Resurgence in the Bi-Yang-Baxter Model

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Previous work by Demulder, Dorigoni, Thompson (2016) and Cherman, Dorigoni, Dunne, Ünsal (2014).
Outline

1. Yang-Baxter deformations
2. Uniton solutions
3. Dimensional Reduction to Schrödinger System
4. Borel Analysis
5. Seiberg-Witten Curve
Non-linear, integrable (i.e. there exists a flat Lax connection) \( \sigma \)-model on the world-sheet.

Fields are maps from the worldsheet into a Lie group: 
\[ g : \Sigma \rightarrow G = SU(N). \]

Action: 
\[ S = \frac{1}{2\pi t} \int dz^2 \text{Tr} \left( g^{-1} \partial_+ gg^{-1} \partial_+ g \right). \]

\( z_\pm = t \pm ix \) light-cone coordinates on the worldsheet \( \partial_\pm = \partial_{z_\pm} \).

\( \partial_+ g \in \mathfrak{g} \) Lie algebra valued.

\( G^L \times G^R \) global symmetry.
Yang-Baxter Deformations

- Let $R$ be the Yang-Baxter (YB) Operator on $\mathfrak{g}$ that satisfies
  \[
  [RA, RB] - R([RA, B] + [A, RB]) = [A, B].
  \]

- On $SU(2)$, in the basis $\sigma_i/\sqrt{2}$:
  \[
  R = \begin{pmatrix}
  0 & -1 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 0
  \end{pmatrix}.
  \]

- bi-Yang-Baxter (bYB) deformed PCM:
  \[
  S = \frac{1}{2\pi t} \int dz^2 \text{Tr} \left( g^{-1} \partial_+ g \frac{1}{1 - \eta R - \zeta R g} g^{-1} \partial_- g \right).
  \]

- $\text{Ad}_g(u) = gug^{-1}$, $R^g = \text{Ad}_{g^{-1}} \circ R \circ \text{Ad}_g$.

- $\eta, \zeta \in \mathbb{R}$ parameters that squish $S^3 \simeq SU(2)$.

- preserves integrability (Klimcik, 2014).
Parametrise $g \in SU(2)$ by Hopf coordinates:

$$g = \begin{pmatrix} \cos \theta e^{i\phi_1} & i \sin \theta e^{i\phi_2} \\ i \sin \theta e^{-i\phi_2} & \cos \theta e^{-i\phi_1} \end{pmatrix}.$$ 

There exists a solution to the equation of motions in terms of a holomorphic function $f(z)$:

$$\phi_1 = \frac{\pi}{2}, \quad \phi_2 = \pi + \frac{i}{2} \log \left( \frac{f}{f} \right),$$

$$\cos \theta(|f|^2) = \frac{\sqrt{1 + (\eta - \zeta)^2(|f|^2 - 1)}}{\sqrt{(1 + |f|^2)^2 + (\eta - \zeta)^2(1 - |f|^2)^2}}.$$ 

“Uniton”.
Finite, quantised action configuration:

\[ S = \frac{2k}{t(1 + (\zeta + \eta)^2)} S_I, \]

\[ S_I = \frac{(1 + (\zeta + \eta)^2)}{2\zeta\eta} \left[ (\zeta - \eta)\arctan(\zeta - \eta) - (\zeta + \eta)\arctan(\zeta + \eta) \right]. \]

\( k \in \mathbb{R} \) order of the polynomial \( f(z) \).

→ Analogous complex uniton.
Compactify along the $x$-direction by a twisted boundary condition

$$g(t, x + L) = e^{iH_L} g(t, x) e^{-iH_R}, \quad H_L = H_R = \begin{pmatrix} \pi/2 & 0 \\ 0 & \pi/2 \end{pmatrix}.$$ 

→ Integrate out heavy Kaluza-Klein modes.

→ Obtain an effective Quantum Mechanics for $\theta$. 

Dimensional Reduction
Schrödinger System

- Schrödinger Equation for $\theta$:

$$H = \frac{g^2}{4} p_\theta^2 + \frac{1}{g^2} V(\theta), \quad g = t(1 + (\zeta + \eta)^2),$$

with

$$V(\theta) = \frac{\pi^2 \text{sn}^2(\theta)(1 + (\zeta - \eta)^2 \text{sn}^2(\theta))}{\text{dn}^2(\theta)}, \quad m = \frac{4\zeta \eta}{1 + (\zeta + \eta)^2}.$$  

- $\text{sn}(z)$, $\text{cn}(z)$ and $\text{dn}(z)$ are the **Jacobi elliptic functions** depending on the elliptic modulus $m$.

- $\zeta = 0 \implies m = 0 \implies \text{sn}(z) = \sin(z), \text{dn}(z) = 1.$
Use the WKB method to obtain a perturbative expansion for the ground state energy:

\[ E(g^2) = \sum_{n=0}^{\infty} a_n g^{2n} \quad a_n = a_n(\zeta, \eta). \]

This series is divergent: \( a_n \sim n! \). **asymptotic expansion**.

Borel transform:

\[ B[E](g^2) = \sum_{n=0}^{\infty} \frac{a_n}{n!} g^{2n}. \]

Borel resummation

\[ S[E](g^2) = \int_0^{\infty} dz B[E](g^2) e^{-z/g^2} \sim E(g^2). \]
Poles in the Borel plane obstruct this integral along certain rays and are due to **non-perturbative** effects: Unitons!

Non-perturbative effects go as $e^{-2S_i/g^2}$.

Use Bender-Wu package (Sulejmanpasic and Ünsal, 2016) to obtain 150-200 orders of perturbation theory (≈ 10 minutes per plot).

Use Padé approximant to find poles and accumulation points.
Poles in the Borel Plane (1)
Poles in the Borel Plane (2)

\[ \eta = \frac{2}{5} e^{i \theta}, \quad \zeta = \frac{2}{5} e^{i \theta} \]

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\[ \eta = \frac{2}{5} e^{\frac{i (3 \pi)}{4}}, \quad \zeta = \frac{2}{5} e^{\frac{i (3 \pi)}{4}} \]
Consider the quadratic differential of the low-energy effective theory of $\mathcal{N} = 2$ Seiberg-Witten theory.

- $SU(2)$ with $N_f = 4$ and pairwise equal masses.
- Trigonometric limit of $SU(2) \times SU(2)$ elliptic quiver theory with $N_f = 2$.
- Coincides with our WKB-differential!
- BPS states = Unitons?
- Connection with integrable Hitchin system through Gaudin Models??
Questions