A Bogomol'nyi equation for Magnetic Skyrmions

Bruno Barton-Singer

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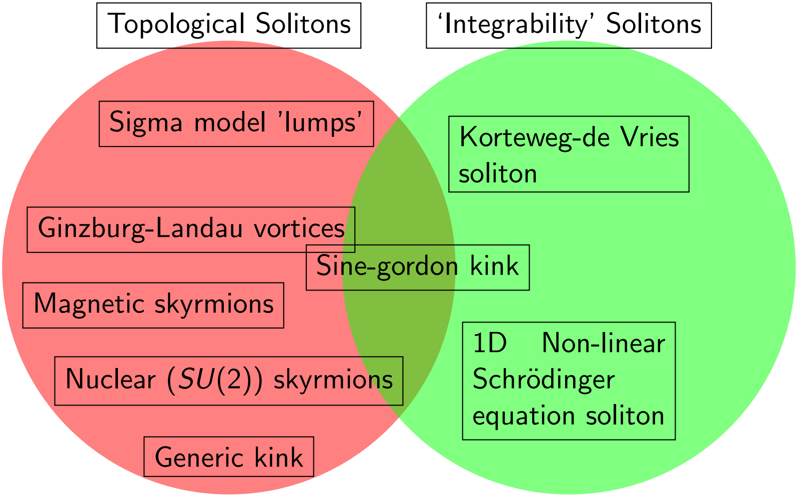
- These are 'conserved charges' of the theory- no corresponding symmetry!

What are they not? A side-note on bad naming



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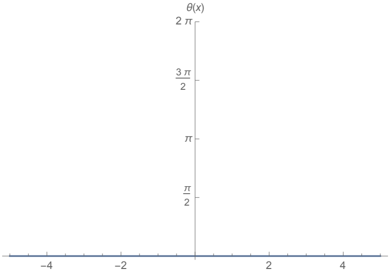
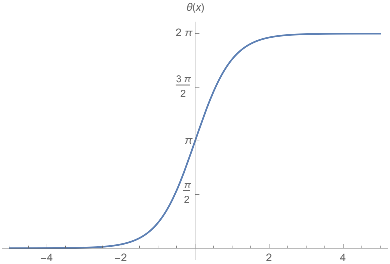
  - It gives us a lower bound on the energy, related to the topology
  - It gives us an equation to solve to find minimizers- which is in general easier to solve than the Euler-Lagrange equation

Example: the 1D kink

We have the static energy functional for a S^1-valued field $\theta(x)$:

$$E = \int \{ \tfrac{1}{2} \theta'(x)^2 + V(\theta) \} dx$$





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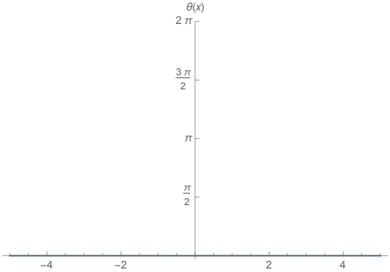
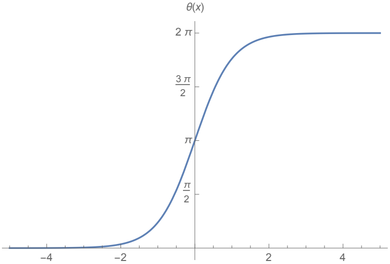
after some blackboard work:

$$E = \int \tfrac{1}{2}(\theta'(x) - \sqrt{2V})^2 dx + Q \int_0^{2\pi} \sqrt{2V} d\theta$$





Chiral magnets and magnetic skyrmions

Take a magnet with a crystal structure that breaks parity. This allows an antisymmetric 'Dzyaloshinskii-Moriya' exchange term in the microscopic hamiltonian:

$$E[\{\mathbf{S}_i\}] = -J\sum_{<ij>}\mathbf{S}_i\cdot\mathbf{S}_j + \sum_{<ij>}\mathbf{D}_{ij}\cdot(\mathbf{S}_i\times\mathbf{S}_j) - \sum_i \mathbf{B}\cdot\mathbf{S}_i$$

Going to the continuum level, this gives the following energy functional:

$$E[n] = \int_{\mathbb{R}^2}\left(\frac{1}{2}(\nabla\mathbf{n})^2 + k\mathbf{n}\cdot(\nabla\times\mathbf{n}) + h_z(1-n_3) + h_a(1-n_3^2)\right)$$

The bogomol'nyi argument for magnetic skyrmions

Two steps:

1. We first reinterpret the model as a gauged sigma model with covariant derivative $D_i\mathbf{n} = \partial_i\mathbf{n} + \mathbf{A}_i \times \mathbf{n}$, $\mathbf{A}_i = -k\mathbf{e}_i$, and associated field strength $\mathbf{F_{12}} = k^2\mathbf{e}_3$:
$E[n] = \int_{\mathbb{R}^2} \left(\frac{1}{2}(D\mathbf{n})^2 + V(\mathbf{n})\right)$
2. When $h_a = -\frac{k^2}{2}$, $h_z = k^2$, we can rewrite the above as
$E[n] = \int_{\mathbb{R}^2} \left(\frac{1}{2}(D_1\mathbf{n} + \mathbf{n} \times D_2\mathbf{n})^2 + \mathbf{n} \cdot (\partial_1\mathbf{n} \times \partial_2\mathbf{n}) + b.t.\right)$

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For the case considered, where $\mathbf{A}_i = -k\mathbf{e}_i$, this becomes:

$$\partial_{\bar{z}} w = \frac{i}{2}kw^2 \implies \partial_{\bar{z}}\frac{1}{w} = -\frac{i}{2}k$$

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Which is solved by
$w(z, \bar{z}) = -\frac{i}{2} k\bar{z} + f(z)$
with $f(z)$ a smooth function of z. For finite degree, $f(z)$ must be a ratio of two polynomials and the degree of the solution can be found from the degrees of these polynomials

Conclusions and further work

- Reference: BBS, Calum Ross and Bernd J Schroers, 'Magnetic Skyrmions at Critical Coupling'
- This method from mathematical physics can be used to find explicit analytical solutions of the Euler-Lagrange equations for 'real world' topological solitons.
- What's more, some of the more outlandish solutions have now been seen numerically away from this 'critical point' (upcoming paper with Kiselev, Kuchkin and collaborators)
- On the theory side, these solutions could be used to investigate the dynamics of skyrmions, or their quantization. No progress yet however.