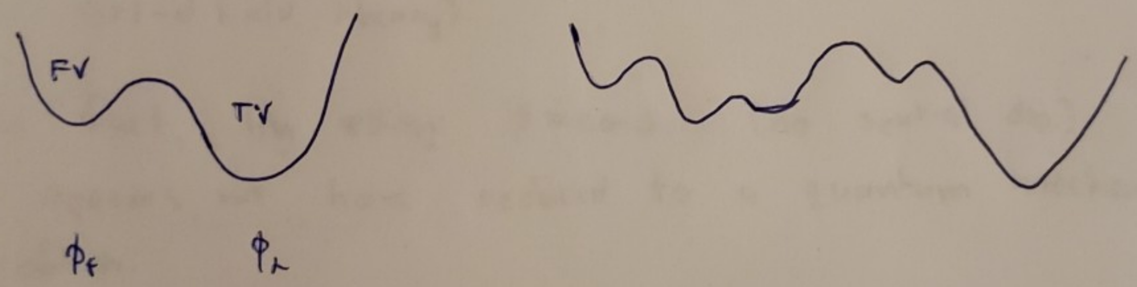
False Vacuum Decay

Let's consider a field theory:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \qquad ds^2 = -dt^2 + d\vec{x}^2$$

with a $V(\phi)$ of the form



More generally, we could have many fields, and some complicated potential.

We want to know what happens when some large region of space is trapped in ϕ_F.

Naive approach --

$\phi(x, t_0) = \phi_{FV}$ is just a constant. and $\dot{\phi}(t=0) = 0$

At the classical level w/ uniform field.

$$\ddot{\phi} + \frac{\partial V}{\partial\phi} = 0 \quad \text{and} \quad \frac{\partial V}{\partial\phi}\Big|_{\phi_{FV}} = 0 \quad \Rightarrow \quad \phi(x,t) = \phi_{FV}$$

So, the field will be trapped at the false vacuum forever.

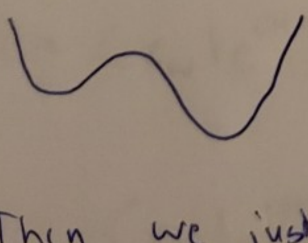
However, energetics suggest the field really wants to decay to the lower energy state ϕ_{tv}.

How does this happen?

First, consider a slightly different problem.
Instead of field theory, we can think about QM (i.e. 0+1-d field theory).

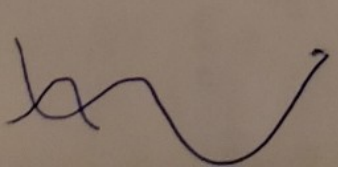
In fact, by setting $\phi = \text{const.}$ (no spatial dep.) it appears we have reduced to a quantum mechanics problem.
(there's a caveat I'll get to in a second.

Say we have a particle in a well

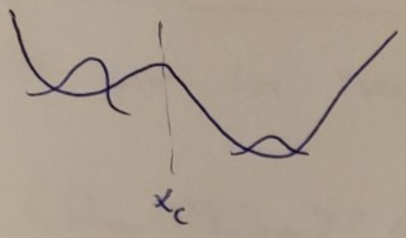


Then we just solve the wavefunction

$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$ w/ an initial Gaussian (approximately) in ~~the~~ one well



At a later time, some probability will have leaked into the other well.



$$P(x \in \text{true}) \simeq \int_{x_c}^{\infty} dx \, |\psi(x)|^2$$

we will also find

$$P_{\text{undecayed}} = 1 - P(x \in \text{true}) \sim e^{-\Gamma t}$$ at intermediate times

Now, how do we compute Γ?

① solve for $|\psi\rangle$ and extract Γ

② via path integral.

Path integral:

Roughly,

$$Z = \langle x_{fv} | e^{iHt} | x_{fv} \rangle = \int_{x(0)=x_{fv}}^{x(T)=x_{fv}} \mathcal{D}x \, e^{\frac{i}{\hbar} S[x]}$$

$$\rightarrow \quad \langle x_{fv} | e^{-HT} | x_{fv} \rangle = \int_{x(0)=x_{fv}}^{x(T)=x_{fv}} \mathcal{D}x \, e^{-S_E[x]}$$

we can write this as

$$Z = \sum_E e^{-ET} \psi_E(x_i) \psi_E^*(x_f)$$ and as $T \to \infty$

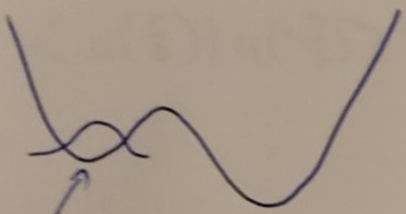
$$E_0 = - \lim_{T\to\infty} \frac{1}{T} \ln Z$$ gives lowest energy state.

Path Integral and Tunnelling..

We want the "vacuum-to-vacuum" amplitude.

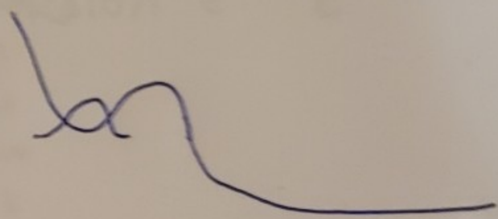
$$\langle \Omega_{vac}(t)|\Omega_{vac}\rangle = \langle \Omega_{vac}|e^{-i\int H dt}|\Omega_{vac}\rangle$$

in QM, let $|a\rangle$ be state near fv.



$\langle x|a\rangle$

or



Let's denote the energy eigenstates by $|n\rangle$.

Then

$$\cancel{|a\rangle = \sum |n\rangle\langle a|n\rangle}$$

$$|a\rangle = \sum |n\rangle\langle n|a\rangle$$

$$\Rightarrow \langle a|e^{-iHT}|a\rangle = \sum e^{-iE_nT}|\langle a|n\rangle|^2$$

Now, to make progress, we "Wick rotate", i.e. work w/ imaginary time

$$\tau = it$$

$$\Rightarrow \langle a|e^{-iHT}|a\rangle = \sum e^{-E_nT}|\langle a|n\rangle|^2$$

If I let $T \to \infty$, then I will pick out only the lowest energy state, E_0.

Now, there is one more key point. Suppose ~~E_0~~ $E_0 = e_0 - i\frac{\Gamma}{2}$ has an imaginary part. (i.e. $\Gamma = -2\,\mathrm{Im}\,E_0$)

In the large T (i.e. large τ limit), we then have

$$\langle a(\tfrac{T}{2}) | a(-\tfrac{T}{2}) \rangle \simeq |\langle a|0\rangle|^2 e^{-E_0 T} \rightarrow |\langle a|0\rangle|^2 e^{-ie_0 T} e^{+i\frac{\Gamma}{2}(iT)}$$

real time $t = \frac{T}{i}$

$$= |\langle a|0\rangle|^2 e^{-ie_0 T} \underbrace{e^{-\frac{\Gamma}{2}T}}$$

amplitude decays in time.

$\therefore$ We interpret an imaginary contribution to E_0 as a decay rate.

In particular

$$P_{\text{undecayed}} = |\langle a(\tfrac{T}{2}) | a(-\tfrac{T}{2}) \rangle|^2 \sim e^{-\Gamma T}$$

Now, how do we find E_0? In particular the imaginary part.

Now back to field theory!

We might expect we can just port the whole calculation over to our field theory.

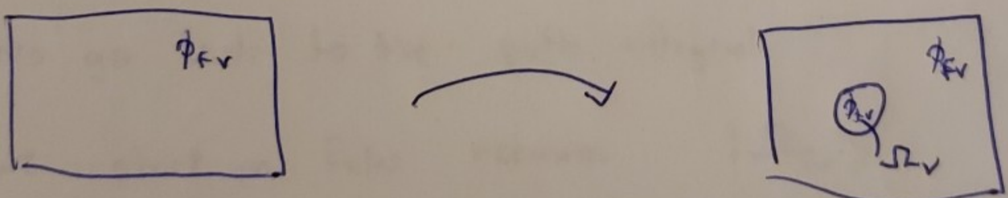
<u>However</u>, in field theory the energy is

$$\propto \underbrace{L^d}_{\text{volume}} (\Delta V) \rightarrow \infty \quad \text{for } L \to \infty$$

So we can't just tunnel the entire field to the ~~false~~ true vacuum and conserve energy.

<u>Instead</u>,

we can tunnel a finite region of volume $\propto \Omega_v$ with $\Delta E = (\Delta V) \propto \Omega_v < 0$



~~Also~~ The field must also be continuous, so in the transition "wall" ϕ has spatial dependence

$$\Rightarrow \cancel{E}\ \rho = \underbrace{\frac{\dot{\phi}^2}{2}}_{KE} + \underbrace{\frac{(\nabla\phi)^2}{2} + V(\phi)}_{\text{"PE"}} \quad \text{has a gradient energy contribution}$$

surface tension is

$$\sigma = \int (\nabla\phi)^2 \, dl \quad \text{along a radial line.}$$

and energy in wall is σA

we can obtain a configuration w/

$\sigma A - \rho \Omega_v = 0$ to conserve energy

$$\Rightarrow \frac{\sigma}{\rho} = \frac{\Omega_v}{A} = (\#) R_{bub} = \frac{\# R_{bub}}{d} \text{ w/ } \# \text{ depending on \# of dimensions}$$

—

So, instead we expect a region of size

$$\boxed{R_{bub} = d\frac{\sigma}{\rho}}$$ to nucleate instead.

—

ok, now, 1 how do we find σ

2 how do we find the decay rate.

—

Let's go back to the path integral.

we start in false vacuum $|\Omega_{fv}\rangle$

it evolves

$T e^{iHt} |\Omega_{fv}\rangle$ to satisfy Schrödinger eqn.

time-ordering

at time t, we want to know $|\langle \Omega(t) | \Omega(t_0) \rangle|^2$

which is the prob. to still be in the false vacuum

Now, how to compute this matrix element.

Note: ϕ is no longer homogeneous, so it isn't just quantum mechanics

—

Now, it is convenient to use the following trick. We analytically continue to complex time

$\tau = it$

$$iS_{real} \to -S_E$$

$$\frac{\partial^2}{\partial t^2} \to -\frac{\partial^2}{\partial \tau^2} \qquad t \to -i\tau$$

$$\Rightarrow iS = i\int d^dx\, dt \left(\frac{\dot{\phi}^2}{2} - \frac{(\nabla\phi)^2}{2} - V(\phi)\right)$$

$$\to -S_E = -\int d^dx\, d\tau \left(\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right)$$

—

So, we now have a high-dimensional integral of the form

$$\int \prod d\phi_i \, e^{-\frac{1}{\hbar}S_E[\vec{\phi}]}$$ if we work on a lattice say

and take $\hbar \to 0$ limit.

How do we evaluate such integrals?

Heuristically, want to find points w/ zero derivative.

If this is a minimum, then clearly dominates path

However, the derivative of the action is just the Euclidean EOM

$$\boxed{\nabla_E^2 \phi - \frac{\partial V}{\partial \phi} = 0}$$

Want solutions of this eqn.

But, elliptic $\Rightarrow$ need boundary conditions.

Since we start in FV, as $r_E \to \infty$, want

$$\phi(r_E = \infty) = \phi_{FV}$$

Ansatz, $\phi = \phi(r_E)$ only

(can prove under some circumstances)

$$\Rightarrow \boxed{\frac{\partial^2 \phi}{\partial r_E^2} + \frac{d}{r}\frac{\partial \phi}{\partial r_E} - \frac{\partial V}{\partial \phi} = 0}$$

$$\phi(\infty) = \phi_{FV}$$

$$\left.\frac{\partial \phi}{\partial r_E}\right|_{r_E = 0} = 0 \leftarrow \text{smooth solution.}$$

solve this equation

Bounce and Decay Parametrics

Recall $\hbar = c = M_p = 1$

Let's write our potential

$$V = m^2 \phi_0^2 f\left(\frac{\phi}{\phi_0}\right) \qquad \text{where } f(x) \text{ is some } \mathcal{O}(1) \text{ function w/ } \mathcal{O}(1) \text{ derivatives}$$

$$\Rightarrow V'' = m^2 f''\left(\frac{\phi}{\phi_0}\right)$$

In d space dimensions

$$[V] = M^{d+1} \quad \text{and} \quad [\phi] = [\phi_0] = M^{\frac{d-1}{2}} \quad \text{and} \quad [m] = M$$

The action is

$$S = \int d^d x \, d\tau \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right)$$

We use dimensionless coordinates

$$\bar{x} = \mu x, \quad \bar{\tau} = \mu\tau \quad \text{and} \quad \bar{\phi} = \frac{\phi}{\phi_0}$$

$$S = \cancel{\Omega_{d+1}} \frac{\phi_0^2}{\mu^{d+1}} \mu^2 \int d^d \bar{x} \, d\bar{\tau} \left[\frac{1}{2}\left(\frac{\partial\bar{\phi}}{\partial\bar{\tau}}\right)^2 + \frac{1}{2}(\bar{\nabla}\bar{\phi})^2 + \frac{m^2\phi_0^2}{\mu^2\phi_0^2} f(\bar{\phi}) \right]$$

choosing $\mu = m$, and assuming Euclidean spherical symmetry

$$S = \Omega_{d+1} \overbrace{\left(\frac{\phi_0^2}{m^{d-1}}\right)}^{\text{dimensionless ratio}} \int d\bar{r}_E \, \bar{r}_E^{\,d} \left(\frac{1}{2}\left(\frac{\partial\bar{\phi}}{\partial\bar{r}_E}\right)^2 + f(\bar{\phi}) \right)$$

Now, if

$$f(x) = \sum \lambda_n x^n \left(\frac{\phi_0}{\Lambda}\right)$$

w/ λ_n a collection of dimensionless coefficients, then for all $\lambda_n \sim \mathcal{O}(1)$, we expect the bounce to have width $\sim m^{-1}$

Then

$$\mathcal{B} \equiv \int d\bar{r}_E\, \bar{r}_E^{\,d} \left(\frac{1}{2}\left(\frac{\partial \bar{\phi}_{bounce}}{\partial \bar{r}_E}\right)^2 + f(\bar{\phi}_{bounce})\right) \sim 1$$

so that

$$S_{bounce} - S_{FV} \simeq \Omega_d \left(\frac{\phi_0^2}{m^{d-1}}\right) B \sim \left(\frac{\phi_0^2}{m^{d-1}}\right)$$

where $B \equiv \int d\bar{r}_E\, \bar{r}_E^{\,d} \left(\frac{1}{2}\left(\frac{\partial \bar{\phi}_{bounce}}{\partial \bar{r}_E}\right)^2 + f(\bar{\phi}_{bounce}) - f(\bar{\phi}_{FV})\right)$

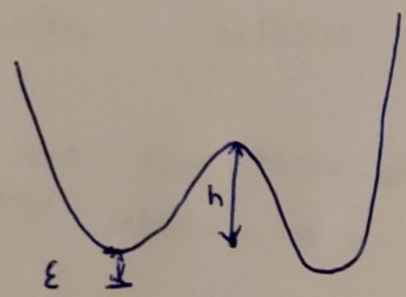
More on thin-wall

DRAFT!!

~~Now do thin-wall to~~ estimate action.

In the general case, this nonlinear BVP must be solved numerically. However, there is a (very informative) special case where further analytic progress can be made.

Consider a ~~well~~ potential with nearly degenerate local minima



What do we mean by this?

ϕ_{tv}, ϕ_{fv}: true & false vacua

ϕ_{max}: $V'(\phi_{max}) = 0$

Let $\mu^2 = -V''(\phi_{max})$. If $\Delta\phi = |\phi_{tv} - \phi_{fv}|$, then the barrier height is

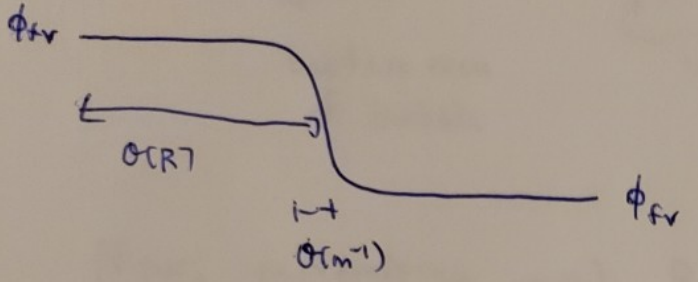
$$\boxed{h \sim \mu^2 (\Delta\phi)^2}$$

so $\cancel{h<<}$ $\varepsilon << h \Rightarrow \cancel{\frac{\mu^2(\Delta\phi)^2}{\varepsilon}}$ $\boxed{\frac{\varepsilon}{\mu^2(\Delta\phi)^2} << 1}$

Ok, now to reexpress in terms of tension.

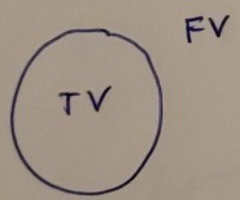
More on thin-wall

In this limit, the instanton looks like



with $\quad mR \gg 1$

We can thus think of the bubble as a thin-wall connecting a TV interior to a FV exterior



The wall has a surface tension σ. To estimate this, we notice that the ~~wall with~~ bubble profile will be almost the same as an equivalent domain wall in 1D interpolating between degenerate vacua.

$$\cancel{\sigma = \int_0^\infty (\partial_r\phi)^2 dr = \int}$$

$$\sigma = \int dr \left(\tfrac{1}{2}(\phi')^2 + V(\phi) - V_{fv}\right)$$

$$= \int dr\, (\phi')^2$$

$$= \int_{\phi_{tv}}^{\phi_{fv}} d\phi \sqrt{V(\phi) - V_{fv}}$$

The action is then, where I subtracted $S_{fv} = \int d^d x \, V_{fv}$

$$S \simeq \underbrace{\sigma \Omega_d R^d}_{\text{surface area of bubble.}} - (V_{fv} - V_{tv}) \underbrace{\frac{\Omega_d}{d+1} R^{d+1}}_{\text{volume of bubble}}$$

Now, extremising w.r.t. R

$$\frac{\partial S}{\partial R} = d \sigma \Omega_d R^{d-1} - (V_{fv} - V_{tv}) \Omega_d R^d = 0$$

$$\Rightarrow \boxed{R = d \frac{\sigma}{V_{fv} - V_{tv}}}$$

and

$$\cancel{S_{bounce} = (V_{tv} - V_{fv}) \Omega_d R^d \left(\frac{d\sigma}{V_{tv} - V_{fv}} R^{-1} - 1 \right)}$$

$$\cancel{= \Omega_d (V_{tv} - V_{fv}) d^d \left(\frac{\sigma}{V_{tv} - V_{fv}} \right)^d}$$

$$S_{bounce} = (V_{fv} - V_{tv}) \Omega_d R^{d+1} \overbrace{\left(\frac{\sigma}{V_{fv} - V_{tv}} R^{-1} - \frac{1}{d+1} \right)}^{= \frac{1}{d} - \frac{1}{d+1} = \frac{1}{d(d+1)}}$$

$$= \frac{(V_{fv} - V_{tv})}{d} \frac{\Omega_d}{d+1} R^{d+1} = \frac{(V_{fv} - V_{tv})}{d} V_{bubble}$$

$$= \boxed{\Omega_d \left(\frac{d^d}{d+1} \right) \frac{\sigma^{d+1}}{(V_{fv} - V_{tv})^d} = S_{bounce}}$$

thin-wall approximation

e.g.

$d=1$: $\Omega_d = 2\pi$ $\qquad S = \pi \dfrac{\sigma^3}{V_{fv}-V_{tv}}$

$d=2$: $\Omega_d = 4\pi$ $\qquad S = \dfrac{16}{3}\pi \dfrac{\sigma^3}{(V_{fv}-V_{tv})^2}$

$d=3$: $\Omega_d = 2\pi^2$ $\qquad S = \dfrac{27}{2}\pi^2 \dfrac{\sigma^4}{(V_{fv}-V_{tv})^3}$

Note in particular, that

$$E_{bubble} = \Omega_{d-1}\sigma R^{d-1} + \frac{\Omega_{d-1}}{d}(V_{fv}-V_{tv})R^d$$

$$= \frac{\Omega_{d-1}}{d} R^d \left(d \frac{\sigma}{V_{fv}-V_{tv}} R^{-1} - 1 \right)(V_{fv}-V_{tv}) = 0$$

so that energy is conserved, as required.

Nonequilibrium QFT

In general, we want to compute operator expectation values in QFT

$$\langle \hat{\mathcal{O}} \rangle \equiv \mathrm{Tr}(\hat{\rho}\hat{\mathcal{O}})$$ where ① $\hat{\mathcal{O}}$ is some operator
② $\hat{\rho}$ is the density operator, so $\mathrm{Tr}\hat{\rho} = 1$

For a pure state $|\psi\rangle$

$$\hat{\rho} = |\psi\rangle\langle\psi| \quad \text{and} \quad \mathrm{Tr}\hat{\rho}^2 = 1$$

More generally

$$\hat{\rho} = \sum P_i |\psi_i\rangle\langle\psi_i|$$ for a collection of states $|\psi_i\rangle$ and $\sum_i P_i = 1$.

e.g. a thermal state, let $|E_i\rangle$ be energy eigenstates

$$\Rightarrow \hat{\rho} = \frac{\sum e^{-\beta E_i} |E_i\rangle\langle E_i|}{Z}$$ w/ Z the partition function.

Now, we need a scheme to allow for numerical evaluation.

Goal: Rewrite in a way that statistical field theory techniques are available.

What do I mean by this?

Easy Numerical Tasks:

(A) Doing time-evolution from an initial state

e.g. $\ddot{\phi} - \nabla^2\phi + V'(\phi) = 0$ ⟸ Relativistic Scalar Field

← Nonrelativistic Field.

(B) Generating samples of a Gaussian Random Field

- Direct sampling of k-modes
- Methods to correctly sample real-space 2-pt. function

(C) Computing nonlinear response (in time) of a field to injected noise.

Can I assemble these ingredients into an approximation scheme for QFT?

YES!!

Heuristic Motivation:

Let's imaging we have a field

$$\hat{\phi} = \underbrace{\bar{\phi}}_{\text{"classical background"}} + \underbrace{\delta\hat{\phi}}_{\text{"quantum fluctuations"}}$$

Suppose that ~~we~~ the $\delta\hat{\phi}$'s are "small", so we can expand the action to quadratic order.
The fluctuations then obey linear equations of motion

$$S[\hat{\phi}] = S[\bar{\phi} + \delta\hat{\phi}] \cancel{\simeq S[\bar{\phi}] + \int}$$

$$= \int d^{d+1}x \left(\mathcal{L}[\bar{\phi}] + \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta^2\mathcal{L}}{\delta\phi^2}[\bar{\phi}]\delta\phi^2 + \cdots \right)$$

(there are also $\delta\dot{\phi}$ terms here).

Since the action is quadratic, $\delta\hat{\phi}$ is a sum of Fourier modes that each act as a harmonic oscillator in the ground state, which is Gaussian

$$\delta\hat{\phi} = \sum \hat{a}_k \sqrt{P_k}\, e^{i\vec{k}\cdot\vec{x}}$$

$$\delta\dot{\hat{\phi}} = \sum \hat{b}_k\, \omega_k \sqrt{P_k}\, e^{i\vec{k}\cdot\vec{x}}$$

w/ $P_k = \cancel{\frac{1}{\sqrt{2\omega}}}\frac{1}{2\omega_k}$ for the Minkowski vacuum.

In this limit, we can replace the $\hat{a}_k$ and $\hat{b}_k$ operators w/ complex Gaussian random deviates to exactly reproduce all QFT correlators by evolving w/ the (linearised) classical EOM

Now, if the fluctuations become large, then $[\delta\hat{\phi}, \delta\dot{\hat{\phi}}] \to 0$, and we are in the classical wave limit. If nonlinearities turn on, we expect to stay in the classical wave limit.
Similarly, for weak interactions, we may expect this approach to be almost correct

Formalisation

Normally, we think of QM in terms of wavefunctions,

$$\psi(x) = \langle x|\psi\rangle \quad \text{or} \quad \hat{\varphi}(p) = \langle p|\psi\rangle$$

However, ~~this~~ expressed in this form, we have no access to phase space, as we only get "x" or "p" information as a complete description.

In a field theory context, say we want to solve

$$\ddot{\phi} - \nabla^2\phi + V'(\phi) = 0$$

As an IVP, we need both the initial values of ϕ and $\dot{\phi}$.

In NR theory, we instead have $\psi(x) \in \mathbb{C}$.

If we write $\psi = \sqrt{\rho}\, e^{i\phi}$ then ρ & ϕ act as position and momenta (i.e. ϕ & $\dot{\phi}$)

Here, we need initial ρ and ϕ values to solve the nonlinear Schrödinger equation.

So the first thing we need is a phase space rep. of QFT.

A very useful representation is the Wigner function

$$W(x,p) \equiv \int \langle x+\tfrac{z}{2}|\hat{\rho}|x-\tfrac{z}{2}\rangle e^{-ipz}\, dz$$

This satisfies some important properties

1. $\int \frac{dx\,dp}{2\pi} W(x,p) = 1$

and

2. $\langle \hat{O}_W \rangle = \int O_W(x,p)\, W(x,p) \frac{dx\,dp}{2\pi}$

where O_W is the Wigner symbol of $\hat{O}$ (roughly symmetrized operator)

$W(x,p)$ thus behaves like a <u>probability density on phase space.</u>

But, W is not generally +ve ~~definite~~ semidefinite.

However, $W \geq 0$ for a Gaussian wavefunction. (such as the ground state of the Harmonic oscillator).

It is therefore of interest to evolve W.

Its equation of motion follows from the Schrödinger eqn.

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}|\Psi\rangle \implies i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}]$$

~~After some manipulation~~

For simplicity, let's assume

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \longleftrightarrow \hat{H} = \int d^dx \left[\frac{\dot{\phi}^2}{2} + \frac{(\nabla\phi)^2}{2} + V(\phi)\right]$$

Can generalise to

$$\mathcal{H} = P(\hat{x}) + V(\hat{\phi}) \quad \text{or} \quad \mathcal{H}(\hat{x}, \hat{\phi}) \quad w/ \quad \hat{x} = \frac{\partial_\mu \hat{\phi} \partial^\mu \hat{\phi}}{2}$$

The Schrödinger eqn. gives

$$\frac{\partial W(x,p)}{\partial t} + \int d^dx \left(\Pi \frac{\delta}{\delta\phi} + \nabla^2\phi \frac{\delta}{\delta\Pi} - \frac{2}{i\hbar} V(\phi) \sin\left(\overleftarrow{\nabla}_\phi \frac{i\hbar}{2} \frac{\overrightarrow{\delta}}{\delta\Pi}\right)\right) W = 0$$

We can now expand in $\hbar$

$$\left[\frac{\partial}{\partial t} + \underbrace{\int d^dx \left(\Pi \frac{\delta}{\delta\phi} + \left(\nabla^2\phi - \frac{\partial V}{\partial\phi}\right)\frac{\delta}{\delta\Pi}\right)}_{\text{classical evolution}} + \underbrace{\mathcal{O}\left(\hbar^2 \frac{\partial^3 V}{\partial\phi^3} \frac{\delta^3}{\delta\Pi^3}\right)}_{\text{quantum noise}}\right] \underbrace{W}_{\text{initial state}} = 0$$

This gives us the evolution equation.

In particular, to $\mathcal{O}(\hbar^2)$, it has the form of a Liouville equation.

full material derivative

$$\frac{\partial W}{\partial t} + A(\phi,\pi)\frac{\delta}{\delta\phi}W + B(\phi,\pi)\frac{\delta}{\delta\pi}W = 0 \quad = \frac{d}{dt}W$$

How do we solve this type of equation?

Method of Characteristics

(i) take a point ϕ_0, π_0 in phase space, and let

$W_0 \equiv W(\phi_0, \pi_0)$

(ii) Now evolve $\phi(t|\phi_0,\pi_0)$ and $\pi(t|\phi_0,\pi_0)$ using

$$\frac{d\phi}{dt} = A(\phi,\pi)$$

from the given initial conditions.

$$\frac{d\pi}{dt} = B(\phi,\pi)$$

Then, if we let

$$W(\phi(t|\phi_0,\pi_0), \pi(t|\phi_0,\pi_0); t) = W_0 = W(\phi_0,\pi_0; t_0)$$

we have

$$\frac{d}{dt}W(\phi(t|\phi_0,\pi_0), \pi(t|\phi_0,\pi_0); t) = \frac{\partial W}{\partial t} + \frac{d\phi}{dt}\frac{\delta W}{\delta\phi} + \frac{d\pi}{dt}\frac{\delta W}{\delta\pi}$$

so that ~~this~~ if we use $\frac{d\phi}{dt} = A$ and $\frac{d\pi}{dt} = B$, we get a solution to the Schrödinger eqn.

Now, from our derivation

$$A(\phi,\pi) = \pi = \frac{\delta \mathcal{H}}{\delta \pi}$$

$$B(\phi,\pi) = -\nabla^2\phi - \frac{\partial V}{\partial \phi} = -\frac{\delta \mathcal{H}}{\delta \phi}$$

i.e. the classical equations of motion

So that to $\mathcal{O}(\hbar^2)$, we can evaluate W by

1. Sampling the initial phase space ϕ, π
2. Evolving this initial state using the classical EOM
3. Repeat 1 & 2 to compute the path integral as a classical statistical average.

Uses the "easy" numerical techniques

Sometimes we can simplify this

(A) If we have a statistically homogeneous field theory that satisfies ergodicity, then we can just use a sufficiently large volume to do our quantum average.
(for example, a Gaussian random field)

However, if the dynamics becomes highly nonGaussian, then ergodicity may not hold or we may require a numerically infeasible volume to overcome ~~"cosm~~ "sample variance"

Now, we have been performing an expansion in $\hbar$, so we can extend this to include higher corrections.

For this, it is convenient to use a path integral (which we will do shortly).

However, first we can start w/ our EOM for W

Let W_0 satisfy $\frac{dW}{dt} = 0$ (i.e. the semiclassical stochastic approx.)

We assume W can be expanded in $\hbar$

$$W = W_0 + \hbar^2 W_1 + \hbar^{2n} W_2 + \cdots$$

We have

$$\frac{\partial W}{\partial t} + L_0 W + \hbar^2 L_1[W] + \cdots = 0$$

w/

$$L_0[W] = \frac{d\phi}{dt}\frac{\delta W}{\delta \phi} + \frac{d\pi}{dt}\frac{\delta W}{\delta \pi} = \frac{\delta \mathcal{H}}{\delta \phi}\frac{\delta W}{\delta \phi} \cancel{+} - \frac{\delta \mathcal{H}}{\delta \phi}\frac{\delta W}{\delta \pi}$$

Expanding order by order in $\hbar$, we have.

$$\frac{\partial W_0}{\partial t} + L_0 W_0 = 0 \qquad \mathcal{O}(\hbar^0)$$

$$\underbrace{\frac{\partial W_1}{\partial t} + L_0 W_1}_{\text{free evolution}} = \underbrace{-L_1 W_0}_{\text{stochastic kick}} \qquad \mathcal{O}(\hbar^2)$$