS SUPERVISOR



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Chapter 1

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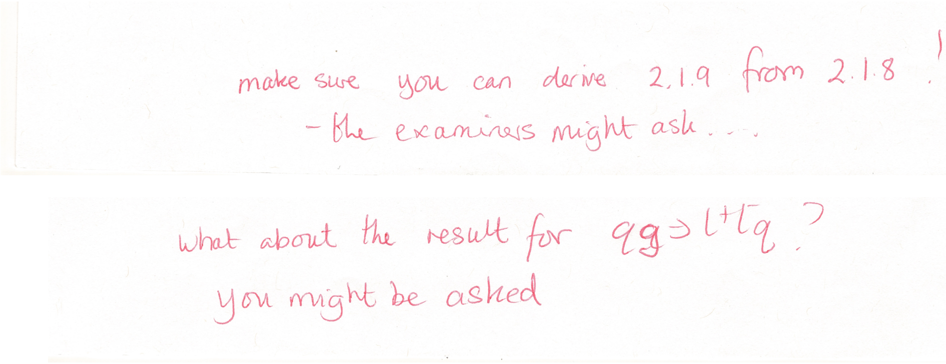
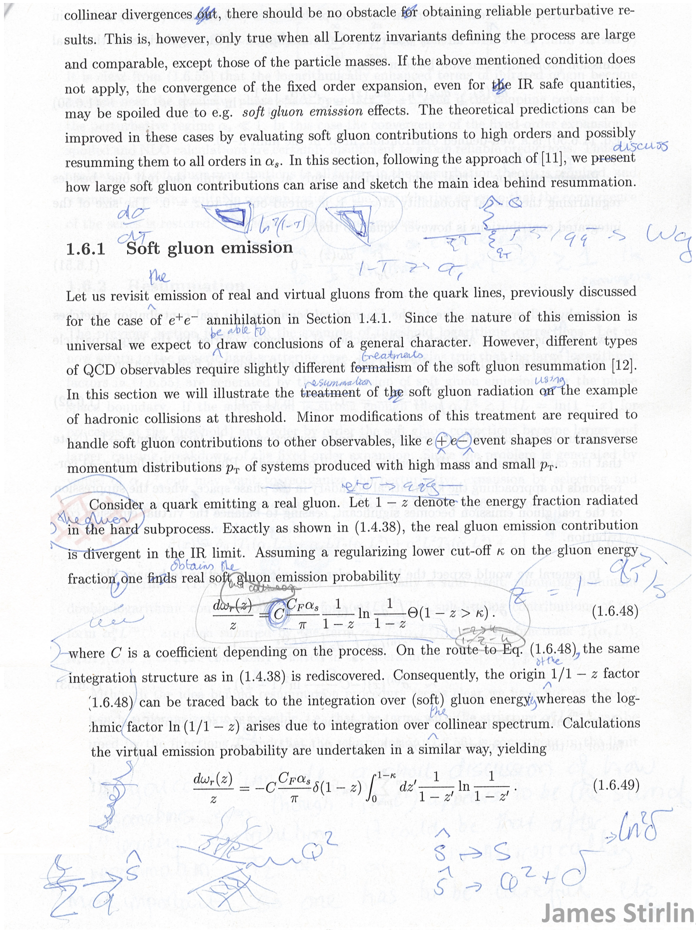


make sure you can derive 2.1.9 from 2.1.8 !
– the examiners might ask . . .

what about the result for $qg \to l^+l^-q$?
you might be asked



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$\frac{d\sigma}{dT}$ $\ln^2(1-T)$ $1-T \to a_1$ q_T $qq \to Wg$

1.6.1 Soft gluon emission

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$e^+e^- \to q\bar{q}g$

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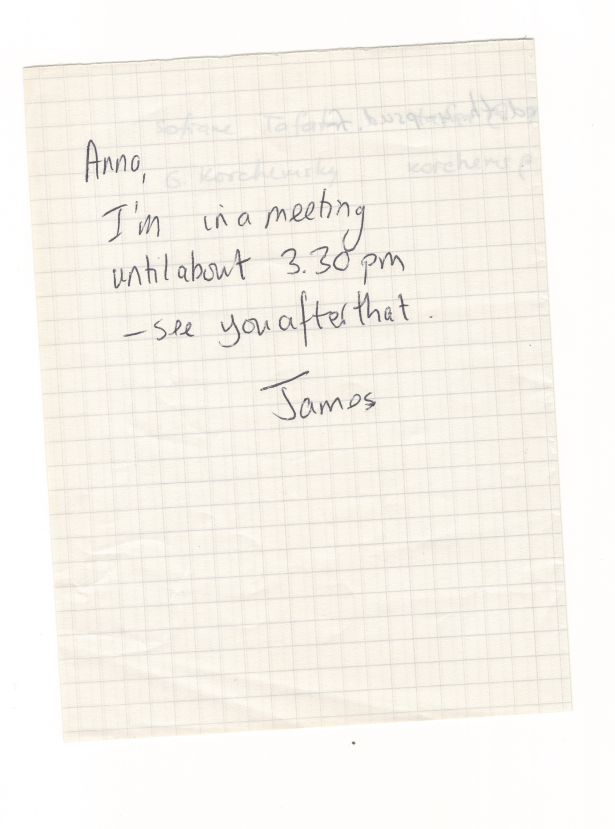
$\to \hat{s}$ Q^2 $\hat{s} \to s$ $\hat{s} \to Q^2 + \delta$ $\to \ln\delta$

JAMES AS SUPERVISOR

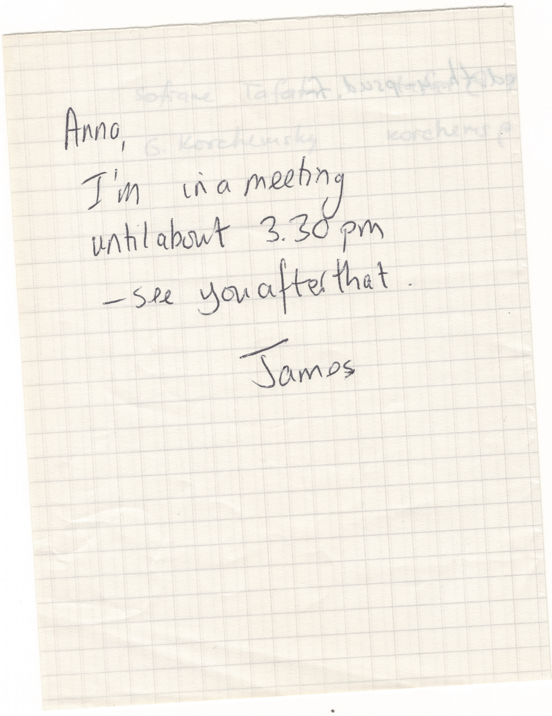


Anna,

I'm in a meeting
until about 3.30 pm
– see you after that.

James

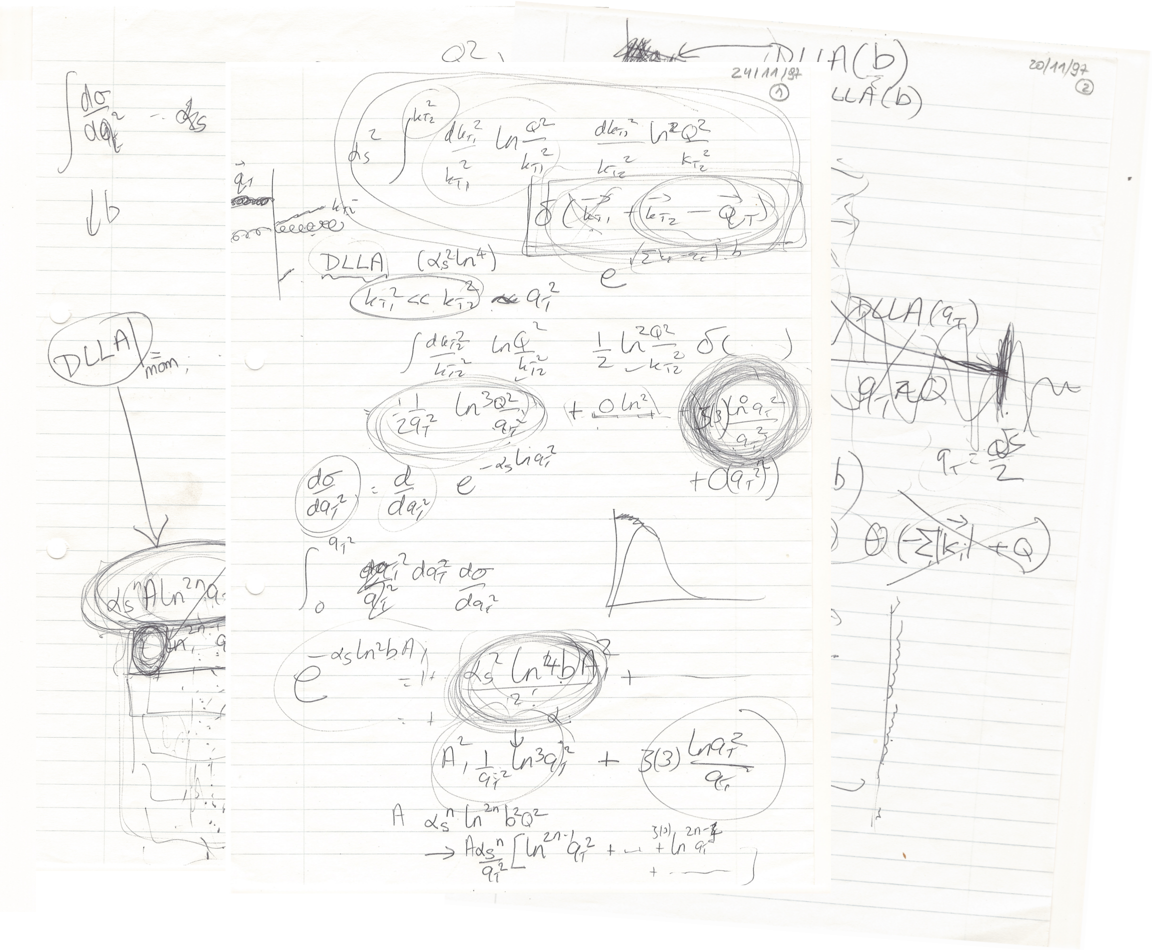
JAMES AS SUPERVISOR



Anno,

I'm in a meeting
until about 3.30 pm
– see you after that.

James



$\int \frac{d\sigma}{dq_T^2} \sim \alpha_s$

$\downarrow b$

DLLA $|_{mom}$

$\alpha_s^n A \ln^{2n} q_T$

24/11/97 ①

Q^2

$\alpha_s^2 \int^{k_{T_2}^2} \frac{dk_{T_1}^2}{k_{T_1}^2} \ln\frac{Q^2}{k_{T_1}^2} \quad \frac{dk_{T_2}^2}{k_{T_2}^2} \ln^2\frac{Q^2}{k_{T_2}^2}$

$\delta(\vec{k}_{T_1} + \vec{k}_{T_2} - \vec{Q}_T)$

$e^{i(\sum k_{T_i} - q_T)\cdot b}$

$\vec{q}_T$ k_{T_1}

DLLA $(\alpha_s^2 \ln^4)$

$k_{T_1}^2 \ll k_{T_2}^2 \approx q_T^2$

$\int \frac{dk_{T_2}^2}{k_{T_2}^2} \ln\frac{Q^2}{k_{T_2}^2} \quad \frac{1}{2} \ln^2\frac{Q^2}{k_{T_2}^2}\, \delta(\quad)$

$\frac{1}{2 q_T^2} \ln^3\frac{Q^2}{q_T^2} + O\ln^2 + \zeta(3)\frac{\ln q_T^2}{q_T^2} + O(q_T^2)$

$\frac{d\sigma}{dq_T^2} = \frac{d}{dq_T^2} e^{-\alpha_s \ln q_T^2}$

$\int_0^{q_T^2} dq_T^2 \frac{d\sigma}{dq_T^2}$

$e^{-\alpha_s \ln^2 b A_1} = 1 + \frac{\alpha_s^2 (\ln^4 b A_1)^2}{2} + \dots$

$A_1^2 \frac{1}{q_T^2} \ln^3 q_T^2 + \zeta(3) \frac{\ln q_T^2}{q_T}$

$A \quad \alpha_s^n \ln^{2n} b^2 Q^2$

$\rightarrow \frac{A \alpha_s^n}{q_T^2} \left[\ln^{2n-1} q_T^2 + \dots + \zeta(3) \ln^{2n-4} q_T^2 + \dots \right]$

20/11/97 ②

DLLA(b)

LLA(b)

DLLA(q_T)

$q_T \approx Q$

$\Theta(-\sum |\vec{k}_i| + Q)$

Thanks to the organizers