Introduction to Non-Equilibrium QFT, and the 2-PI Effective Action

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Motivation

- Understand: foundations of QFT, general/unconstrained setting
- Find: models for Heavy-Ion collisions, post inflationary physics, strongly quenched dilute Bose gases, ...
- Provide: "nice" measurable quantities for experiments to test against in thermal equilibrium

What is Equilibrium classically

Thermodynamics and Statistical (classical) Physics:

- Ignore details: **variables of state**, e.g. volume V, pressure P, temperature T.
- Find Equilibrium: **thermodynamic potential** Φ and $\delta\Phi = 0$
- Equilibrium: variables of state constrained by **equation of state**

What is Equilibrium in Quantum Mechanics?

Quantum Statistical Physics:

- Statistical uncertainty: **density matrix**:

$$\rho_0 = \sum_n p[\psi_n] \cdot |\psi_n\rangle \langle\psi_n|$$

- Equilibrium: $\rho_0 = \rho_\beta \sim e^{-\beta H}$, with $|\psi_n\rangle = |E_n\rangle$ and

$$p[|E_n\rangle] \sim e^{-\beta \cdot E_n}$$

- Expectation values:

$$\langle \mathcal{O}(t) \rangle = tr\{\rho_0(t)\, \mathcal{O}(t)\}$$

What is Equilibrium QFT?

Thermal and Vacuum Quantum Field Theory:

- Rewrite: $\rho_\beta = e^{i(i\beta)H} = e^{-i(\tau-(-\tau))H}$, with $2i\tau = \beta$

- Derive Path integral:

$$\mathcal{Z}_\beta \sim \int_{\phi(+\tau)=\phi(-\tau)} [\mathcal{D}\phi] e^{-i\int_{-\tau}^{+\tau}\int_x \mathcal{L}[\phi]} \qquad (1)$$

- In comparison: "Vacuum" QFT result:

$$\mathcal{Z}_0 \sim \int [\mathcal{D}\phi] e^{-i\int_{-\infty}^{+\infty}\int_x \mathcal{L}[\phi]}$$

Back to square one

- Knowing $\mathcal{Z}[\mathcal{J}]$ $\leftrightarrow$ Knowing all correlation functions

- Example SFT: $\mathcal{G}^n \sim \langle \phi_1 .... \phi_n \rangle$, with notation $\phi_i = \phi(x_i)$

- Consider: $\langle \mathcal{O}(t) \rangle$ and $\rho_0(t) = \mathcal{U}(t, t_0)\, \rho_0\, \mathcal{U}(t_0, t)$, then

$$\langle \mathcal{O}(t) \rangle = tr\{\rho_0\, \mathcal{U}(t_0, t)\, \mathcal{O}(t)\, \mathcal{U}(t, t_0)\}$$

From First Principles: the Keldysh Contour

$$\langle \mathcal{O}(t) \rangle = tr\{\rho_0 \, \mathcal{U}(t_0, t) \, \mathcal{O}(t) \mathcal{U}(t, t_0)\}$$

- Visualizing this "forward-backward" evolution with a time-contour $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$:

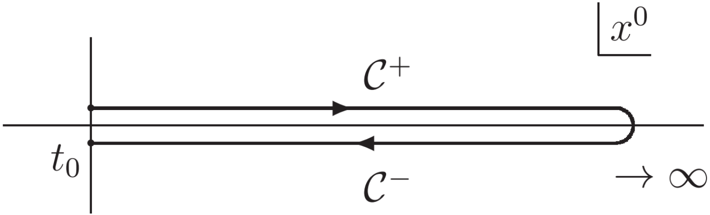

Figure: Taken from [Ber15]

Vacuum QFT from first principles

- Expectation values of Vacuum theory:

$$\langle 0, t| \mathcal{O}(t) |0, t\rangle = \langle 0, -\infty| \mathcal{U}(-\infty, t) \mathcal{O}(t) \mathcal{U}(t, -\infty) |0, -\infty\rangle$$

- Interactions **adiabatically** turned on/off, i.e. in/out Vacua equal up to phase $e^{i\mathcal{L}}$:

$$\mathcal{U}(-\infty, +\infty) |0\rangle = e^{i\mathcal{L}} |0\rangle$$

- This leads to

$$\langle 0, t| \mathcal{O}(t) |0, t\rangle = \frac{\langle 0| \mathcal{U}(-\infty, t) \mathcal{O}(t) \mathcal{U}(t, +\infty) |0\rangle}{e^{i\mathcal{L}}}$$

... have we seen this before?

Compare to "standard" result (Gell-Mann Low Theorem):

$$\langle\Omega|\,\mathcal{O}(t)\,|\Omega\rangle = \frac{\langle 0|\,U(-\infty,t)\,\mathcal{O}(t)\,U(t,+\infty)\,|0\rangle}{\langle 0|\,U(-\infty,+\infty)\,|0\rangle}$$

- Comparing both formulas, we find

$$e^{i\mathcal{L}} = \langle 0|\,\mathcal{U}(-\infty,+\infty)\,|0\rangle$$

$\Rightarrow e^{i\mathcal{L}} \mathrel{\hat{=}}$ Vacuum fluctuations

Where's the Non-Equilibrium?

- Consider: $\rho_0 \sim e^{\mathcal{P}(\phi)}$, and $\mathcal{P}$ polynomial:

$$\mathcal{P}(\phi) = \alpha_1 \cdot \phi + \alpha_2 \cdot \phi\phi + \ldots$$

- Derive Path integral: complete set of states $|\phi_0^{\pm}\rangle$
- ... yielding for a SFT:

$$\mathcal{Z}[\mathcal{J};\rho_0] = \int [\mathcal{D}\phi_0^{\pm}] \langle \phi_0^+ |\, \rho_0 \,| \phi_0^- \rangle \int_{\phi_0^+}^{\phi_0^-} [\mathcal{D}\phi] e^{-i\left(\mathcal{S}_{SFT}[\phi] + \int_{\mathcal{C}} J\cdot\phi\right)}$$

The 1-PI Effective Action

$$\Gamma[\phi] = \mathcal{W}[\mathcal{J}] - \int_x (\mathcal{J} \cdot \phi)(x)$$

- Γ: **Legendre trasform** of $\mathcal{W}$, changing $J \to \phi$
- Source $\mathcal{J}$, macroscopic field $\phi \leftrightarrow B_{ext}$, magnetization M
- Γ: "effective", full quantum action, irreducible building blocks of theory

The 2-PI Effective Action

$$\Gamma[\phi, G] = \Gamma[\phi] - \frac{1}{2}\int_{x,y} \left(G(x,y) + \phi(x)\,\phi(y)\right)\mathcal{R}_2(x,y)$$

- $(G + \phi\phi)$: full propagator; G: connected propagator
- Quantum E.O.M.:

$$\frac{\delta\Gamma}{\delta\phi(y)} = -\mathcal{J}(y) - \int_x \left(\phi(x)\cdot\mathcal{R}_2(x,y)\right)$$

$$\frac{\delta\Gamma}{\delta G(x,y)} = -\frac{1}{2}\mathcal{R}_2(x,y)$$

Some interesting E.O.M.

- Consider: $F(x,y) = \frac{1}{2}\{\phi(x), \phi(y)\}$, "Statistical Function":

$$[\Box_x + M^2(x)]F(x,y) = -\int_{t_0}^{x_0} \mathrm{d}z\ \Sigma^{\rho}(x,z)\,F(z,y) + \int_{t_0}^{y_0} \mathrm{d}z\ \Sigma^{F}(x,z)\,\rho(z,y)$$

- Consider: $\rho(x,y) = \frac{i}{2}[\phi(x), \phi(y)]$, "Spectral Function":

$$[\Box_x + M^2(x)]\rho(x,y) = \int_{x_0}^{y_0} \mathrm{d}z\ \Sigma^{\rho}(x,z)\,\rho(z,y)$$

Bringing it all together...

Remember: general Path Integral

$$\mathcal{Z}[\mathcal{J}] = \int [\mathcal{D}\phi_0^{\pm}] \langle \phi_0^+ | \, e^{\mathcal{P}} \, | \phi_0^- \rangle \int_{\phi_0^+}^{\phi_0^-} [\mathcal{D}\phi] e^{-i\left(\mathcal{S}_{SFT}[\phi] + \int_{\mathcal{C}} J \cdot \phi\right)}$$

and polynomial

$$\mathcal{P}(\phi) = \alpha_1 \cdot \phi + \alpha_2 \cdot \phi\phi + \ldots$$

- Interpret α's as sources for operators $\phi, \phi\phi,$ ..., at time t_0

Bringing it all together...

- Interpret α's as sources for operators $\phi, \phi\phi,$ ..., at time t_0

- Absorb sources at t_0 into general sources, e.g.:

$$\mathcal{J}(t) \longrightarrow \mathcal{J}(t) + \alpha_1 \cdot \delta(t_0)$$

- Re-write Path Integral, absorb initial conditions:

$$\mathcal{Z}[\mathcal{J}, \mathcal{R}_2; \alpha_1, \alpha_2, ...] \sim \int [\mathcal{D}\phi] e^{-i\left(\mathcal{S}_{SFT}[\phi] \, + \, \int_{\mathcal{C}} J \cdot \phi \, + \, \frac{1}{2} \int_{\mathcal{C}} \mathcal{R}_2 \cdot \phi\phi \, + ...\right)}$$

Bringing it all together...

Punchline: E.O.M. get initial values

Some interesting E.O.M. (again!)

- Consider: $F(x,y) = \frac{1}{2}\{\phi(x), \phi(y)\}$, "Statistical Function":

$$[\Box_x + M^2(x)]F(x,y) = -\int_{t_0}^{x_0} \mathrm{d}z\ \Sigma^{\rho}(x,z)\,F(z,y) + \int_{t_0}^{y_0} \mathrm{d}z\ \Sigma^{F}(x,z)\,\rho(z,y)$$

- Consider: $\rho(x,y) = \frac{i}{2}[\phi(x), \phi(y)]$, "Spectral Function":

$$[\Box_x + M^2(x)]\rho(x,y) = \int_{x_0}^{y_0} \mathrm{d}z\ \Sigma^{\rho}(x,z)\,\rho(z,y)$$

Thermalization

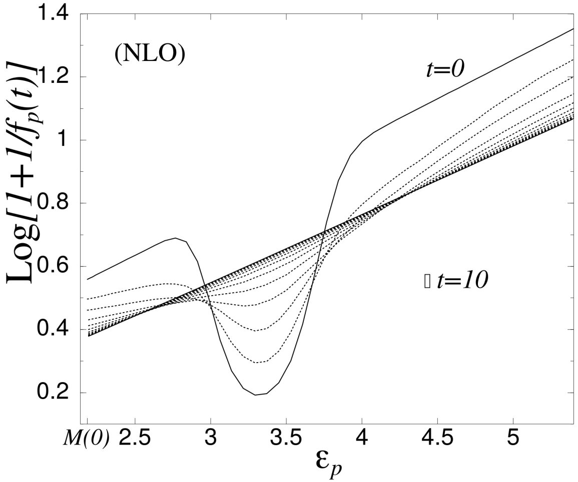

Figure: Taken from [Ber15]

Thank you for your attention,
Questions?

References Theory (with links!)

Ber15 J. Berges: "From Cold Atoms to Cosmology", arXiv:1503.02907

Kam05 A. Kamenev: "Many-Body Theory of Non-Equilibirum System", arXiv:cond-mat/0412296

Kam09 A. Kamenev: "Introduction to the Keldysh Formalism", http://www.capri-school.eu/capri09/lectures/Kamenev_Capri09.pdf

Wei14 T. Weigand: "Quantum Field Theory I+II", Lecture Notes http://www.capri-school.eu/capri09/lectures/Kamenev_Capri09.pdf

Back-up slides

Resolving the time Contour

To recover "intuitive" picture, get rid of contour $\mathcal{C}$, e.g. SFT:

- $\phi^{\pm}(t) \equiv \phi \cdot \chi(t \in \mathcal{C}^{\pm})$
- Resolving $\mathcal{C}$ as follows:

$$\int_{\mathcal{C}} \phi \, \mathrm{d}t = \int_{\mathcal{C}^+} \phi^+ \mathrm{d}t + \int_{\mathcal{C}^-} \phi^- \mathrm{d}t = \int_{t_0}^{\infty} \left(\phi^+ - \phi^-\right) \, \mathrm{d}t$$

- new fields: $\varphi \sim (\phi^+ - \phi^-)$ and $\Phi \sim (\phi^+ + \phi^-)$
- φ: "quantum"/"response", Φ: "classical"/"statistical" field

The n-PI Effective Action

Why stop at single Legendre transform?

$$\mathit{\Gamma}[\phi, G, V^{(3)}, \ldots, V^{(n)}]] = \mathit{\Gamma}[\ldots, V^{(n-1)}] - \int_{x_1, x_2, \ldots, x_n} \left(V^{(n)} \cdot \mathcal{R}_n \right)$$

- $V^{(n)}$, full n-point functions, i.e. correlation functions
- Need **source** terms $\mathcal{R}_n$ in $\mathcal{W}$ that couple to $V^{(n)}$'s