Developments on the concepts of high precision computations

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OUTLINE

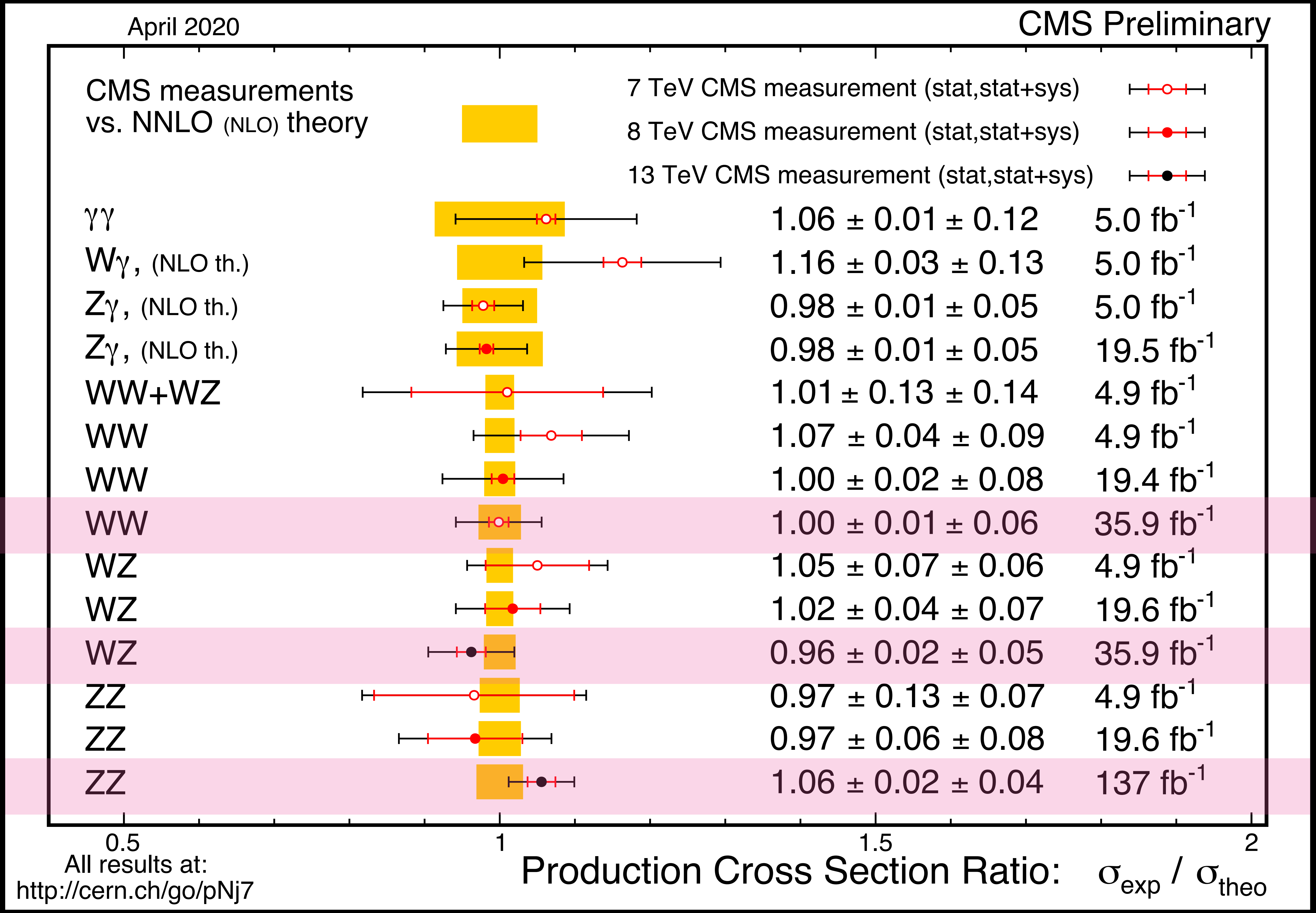
- Motivation for precision calculations
- Sketch of “typical” calculations
- Concepts/properties of gauge theories for perturbation theory

    1. Factorization of Infrared Divergences
    2. Unitarity
    3. The dream of computing in exactly D=4.

- Local factorization at two-loops [*work with R. Haindl, G. Sterman, Z. Yang, M. Zeng*]
- Towards D=4 exactly, in practice
- Outlook

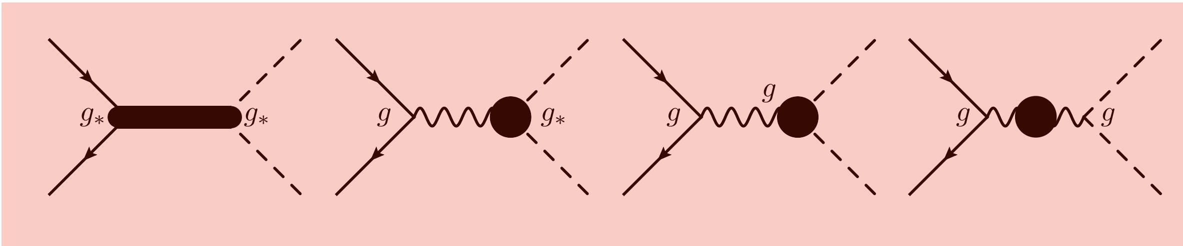
Motivation

- Stunning high precision measurements at the LHC
- Leading to a need for cutting edge detailed (differential) perturbative computations.
- Precision can probe unexplored territories beyond the SM



Current precision in diboson production

Franceschini, Panico, Pomarol, Riva, Wulzer



$$\text{LHC, } 300\,\text{fb}^{-1}:\quad a_q^{(3)} \in [-1.4, 0.9]\,10^{-1}\ \text{TeV}^{-2}$$

$$\text{HL-LHC, } 3\,\text{ab}^{-1}:\quad a_q^{(3)} \in [-4.9, 3.9]\,10^{-2}\ \text{TeV}^{-2}$$

$p_{T,V}$ range	Expected Events
[100-150] GeV	$3100 + 1040\, a_q^{(3)} + 260\, a_q^{(3)\,2}$
[150-220] GeV	$2620 + 1030\, a_q^{(3)} + 140\, a_q^{(3)\,2}$
[220-300] GeV	$937 + 600\, a_q^{(3)} + 230\, a_q^{(3)\,2}$
[300-500] GeV	$544 + 700\, a_q^{(3)} + 560\, a_q^{(3)\,2}$
[500-750] GeV	$86.5 + 260\, a_q^{(3)} + 490\, a_q^{(3)\,2}$
[750-1200] GeV	$16.1 + 120\, a_q^{(3)} + 640\, a_q^{(3)\,2}$

Projected sensitivity to new physics

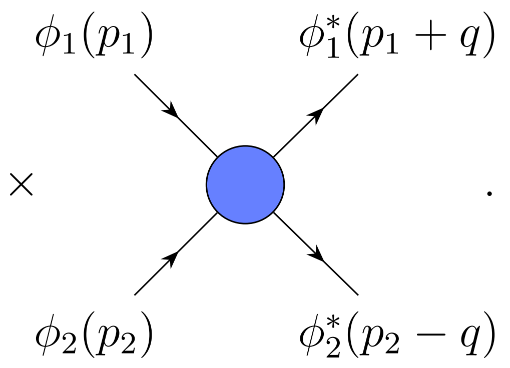
Motivation

- Perturbative quantum field theory techniques turn
  out to be useful beyond collider physics
- A vigorous precision physics program for gravitational
  waves and for measuring the Large Scale Structure
- A surprising use of scattering amplitudes in (mostly) classical physics

Classical impulse from classical limit of amplitudes

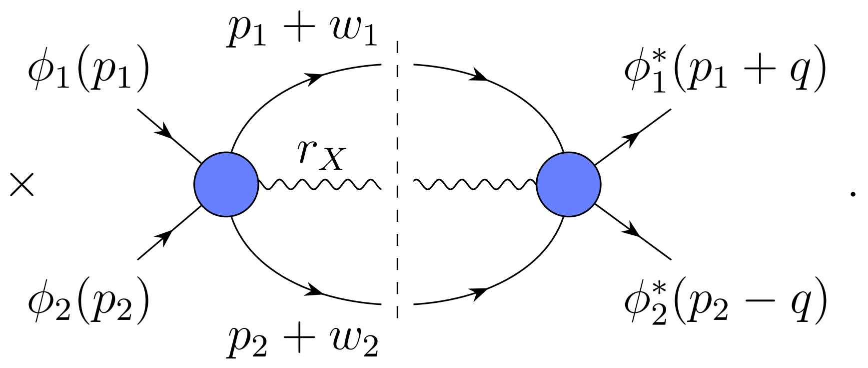
$$I^\mu_{(1)} = \int d\Phi(p_1)d\Phi(p_2)\hat{d}^4q\,\hat\delta(2p_1\cdot q+q^2)\hat\delta(2p_2\cdot q-q^2)\Theta(p_1^0+q^0)\Theta(p_2^0-q^0)$$

$$\times\, e^{-ib\cdot q}\, iq^\mu \times$$



$$I^\mu_{(2)} = \sum_X \int \prod_{i=1,2} d\Phi(p_i)\hat{d}^4 w_i \hat{d}^4 q\;\hat\delta(2p_i\cdot w_i + w_i^2)\Theta(p_i^0+w_i^0)$$
$$\times\,\hat\delta(2p_1\cdot q+q^2)\hat\delta(2p_2\cdot q-q^2)\Theta(p_1^0+q^0)\Theta(p_2^0-q^0)$$
$$\times\; e^{-ib\cdot q/\hbar}\, w_1^\mu\, \hat\delta^{(4)}(w_1+w_2+r_X)$$

$$\times$$

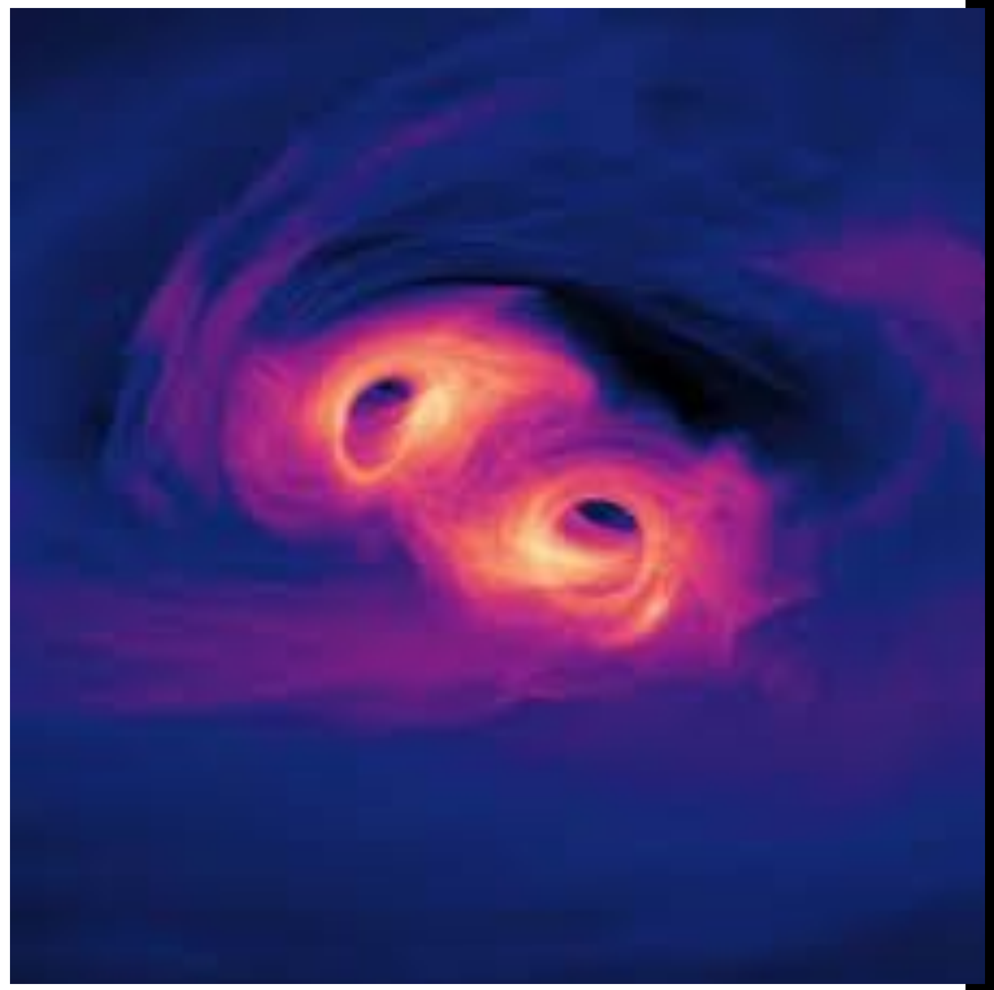


Classical two-body potential in GR

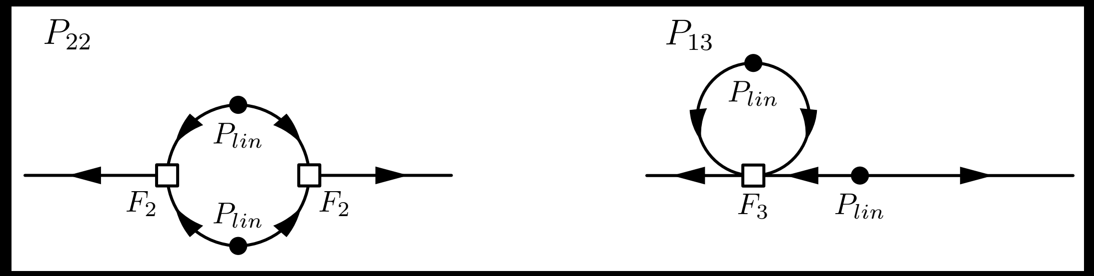
$$H^{3\mathrm{PM}}(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2+m_1^2} + \sqrt{\boldsymbol{p}^2+m_2^2} + V^{3\mathrm{PM}}(\boldsymbol{p},\boldsymbol{r})\,,$$

$$V^{3\mathrm{PM}}(\boldsymbol{p},\boldsymbol{r}) = \sum_{n=1}^{3}\left(\frac{G}{|\boldsymbol{r}|}\right)^n c_n(\boldsymbol{p}^2)\,,$$

$$\frac{\nu^2 m^4}{\gamma^2\xi}\Bigg[\frac{1}{12}\left(3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3\right)-\frac{4\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}$$
$$-\frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)}-\frac{3\nu\sigma\left(7-20\sigma^2\right)}{2\gamma\xi}+\frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3}$$
$$-\frac{\nu^2\left(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2\right)\left(1-2\sigma^2\right)}{4\gamma^3\xi^2}+\frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\Bigg].$$



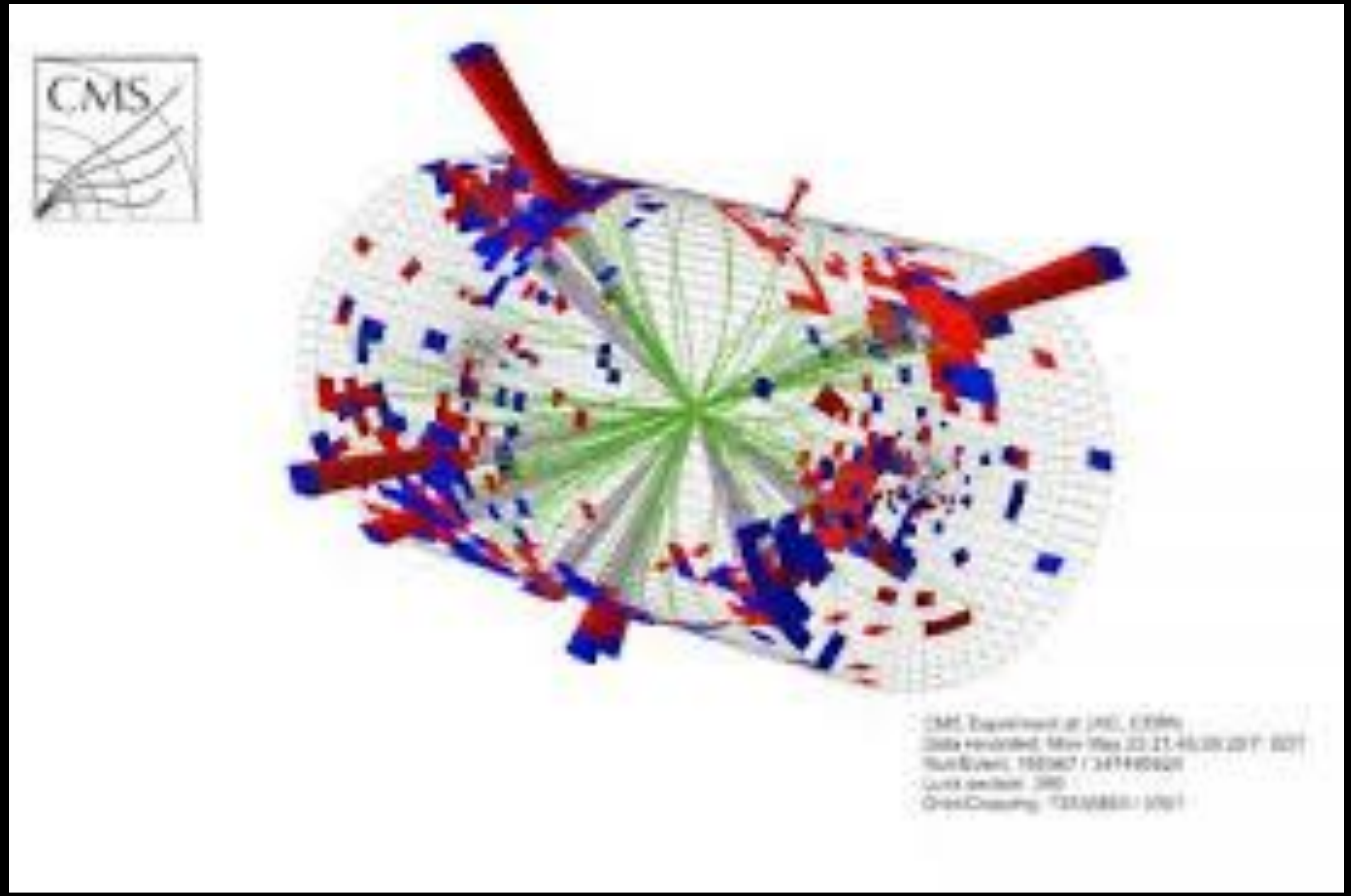
Bern, Cheung, Roiban, Shen, Solon, Zeng;…



$$\mathsf{I}(\nu_1,\nu_2) = \frac{1}{8\pi^{3/2}}\frac{\Gamma(\frac{3}{2}-\nu_1)\Gamma(\frac{3}{2}-\nu_2)\Gamma(\nu_{12}-\frac{3}{2})}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(3-\nu_{12})}\,,$$

Simonovic, Baldauf, Zaldarriaga, Carrasco, Kollmeier; …

Structure of "interesting" events in colliders



- Scattering system from short-distance interactions
- Long-distance described by hadrons

Factorization

$$\sigma\left[\mathcal{O}\right] = f_i \otimes f_j \otimes \sigma_{ij}\left[\mathcal{O}\right] + small$$

Collins, Soper, Sterman

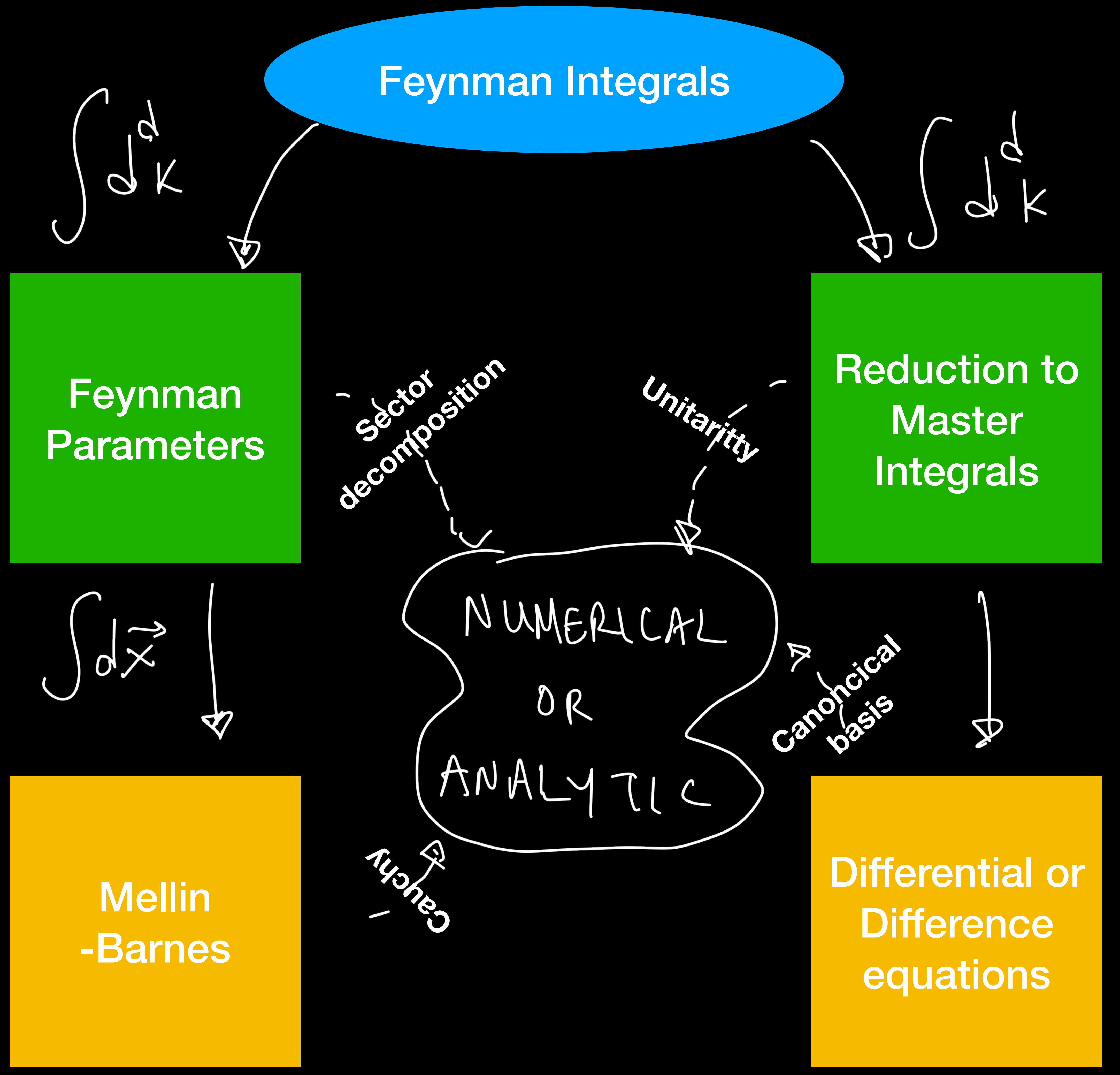
Perturbation theory

$$\sigma_{ij}\left[\mathcal{O}\right] = \int_{\mathcal{O}} dPhaseSpace\ \left|Amplitude\right|^2$$

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \delta\sigma_{ij}^{(1)} + \delta\sigma_{ij}^{(2)} + \delta\sigma_{ij}^{(3)} + \ldots$$

more diagrams and more integrations

amazing computational skills

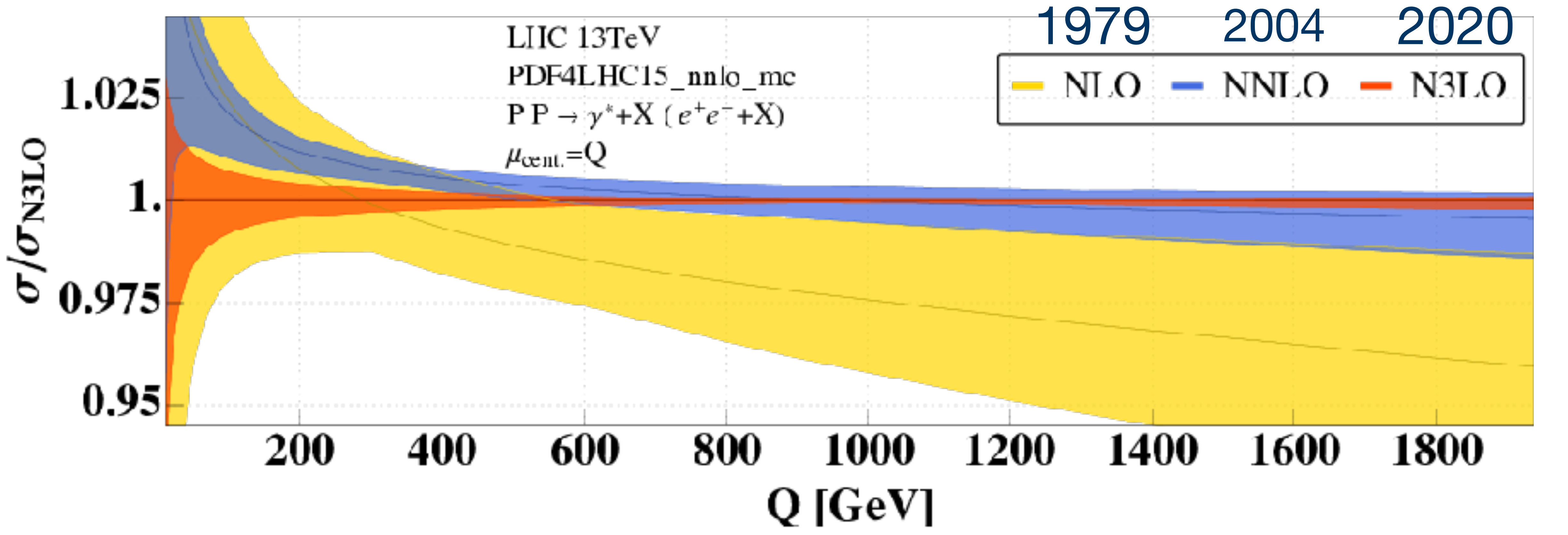


Powerful schemes which have lead to impressive breaktroughs.

The effect of higher orders

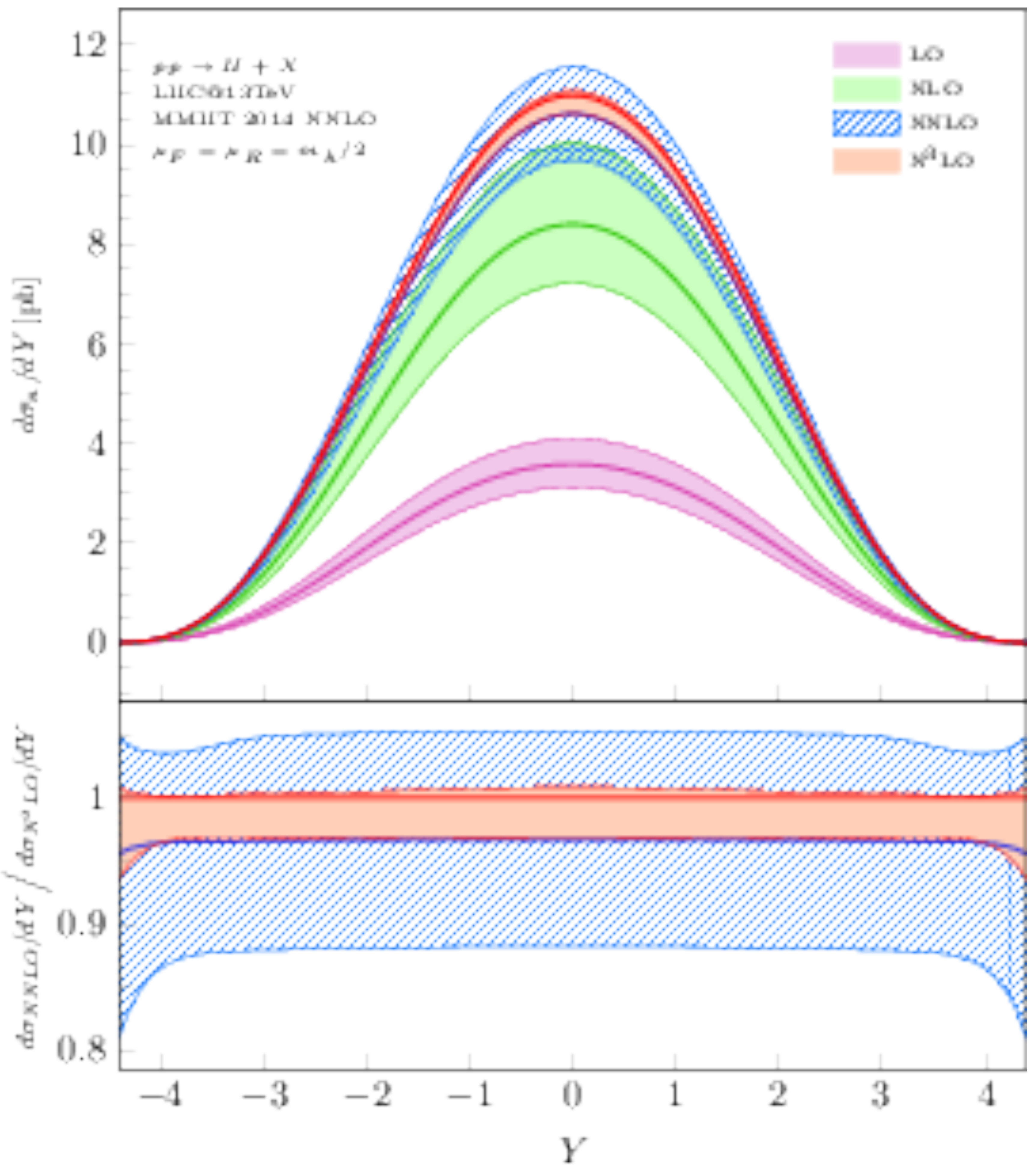
- Each order offers a leap in precision
- Probes the convergence pattern of the series.
- Reduced sensitivity to unphysical scales.

Drell-Yan production at N3LO



Duhr, Dulat, Mistlberger

Higgs rapidity distribution at N3LO



Dulat, Mistlberger, Pelloni

this is how the game is played now…

A wish list…

PROCESSS CLASS	EXAMPLES	STATUS	POSSIBLE Phenomenology motivated GOAL
$2 \to 1$	H,W,Z,WH,ZH	N3LO	N4LO
$2 \to 2$	jet inclusive, diboson, top-pair, photon-jet,…	NNLO	N3LO
$2 \to 3$	ttH,diphton+jet,WW/ZZ/ZW+jet, top-pair+jet,…	NLO	NNLO

Are we ready for such a leap?

- The demands for precision will be even higher…
- The techniques need to become scalable, an order of magnitude more complicated problems
- Strong desire for new solutions which can supersede very ingenious techniques developed over a span of decades.
- Time for reinvention…and thinking now about the problems of the next generation.

Typical calculations

$$\sigma_{ij}\left[\mathcal{O}\right] = \sum_{final-states} \int_{\mathcal{O}} dPhaseSpace\ \left|Amplitude\right|^2 \left.\vphantom{\int}\right\} \begin{array}{l} D = 4 - 2\epsilon \\ \textbf{UV and IR} \\ \textbf{divergences} \end{array}$$

$$Amplitude = \sum_j c_j\, Master_j = \sum_k d_k Polylogs_k(momenta) = Numbers$$

(mostly) Analytic

$$\sum_{final-states} \int_{\mathcal{O}} dPhaseSpace\,|A|^2 = \sum_{final-states} \left[\int_{\mathcal{O}} dPhaseSpace\,|A|^2 \Big|_{sing} + \int_{\mathcal{O}} dPhaseSpace\,|A|^2 \Big|_{reg} \right]$$

Analytic
Universal

Numerical
Monte-Carlo

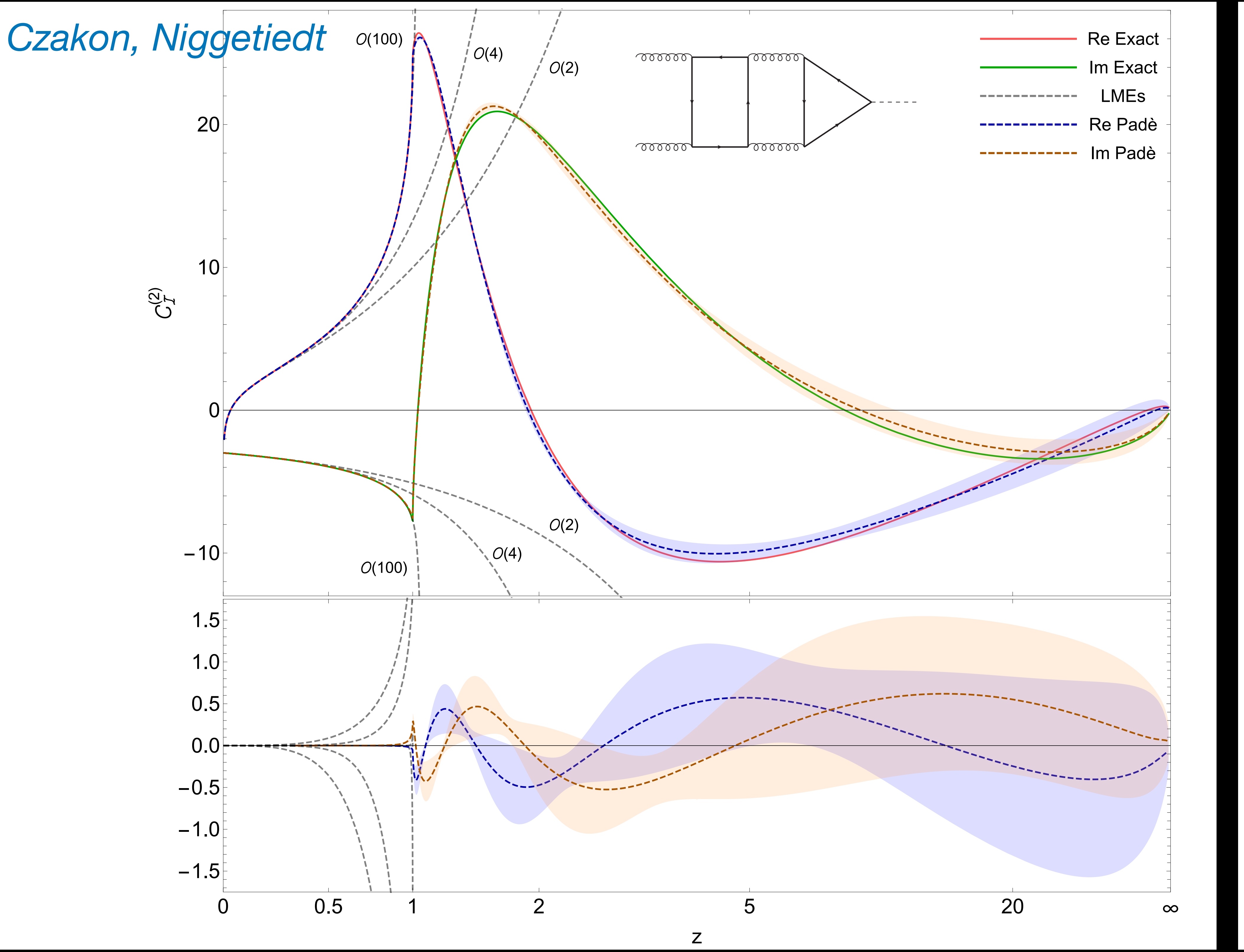
Analytic versus Numerical

- Progress for integration over phase-space of final-states in differential cross-sections has been made by developing numerical methods.
- Progress for loop amplitude integrations has been made mostly with analytic methods.
- The latter are algorithmic. But the computational cost scaling is a, perhaps unsurmountable, challenge for the future.
- We see, in recent years, efforts to replace major pieces of the “analytic”-algorithmic chain for amplitudes with numerics.

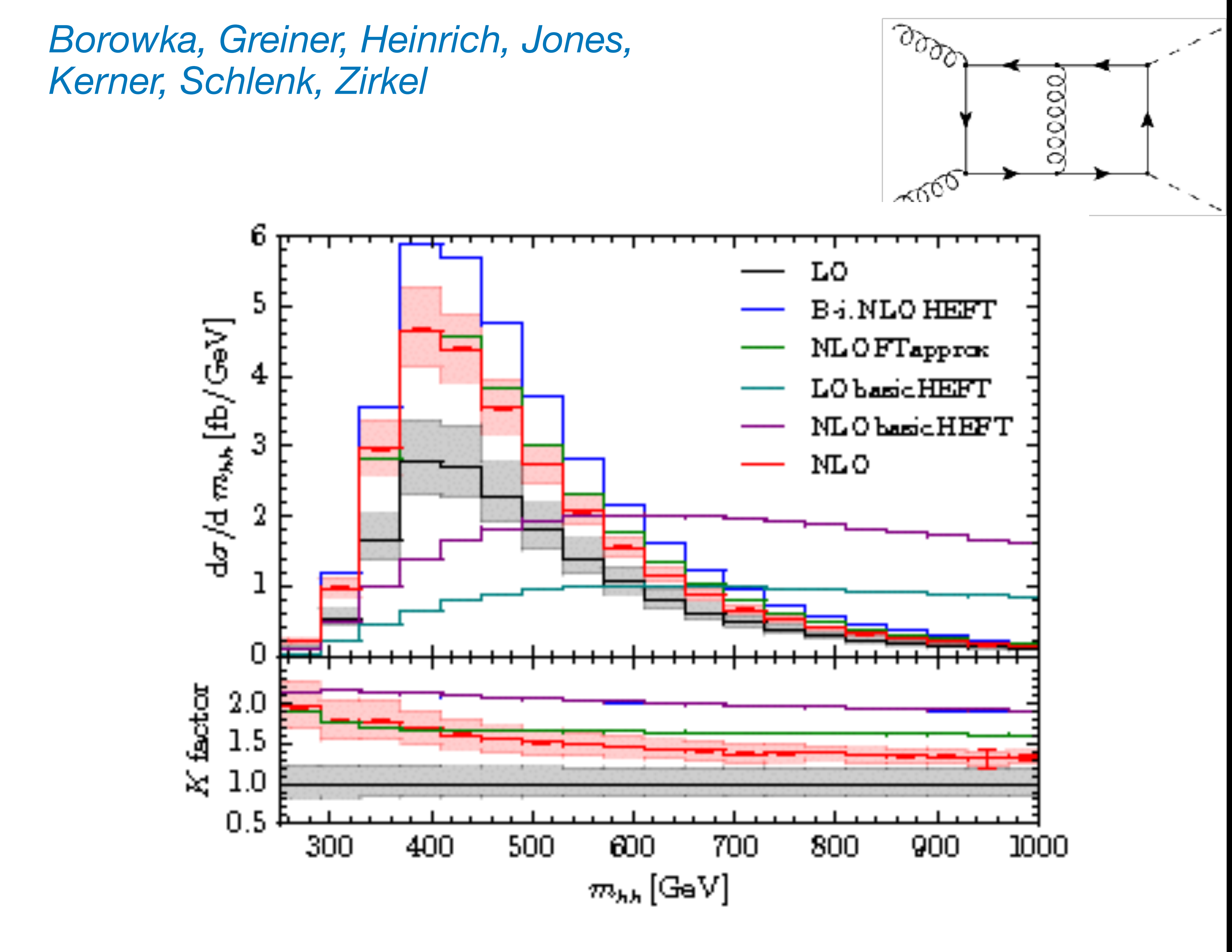
Two and three-loop amplitudes for Higgs production numerically

$$Amplitude = \sum_j c_j\, Master_j = \cancel{\sum_k d_k Polylogs_k(momenta)} = Numbers$$

Solving differential equations numerically



Monte-Carlo integration over infrared treated Feynman parameter integrals (sector decomposition)



Two-loop planar master integrals for $pp \to H + 2jets$

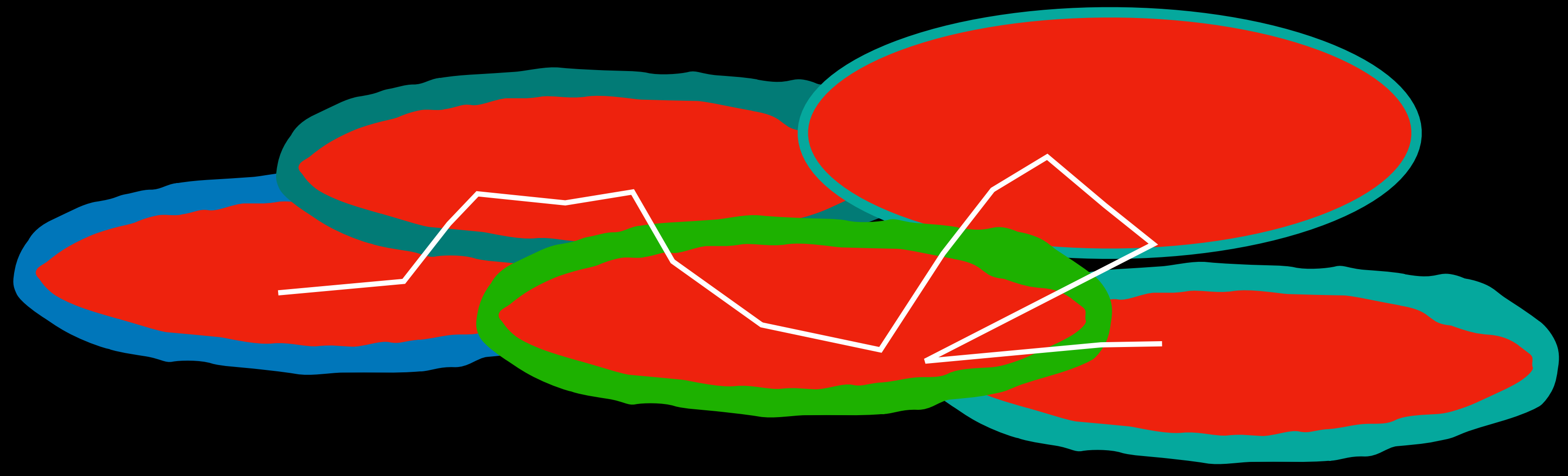
$$Amplitude = \sum_j c_j\, Master_j = \cancel{\sum_k d_k Polylogs_k(momenta)} = Numbers$$

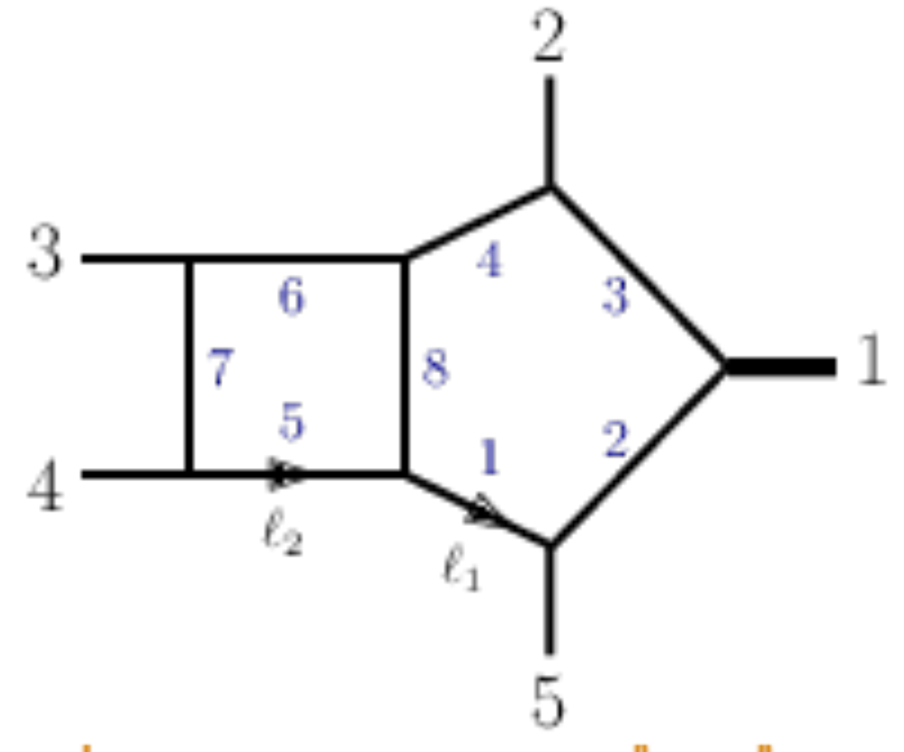
Direct solution of system of differential equations for master integrals along curves connecting any two points.

Points within small/overlapping patches in the space of kinematic

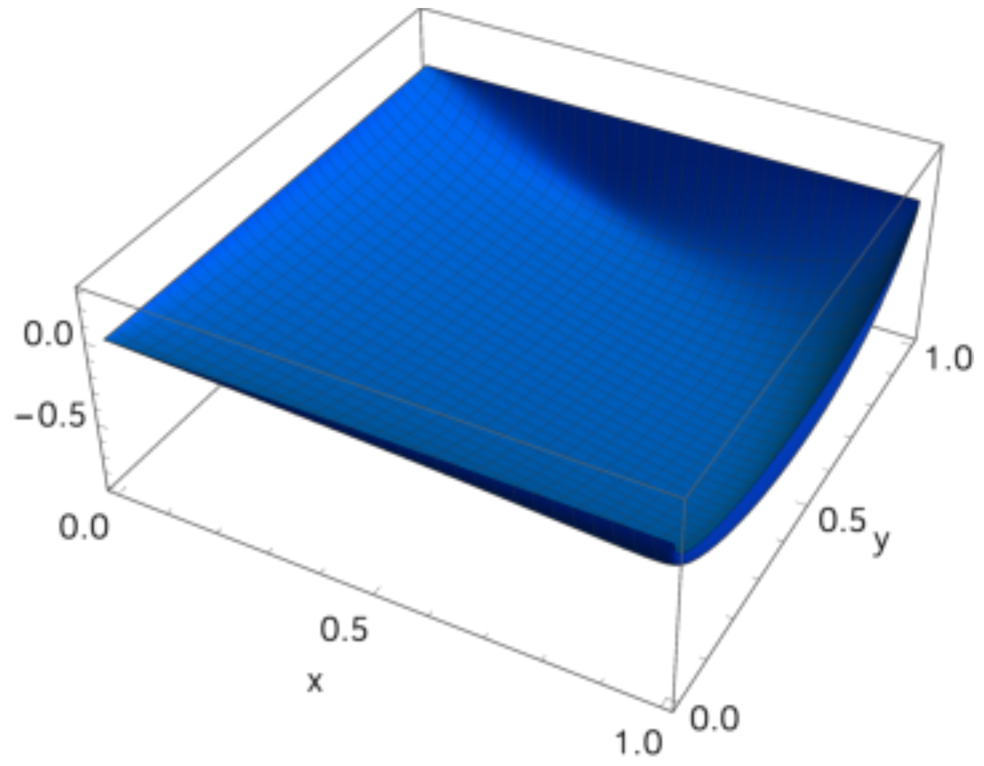
Solution as a controllable series expansion in a single variable!

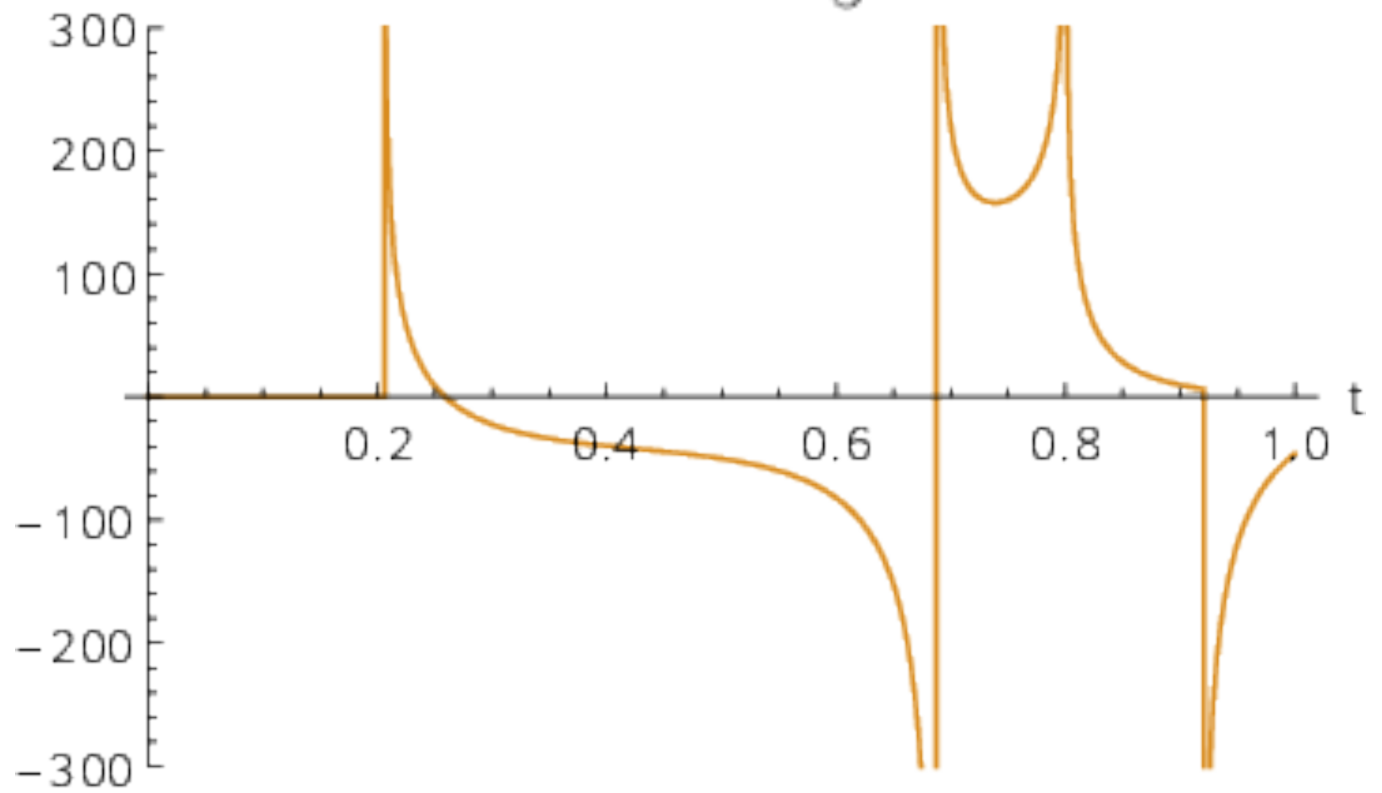
Moriello





Abreu, Ita, Moriello, Page, Tschernow, Zeng





6 kinematic invariants / many thresholds

Are we exploiting all we can?

Notwithstanding the progress and successes in perturbative QCD, and the impressive new mathematical techniques that keep emerging, our methods often seem to slice and dice physical cross-sections in unnatural ways…

Two properties of gauge theories,

1. **Factorization of infrared singularities**
2. **Unitarity**

can be further exploited.

Concepts that can bring further progress
INFRARED FACTORIZATION

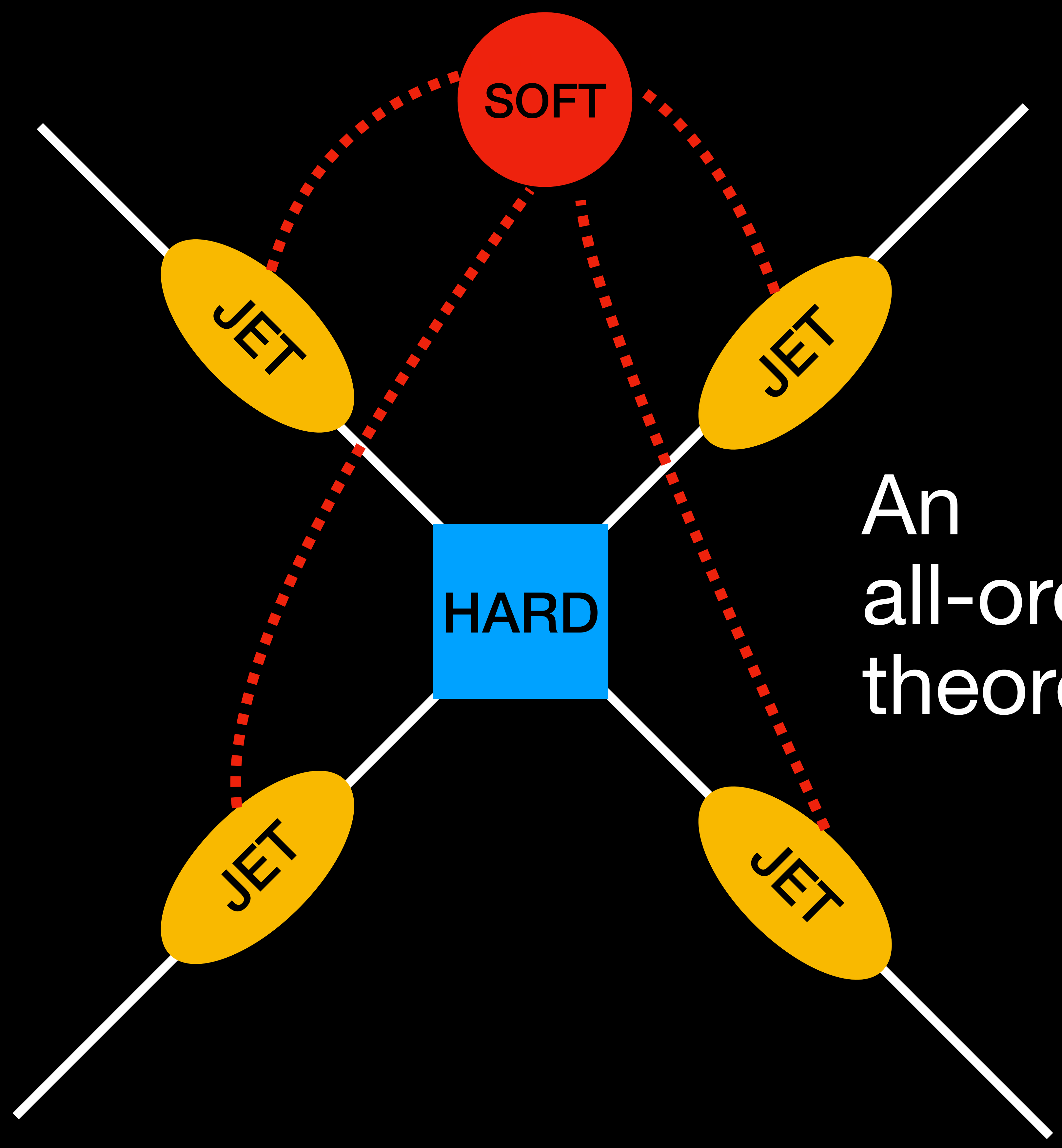
- UV Renormalized scattering amplitudes for well-separated final-states take a simple factorized form

$$Amplitude = \text{hard} \cdot \text{soft} \cdot \prod_i \text{jet}_i \,.$$

- "soft" and "jet" functions contain all divergences.

- These are universal functions. For any new process we should need to compute only the "hard" function.

- So far, we do not have a way to compute the "hard" function directly



An all-orders theorem

Ma; Erdogan, Sterman; Schwartz; Collins

How would we like to use factorization?

$$Amplitude = \cancel{\sum_j c_j Master_j = \sum_k d_k Polylogs_k(momenta)} = Numbers$$

$$= \int [dk]\, \mathscr{A}(k) = \int \mathscr{S} \int \mathscr{J} \int [dk] \frac{\mathscr{A}(k)}{\mathscr{S}(k)\mathscr{J}(k)}$$

integrand of hard function

Analytic

Universal

Numerical process-dependent integration

- From factorisation we could identify, remove and integrate separately the singular parts of amplitudes.
- This procedure is universal…can be applied to any process, irrespectively of the complexity of its final state, always requiring the same number of integrations.

Local infrared factorization

Factorization theorem $\quad Hard = \dfrac{Amplitude}{Soft \cdot Jets}$ $\quad$ valid after integrations are performed

Is it also valid locally, for the integrands? In other words, does a local integrand representation for the hard function exist which is free of singularities?

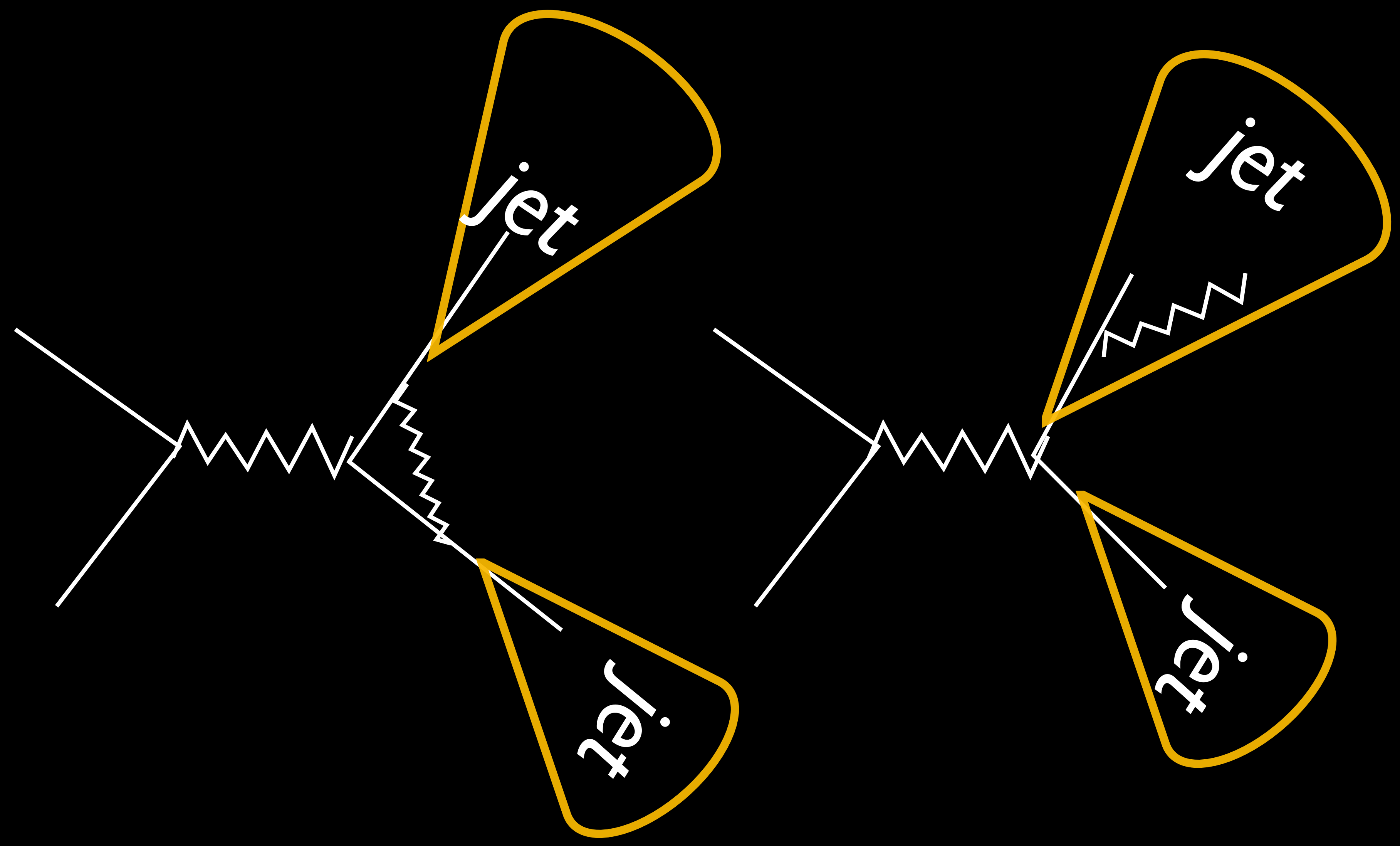
$$\mathscr{H}(k) \overset{?}{=} \frac{\mathscr{A}(k)}{\mathscr{S}(k)\mathscr{J}(k)} \xrightarrow{k \in \text{ IR or UV region}} \text{Finite}$$

Main arguments in an all-orders proof [Ma 2019] are local (power-counting, local subtractions and gauge symmetry)

A fully local formulation of the factorization theorem,
for the purposes of using it as a technique to compute amplitudes, is still missing.

Concepts that can bring further progress
UNITARITY

- Probabilities are finite for observables that can be computed perturbatively (infrared safe observables).
- Divergences cancel in sums of "*partonic*" cross-sections.
- So far, we do not have a way to make these cancelations manifest, before performing very tough integrations.



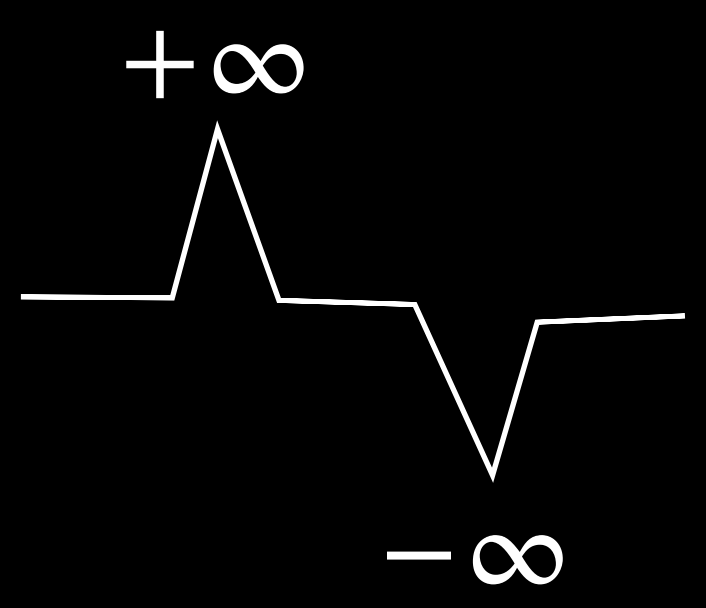
$$\sigma\left[\mathcal{O}\right] = \sum_{final-state} \sigma_{final-state}\left[\mathcal{O}\right]$$

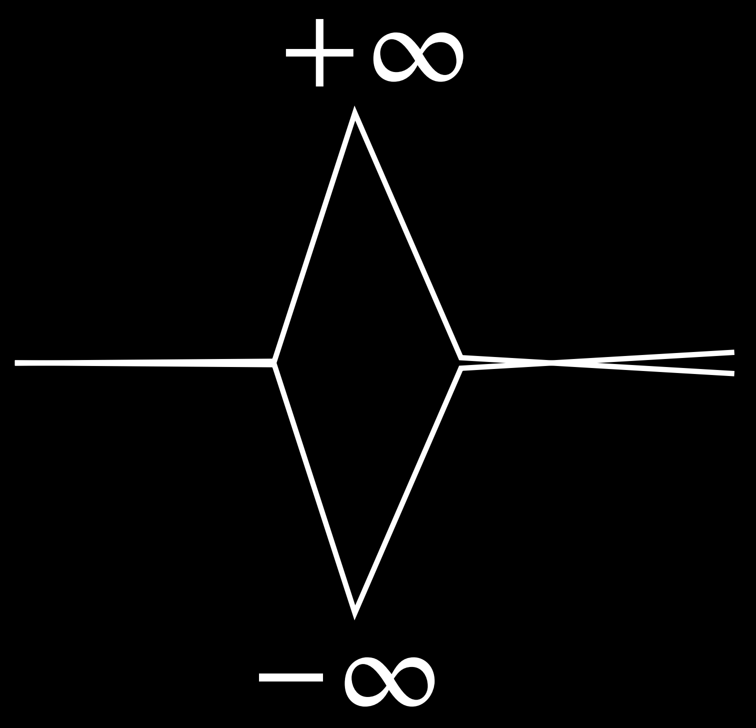
$$= \sum_{final-state}\left(\cancel{\sigma_{final-state}\left[\mathcal{O}\right]\Big|_{div.}} + \sigma_{final-state}\left[\mathcal{O}\right]\Big|_{reg.} \right)$$

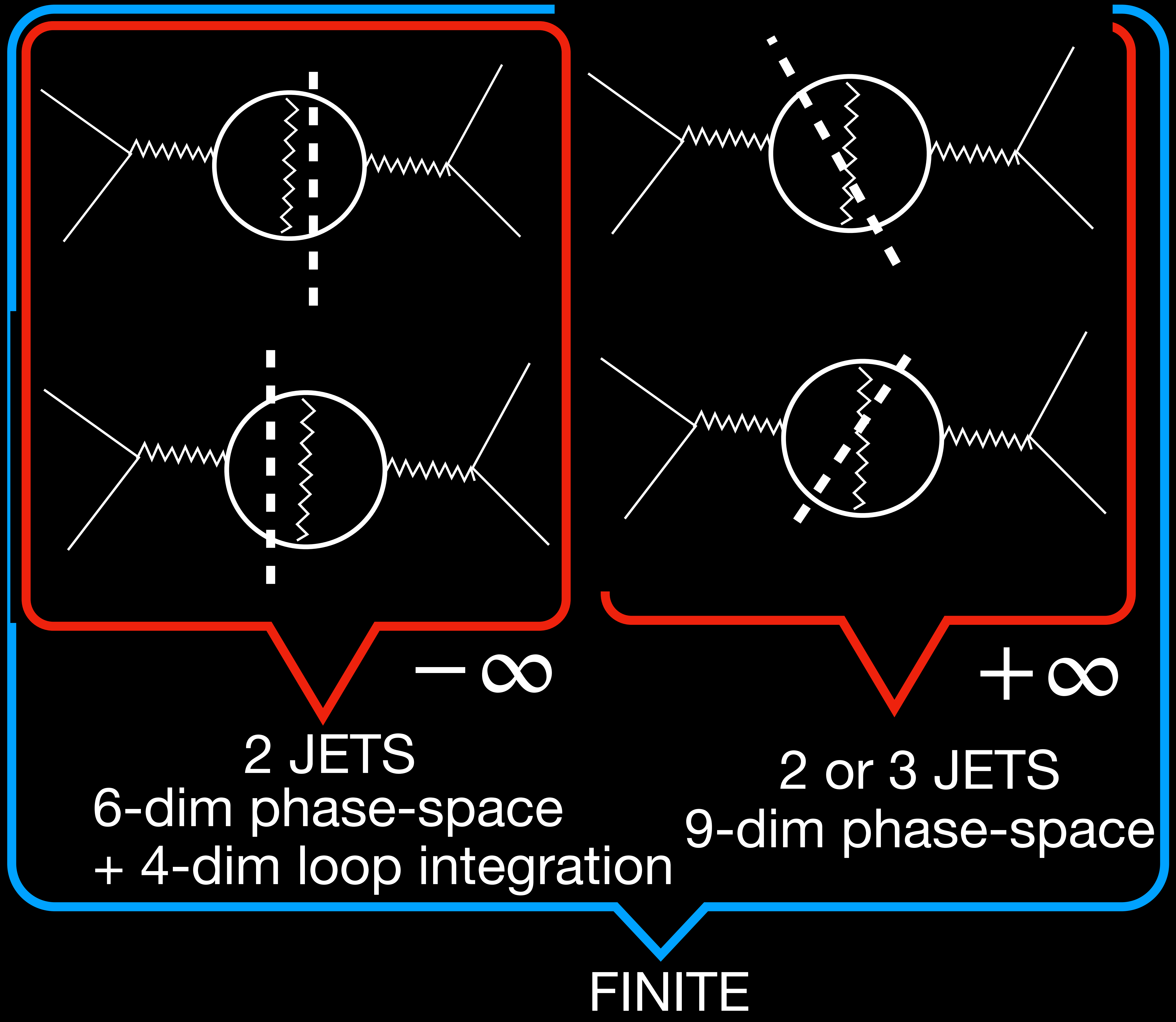
$$= \sum_{final-state} \sigma_{final-state}\left[\mathcal{O}\right]\Big|_{reg.}$$

How would we like to use UNITARITY?

- Recall the optical theorem
- Cross-sections for diverse parton multiplicities can be obtained from cuts of a forward scattering amplitude.
- We would like to have a "local" formulation of the optical theorem.
- Putting together the integration domains of the diverse cuts in a clever way, aligning all of their singularities and cancelling them before integration.







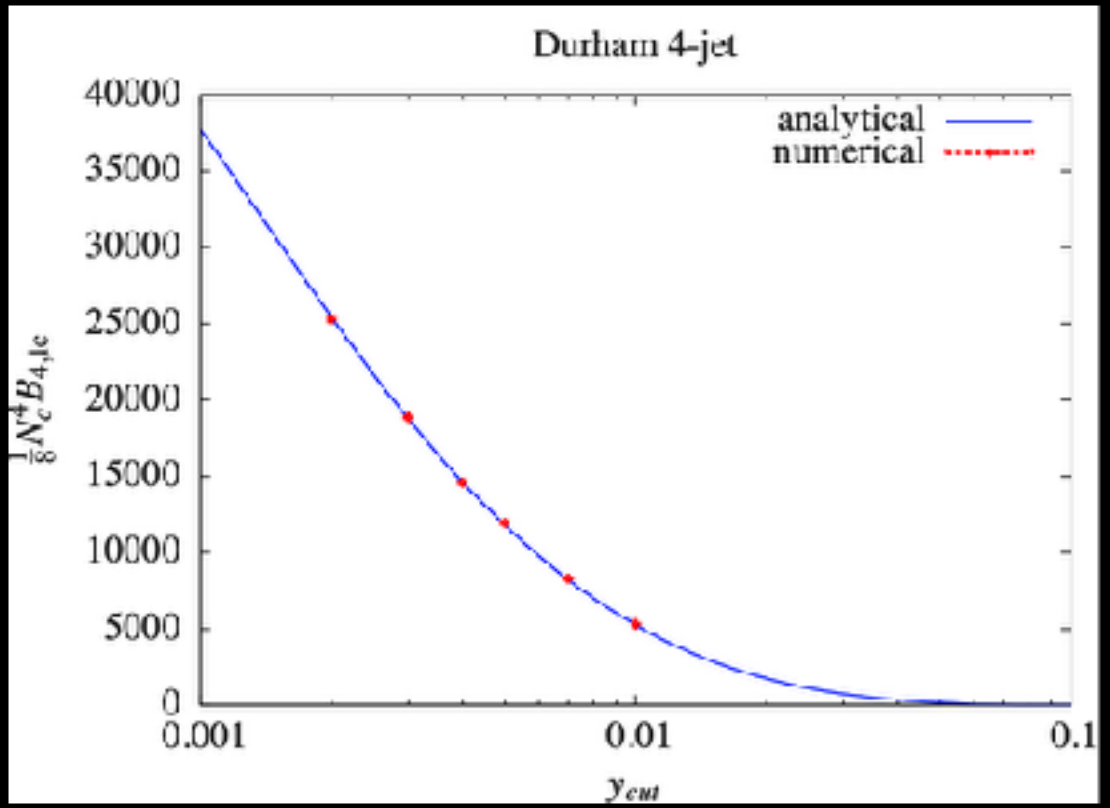
Achieving factorization and unitarity locally

- Let’s dream that we can make both infrared factorization and unitarity completely local.
- Then, we will be able to define cross-section integrands for infrared-safe observables which are integrable in exactly D=4 dimensions.
- No need for regularization (except, maybe, for computing ONCE at each loop order universal finite remnants of the ultraviolet and the infrared regions).

NLO cross-sections computed in exactly D=4

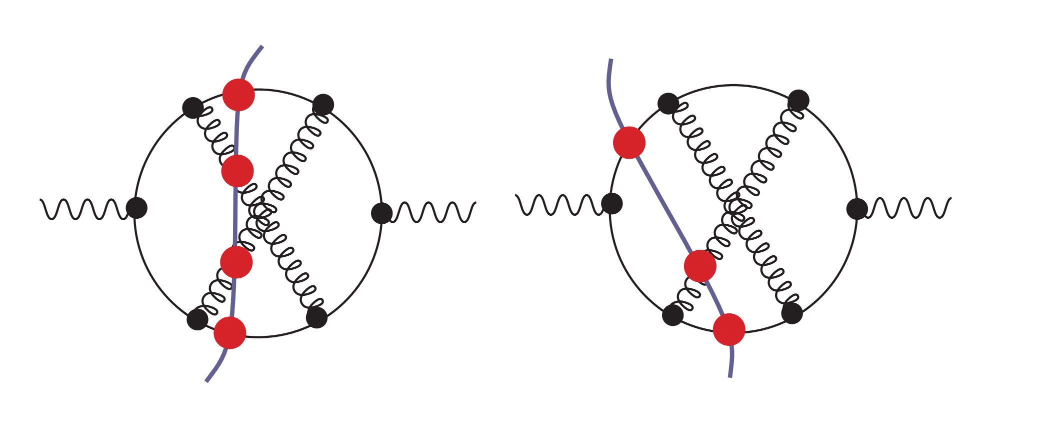
FACTORIZATION+UNITARITY

- Local UV, soft and collinear approximations for one-loop amplitudes to remove their singularities. *Nagy, Soper 2003*
- Local factorization of infrared approximations *Assadsolimani, Becker, Weinzierl 2010*
- Numerical integration methods for the finite remainders and finite intervals at any loop order *Becker, Weinzierl 2012*
- Unitarity to combine singular contributions of loop and phase-space integrations. *Seth, Weinzierl 2006*



Becker, Gotz, Reuschle, Schwan, Weinzierl (2012)

UNITARITY (alone)



D. Soper, 1998

$e^+ + e^- \to 3\,jets$ at NLO

n	numerical
1.5	$4.127 \pm 0.008 \pm 0.025$
2.0	$1.565 \pm 0.002 \pm 0.007$
2.5	$(6.439 \pm 0.010 \pm 0.022) \times 10^{-1}$
3.0	$(2.822 \pm 0.005 \pm 0.009) \times 10^{-1}$
3.5	$(1.296 \pm 0.002 \pm 0.004) \times 10^{-1}$
4.0	$(6.159 \pm 0.011 \pm 0.016) \times 10^{-2}$
4.5	$(3.009 \pm 0.006 \pm 0.007) \times 10^{-2}$
5.0	$(1.501 \pm 0.003 \pm 0.003) \times 10^{-2}$

moments of thrust distribution

A Don Quijotian approach?

…against "giants" (state-of-the-art methods) at NLO, NNLO and N3LO…. (so far)

$$Amplitude = \sum_j c_j\, Master_j = \sum_k d_k Polylogs_k(momenta) = Numbers$$

- Unitarity and Integrand Reduction Methods at one [NLO revolution] and now two-loops [first five point QCD amplitudes]
- Automatization of Integration by Parts reductions [Laporta and many improvements thereafter, e.g. finite fields]

- Differential equations / Canonical basis
- Automation of asymptotic expansions
- Mellin-Barnes / Sector-decomposition

- Symbol and Coproduct
- Systematization of Ellptic polylogaritmsn

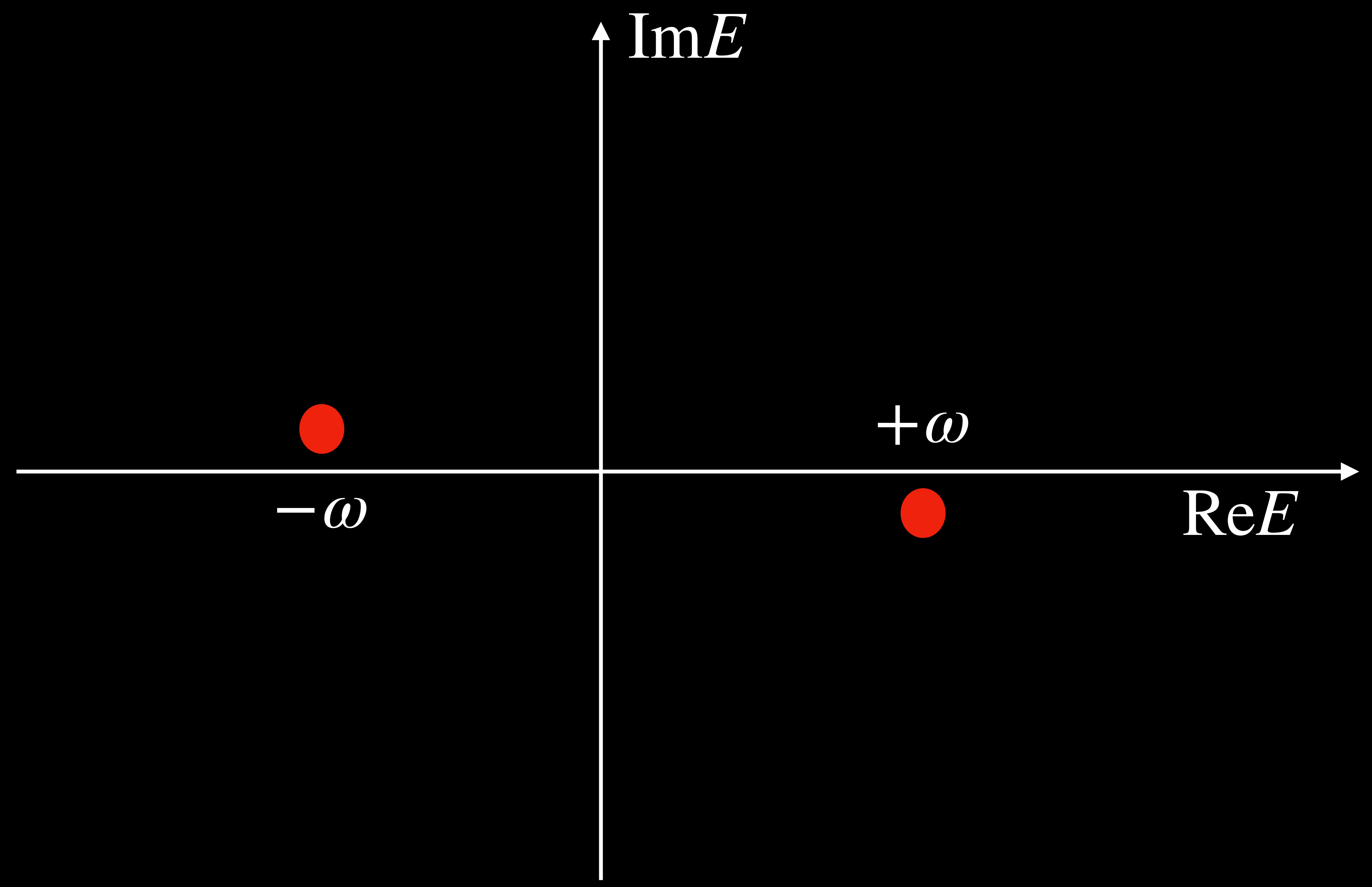
- But, we now have illuminating proofs of amplitude factorization,
- deeper understanding of integrable singularity structure and better numerical integration formalisms
- while established methods somewhat struggling for next generation problems….

Origin of infrared divergences and local factorization

“Infinities” from classical behaviour

$$\int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{E^2 - \omega + i\delta} = \int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{\omega} \left(\frac{1}{E - \omega + i\delta} - \frac{1}{E + \omega - i\delta} \right)$$

- The poles can lie inside the domain of integration.

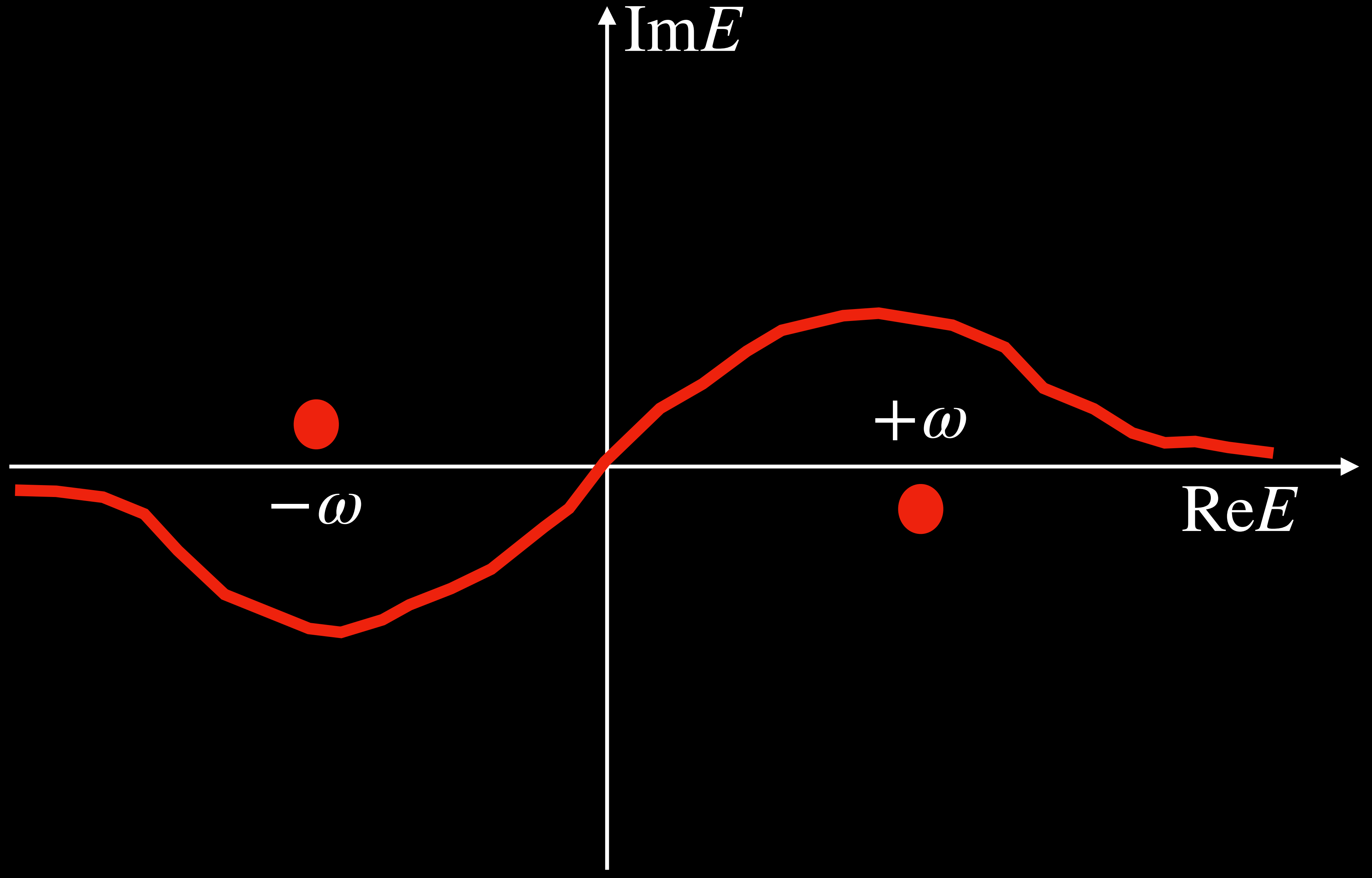


$\omega \to \omega - i\delta$ with $\delta \to 0$

"Infinities" from classical behaviour

$$\int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{E^2 - \omega + i\delta} = \int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{\omega} \left(\frac{1}{E - \omega + i\delta} - \frac{1}{E + \omega - i\delta} \right)$$

- The poles can lie inside the domain of integration.
- If we can deform the path of integration away from the poles, then they lead to no singularities

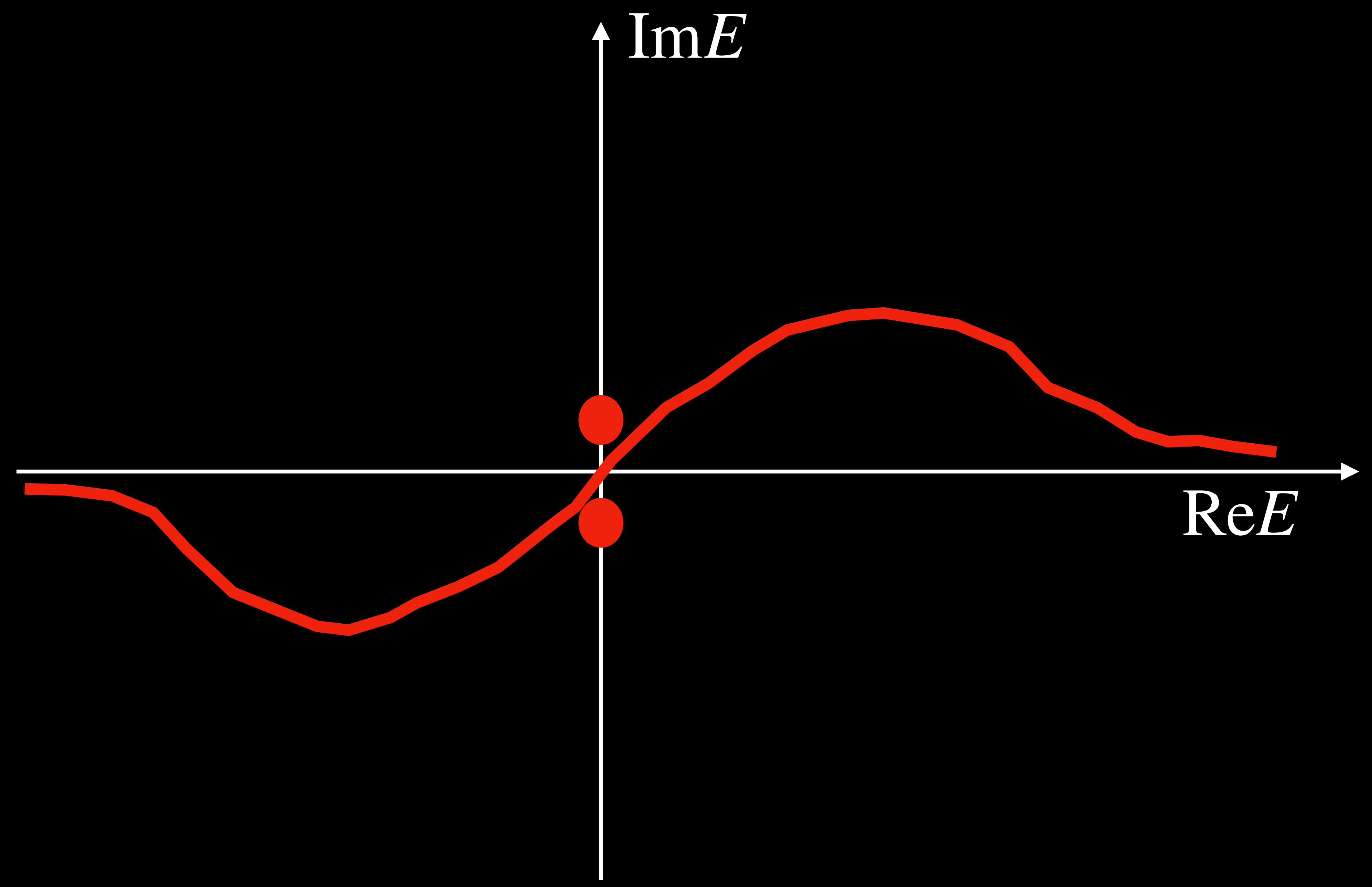


$\omega \to \omega - i\delta$ with $\delta \to 0$

Soft massless particles

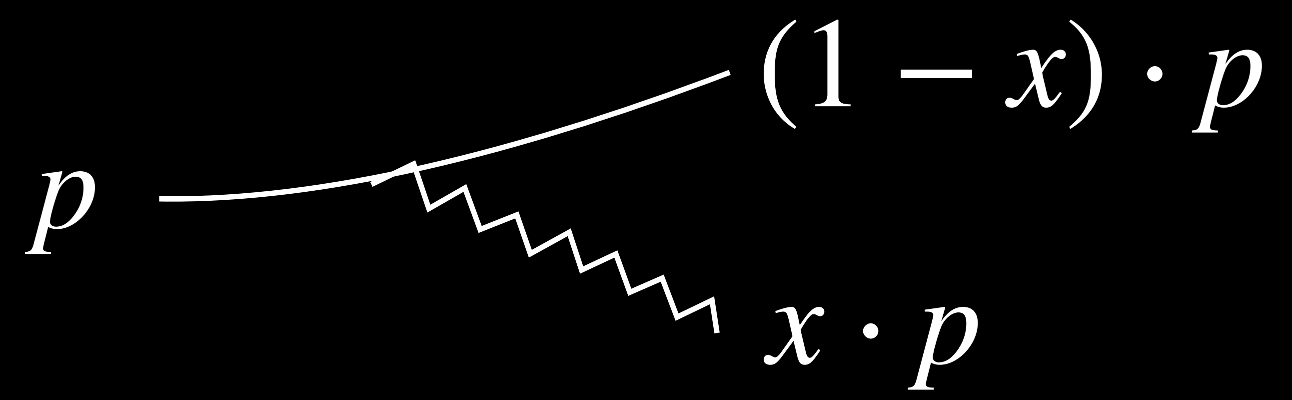
$$\int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{(E+i\delta)\,(E-i\delta)}$$

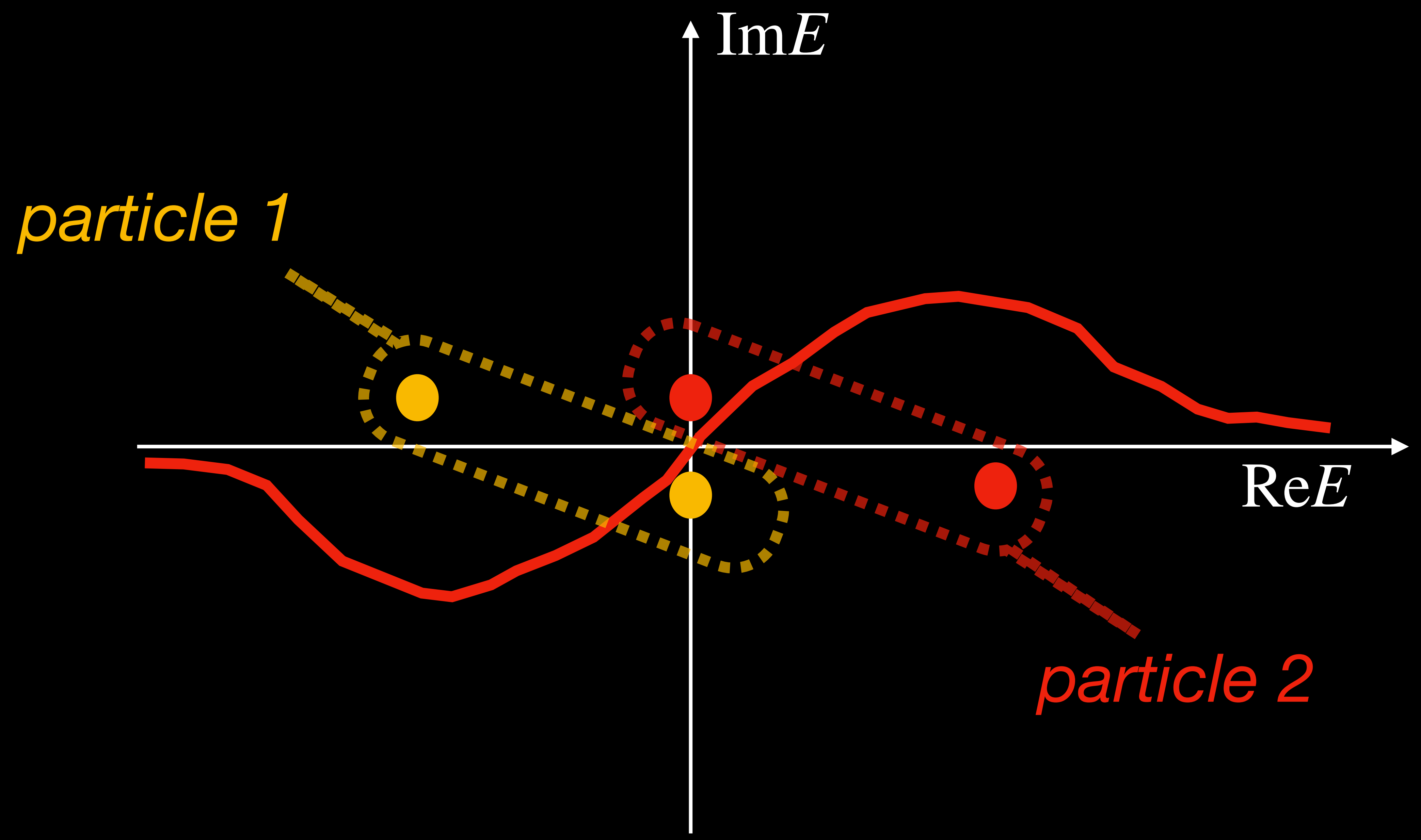
- Poles due to soft massless particles.
- These singularities pinch the integration path from both sides.
- Condition for a TRUE INFINITY



Collinear massless particles

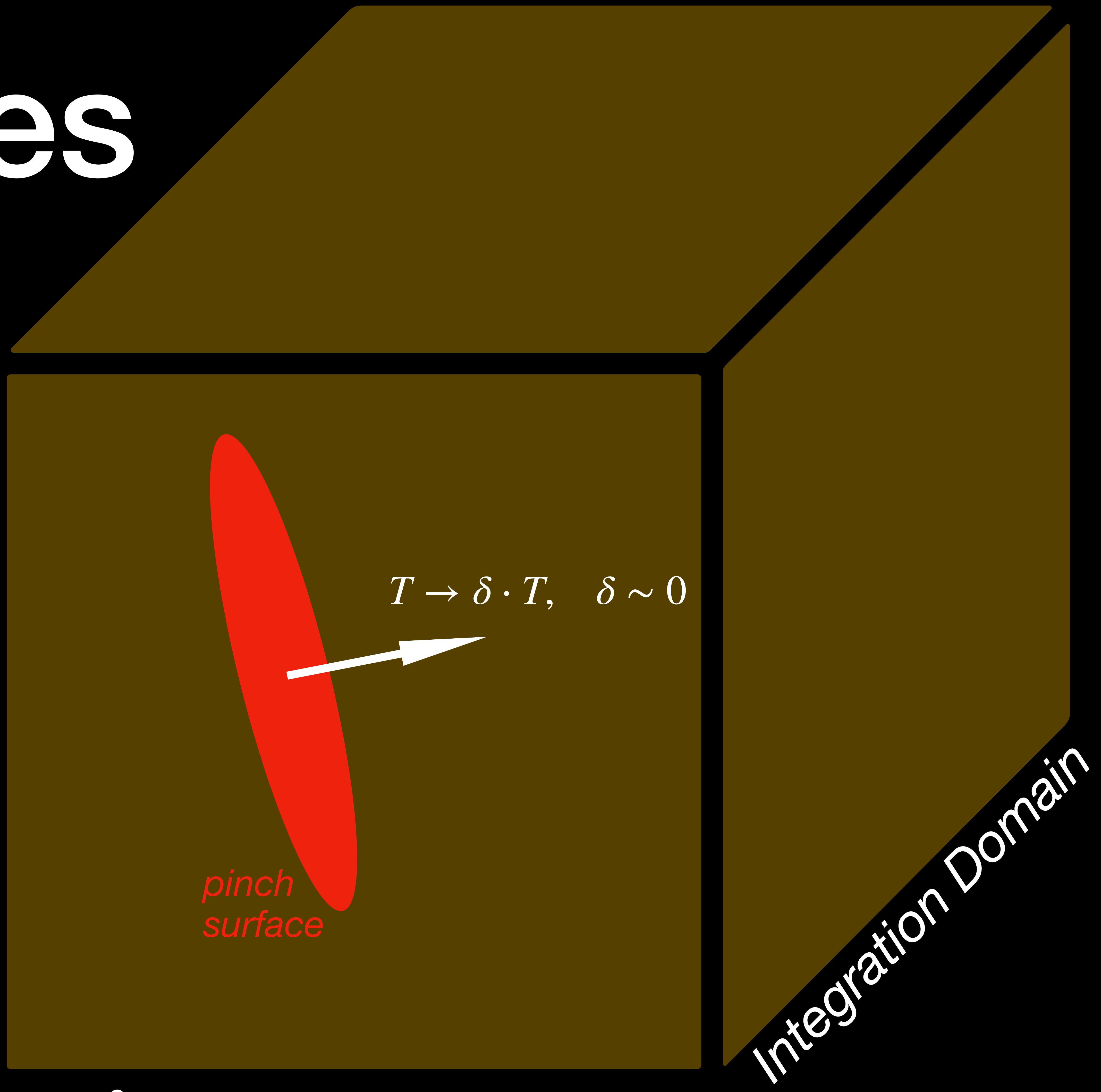
- A second source of infinities due to massless collinear particles.
- A singularity of one particle in the lower half-plane lines up with the singularity of a collinear particle in the higher half-pane.
- The singularities pinch the integration path from both sides.
- We cannot deform the path, a condition for a TRUE INFINITY!





Pinch singularities

- To know if a singularity develops, we need to study the behaviour of the integral in the vicinity of the pinch surface.
- We can calculate a degree of divergence.
- Scale variables which are perpendicular to the pinched surface with a small parameter and calculate the scaling of the integrand as the parameter is driven to zero.



Soft $k^\mu \sim \delta Q, \quad d^4k \sim \delta^4$

Collinear $k = xp + \alpha\eta + \beta p_\perp, \quad x \sim \delta^0, \alpha \sim \delta, \beta \sim \delta^{\frac{1}{2}} \quad d^4k \sim \delta^2$

Integrand: $d^4k\mathcal{I}(k) \sim \delta^n$

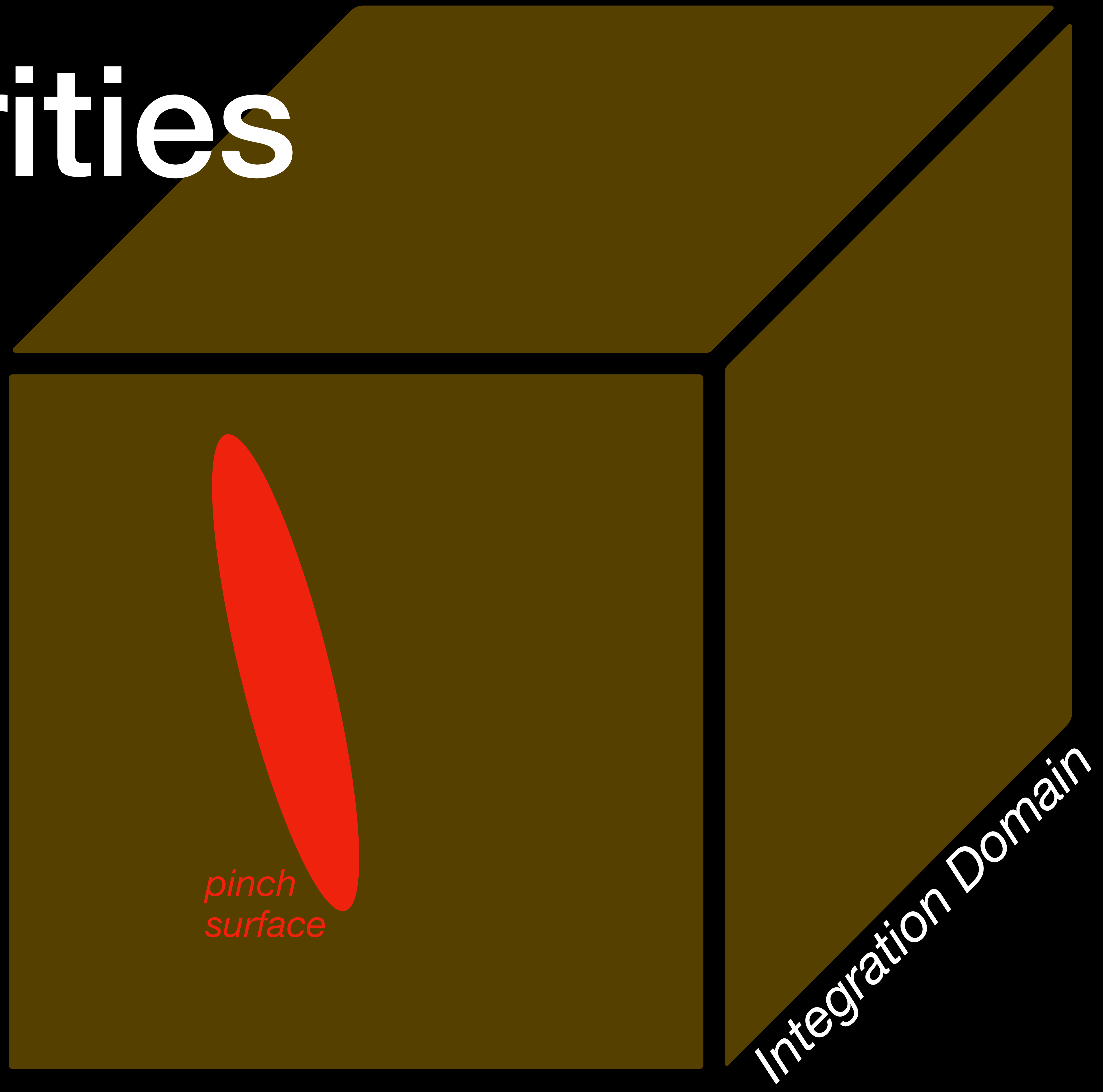
Divergent: $n \leq 0$

Convergent: $n > 0$

Removing singularities

- Once a pinch surface which yields a singularity is identified, then we can remove the singularity with a subtraction.

$$A = \int [dk]\mathcal{F}(k)$$



Removing singularities

- Once a pinch surface which yields a singularity is identified, then we can remove the singularity with a subtraction.

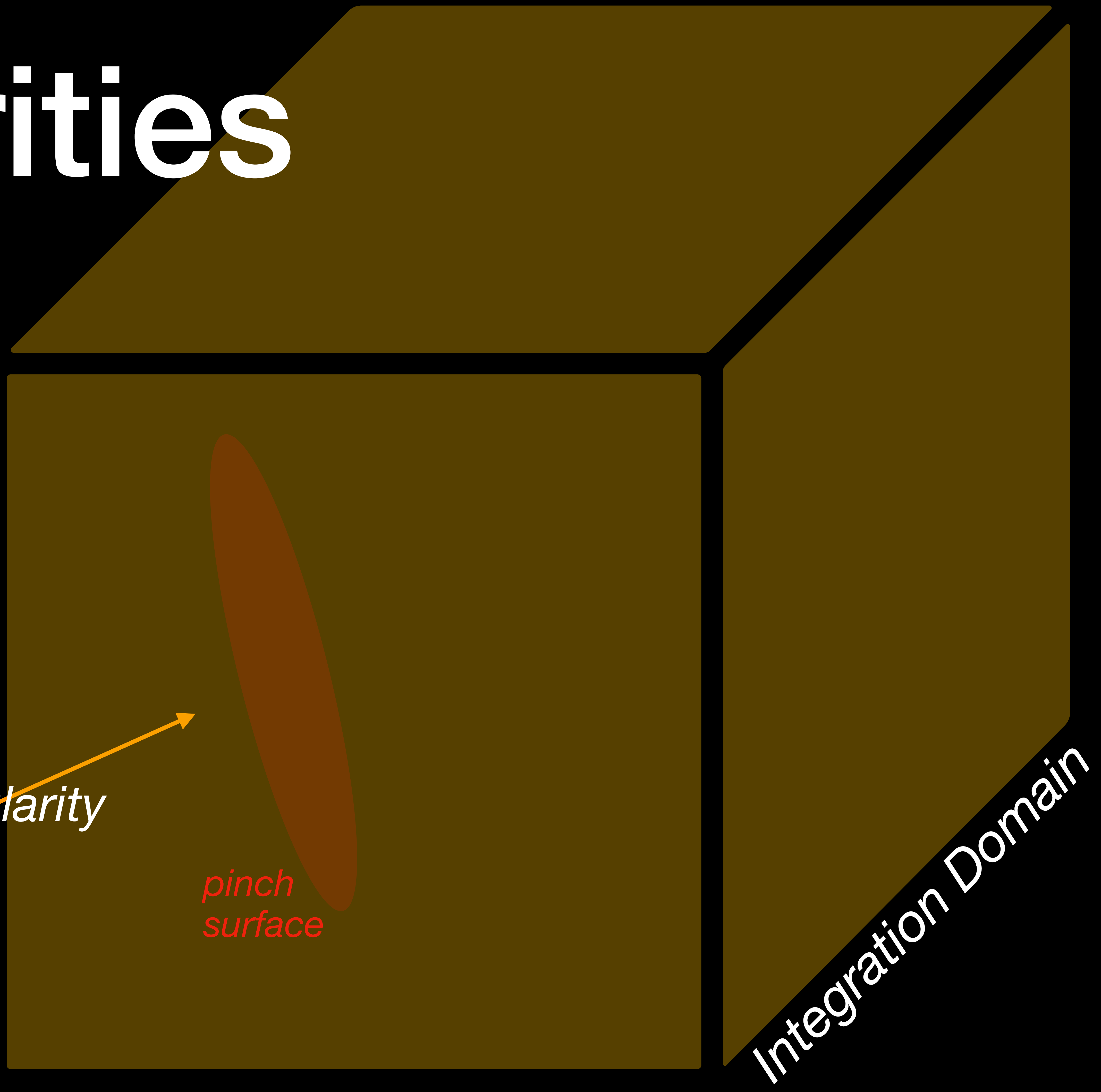
$$A = \int [dk]\ \mathcal{F}(k)$$

$$\to \int [dk]\ \left[\mathcal{F}(k) - t\mathcal{F}(k)\right]$$

no singularity

integrand

approximation of integrand on singular surface



Removing singularities

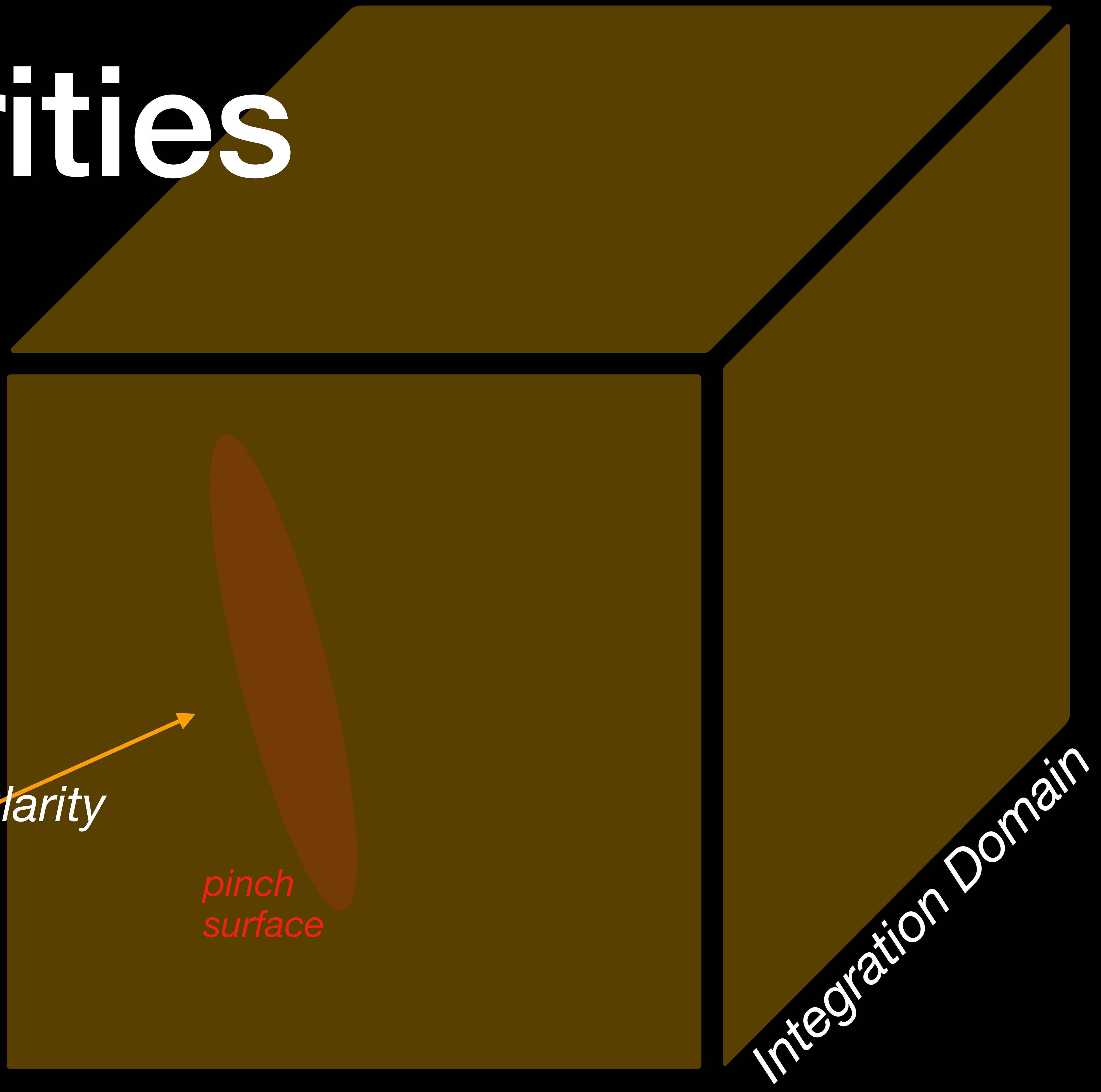
- Once a pinch surface which yields a singularity is identified, then we can remove the singularity with a subtraction.

$$A = \int [dk]\ \mathscr{F}(k)$$

$$= \int [dk]\ \left[\mathscr{F}(k) - t\mathscr{F}(k)\right] \longrightarrow \text{...hard}$$

$$+ \int [dk]\ t\mathscr{F}(k) \longrightarrow \text{...soft or jet}$$

no singularity

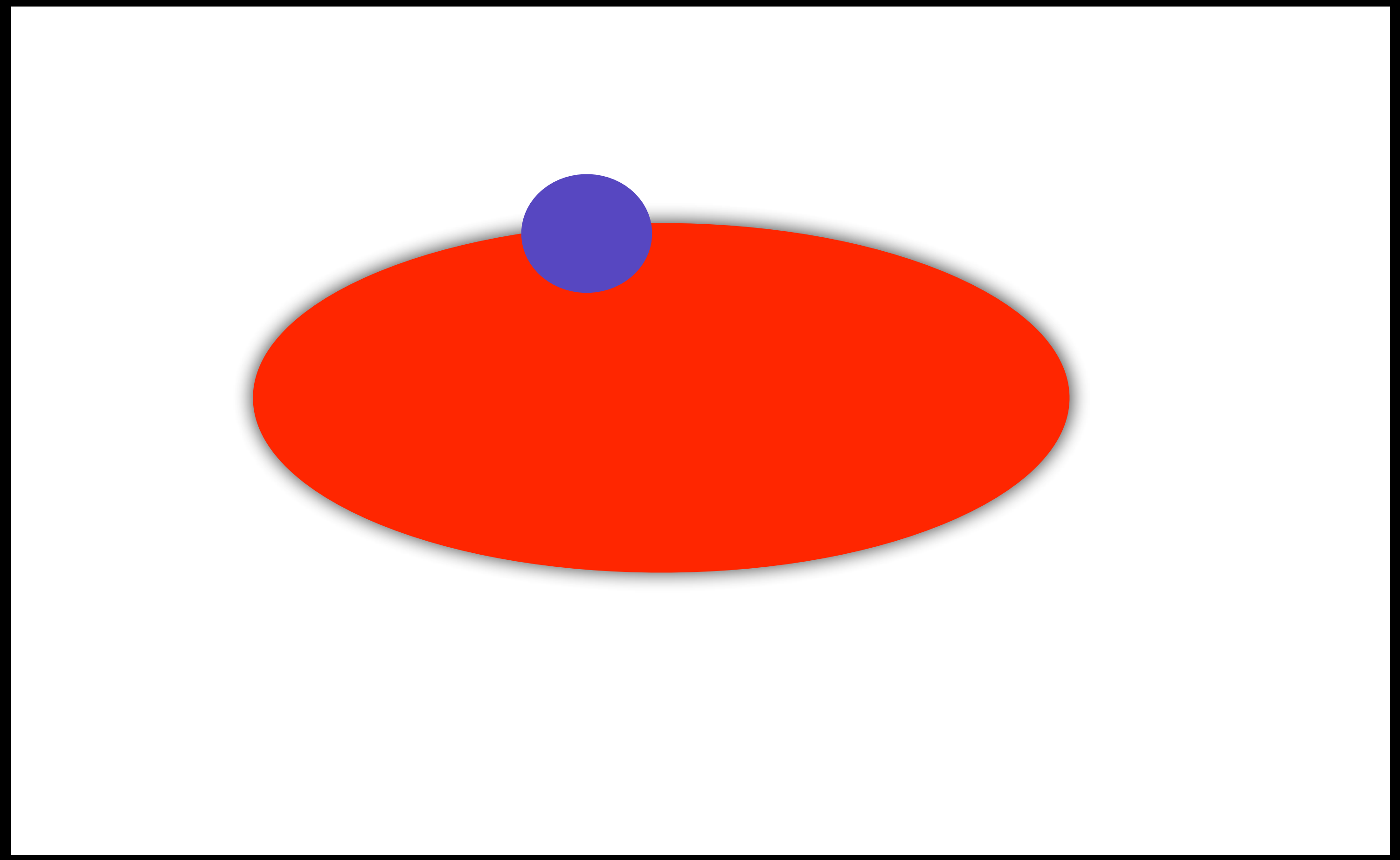


Nested subtractions

Ma; Erdogan, Sterman; Collins; Collins, Soper, Sterman

- Singular regions are interconnected. How can we create systematically an approximation of the loop integrals in all singular regions?

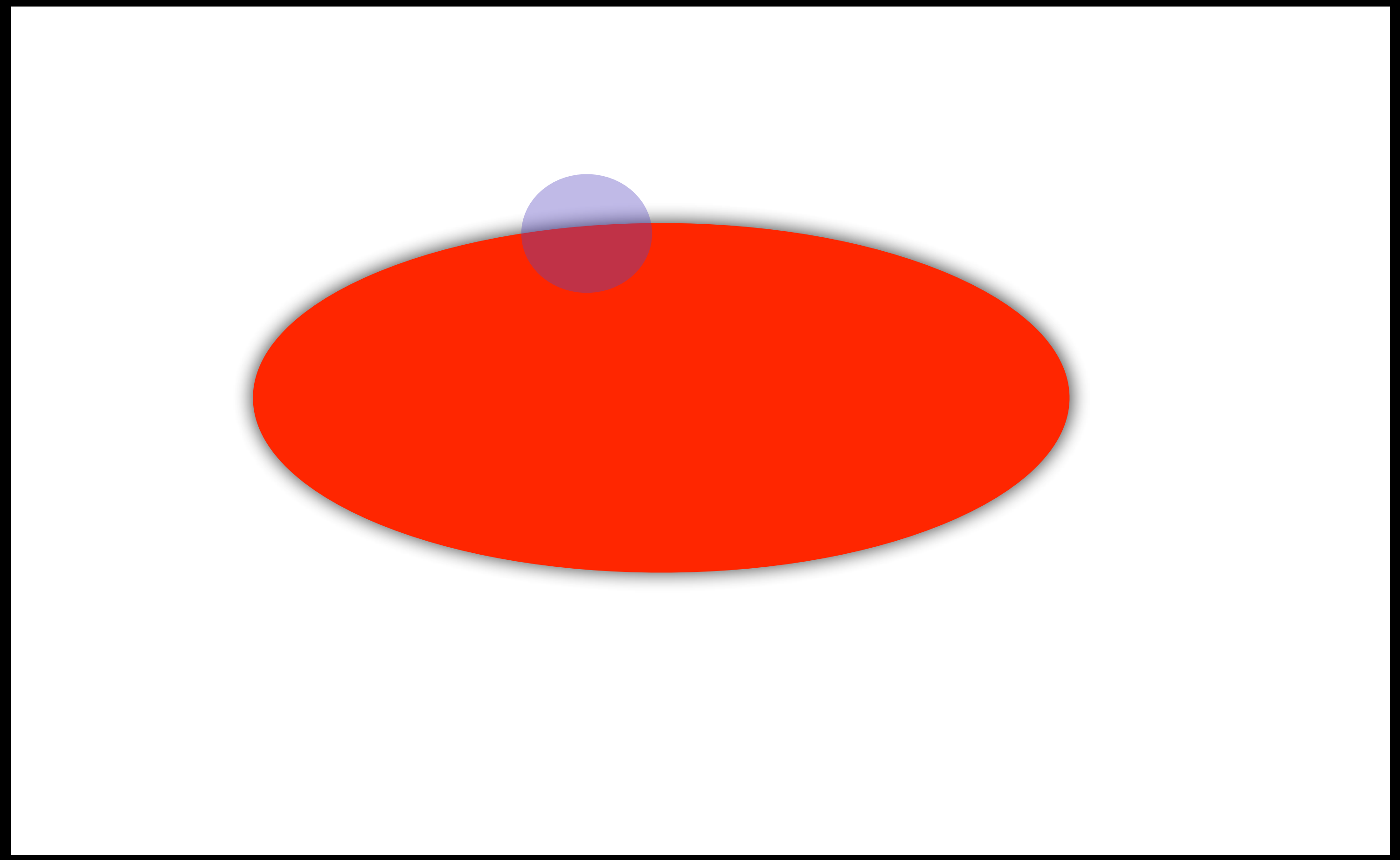
- Order the singular regions by their “volume”



Nested subtractions

Ma; Erdogan, Sterman; Collins; Collins, Soper, Sterman

- Singular regions are interconnected. How can we create systematically an approximation of the loop integrals in all singular regions?
- Order the singular regions by their “volume”
- Subtract an approximation of the integrand in the smallest volume
-



Nested subtractions

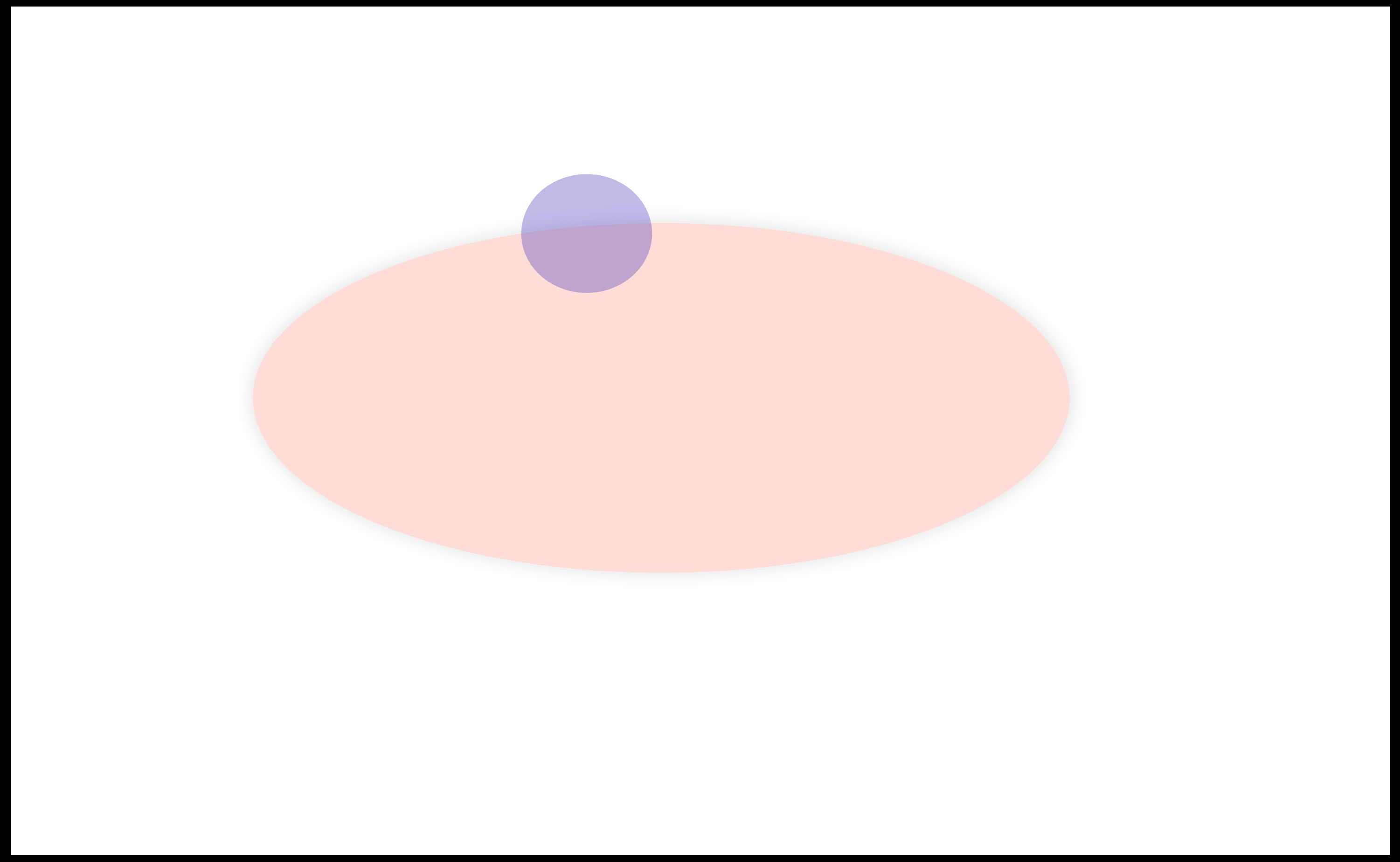
Ma; Erdogan, Sterman; Collins; Collins, Soper, Sterman

- Singular regions are interconnected. How can we create systematically an approximation of the loop integrals in all singular regions?

- Order the singular regions by their “volume”

- Subtract an approximation of the integrand in the smallest volume

- Then, proceed to the next volume and repeat until there are no more singularities to remove.



Nested subtractions

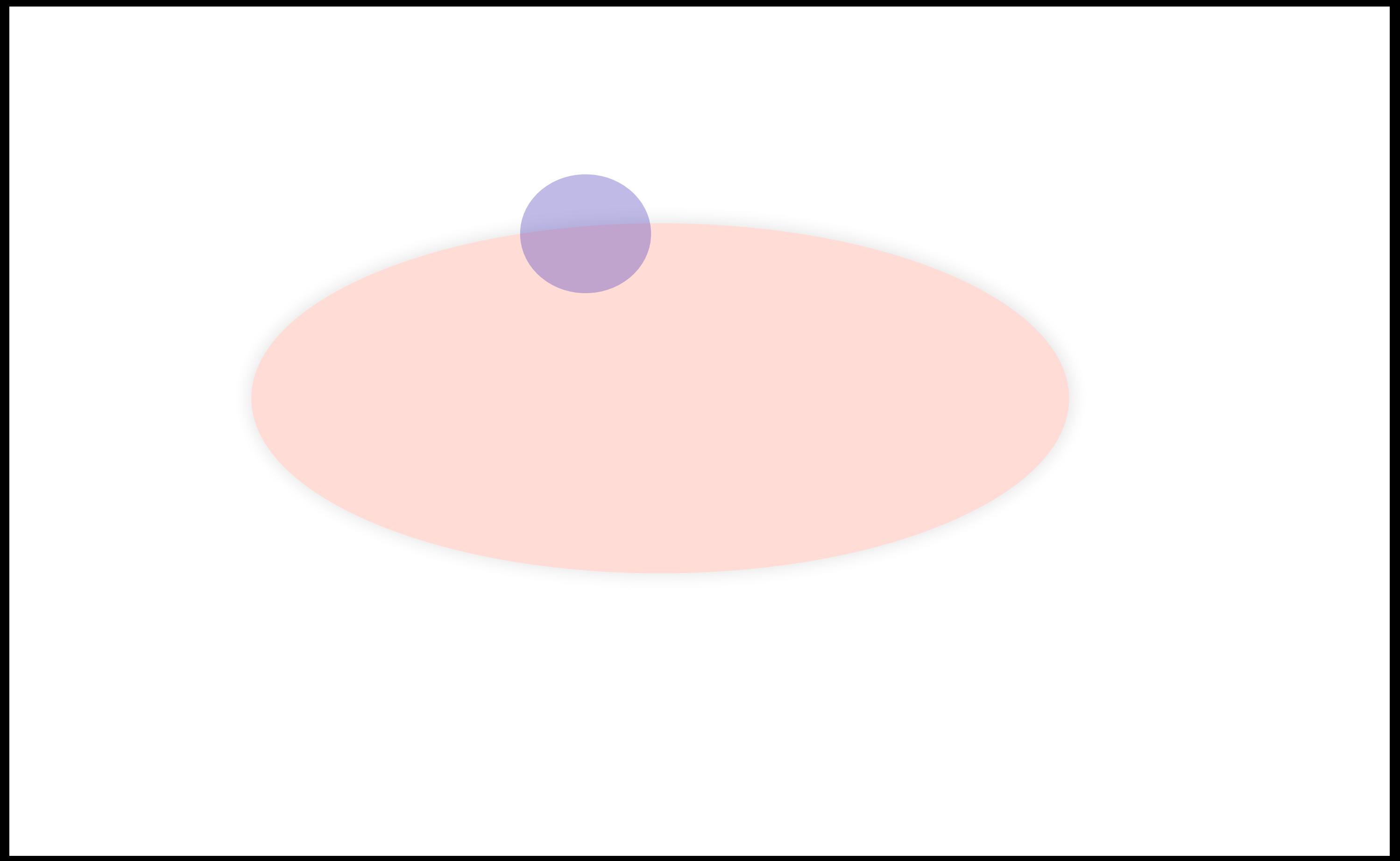
Ma; Erdogan, Sterman; Collins;
Collins, Soper, Sterman

- The procedure of nested subtractions has a solution for the finite remainder at any loop order as a Forest formula (similarly to BPHZ of UV renormalzation)

- It is valid term by term in an amplitude or a Feynman diagram.

- This forest formula structure combined with gauge symmetry, gives rise to the factorization of gauge theory amplitudes in terms of Jets, Soft and Hard fucntions.

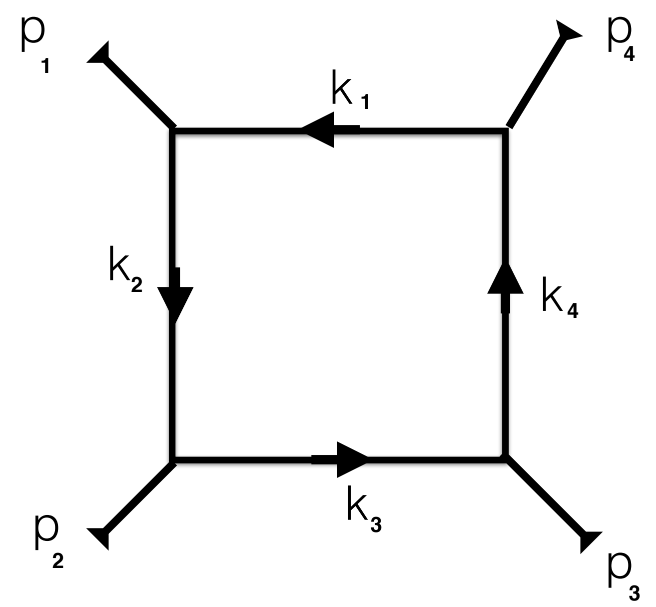
$$R^{(n)}\,\gamma^{(n)} \;=\; \gamma^{(n)} + \sum_{N\in\mathcal{N}[\gamma^{(n)}]}\prod_{\rho\in N}\left(-t_\rho\right)\gamma^{(n)}\,,$$



An one-loop example

- One-loop massless box has both soft and collinear singularities
- A soft singularity occurs in a single point in momentum space (smallest volume). Needs to be subtracted first.
- A collinear singularity occurs in an one-dimensional space (larger volume). Needs to be subtracted after the soft.

$$\mathrm{Box} \equiv \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{1}{A_1 A_2 A_3 A_4},$$

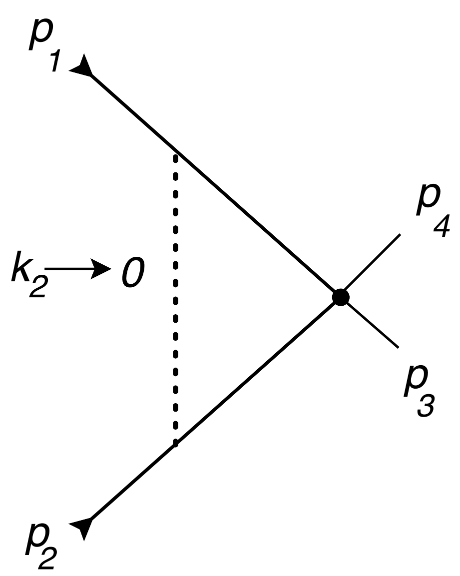


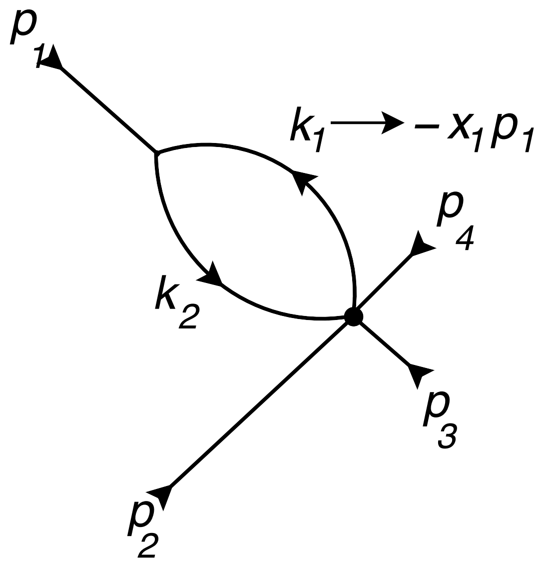
soft singularities

$$\frac{d^d k_2}{A_1 A_2 A_3 A_4} \to \frac{d^d k_2}{(-2k_2 \cdot p_1) k_2^2 (2k_2 \cdot p_2) t} \sim \mathcal{O}(\delta^{d-4})$$

collinear singularities

$$\frac{d^d k_2}{A_1 A_2 A_3 A_4} \to \frac{d^d k_2}{A_1 A_2 s t x_1 (1 - x_1)} \sim \mathcal{O}(\delta^{\frac{d}{2}-2}).$$

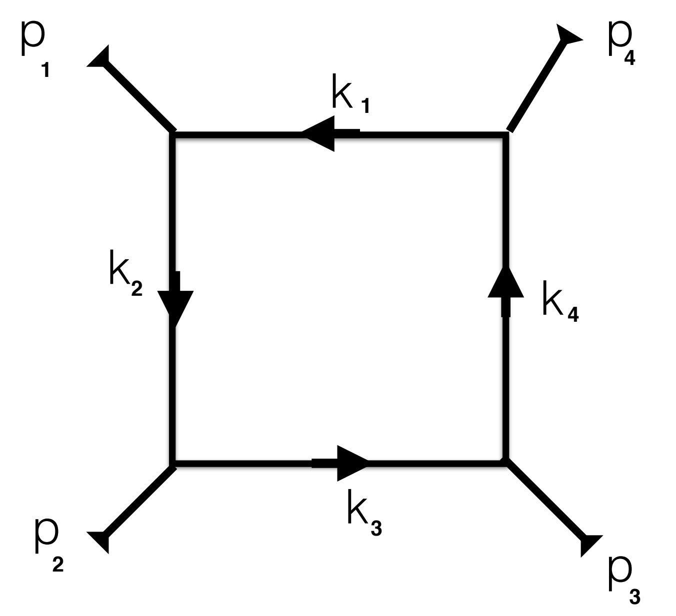




An one-loop example

- Let's focus on the soft-subtractions which come first.
- Need to construct an approximation of the integrand in the soft limits.
- Options are not unique. Can have significant differences in their UV behaviour.

$$\text{Box} \equiv \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{1}{A_1 A_2 A_3 A_4},$$



$$\text{Box}\,(s,t,\epsilon) = \frac{1}{st}\left\{\frac{2c_\Gamma}{\epsilon^2}\left[(-s)^{-\epsilon}+(-t)^{-\epsilon}\right] - \pi^2 - \ln^2\left(\frac{t}{s}\right)\right\} + \mathcal{O}(\epsilon)$$

$$\begin{aligned} t_{S_2} &: A_1 \to -2p_1\cdot k_2 \\ t_{S_2} &: A_2 \to A_2\,, \\ t_{S_2} &: A_3 \to 2p_2\cdot k_2\,, \\ t_{S_2} &: A_4 \to t\,. \end{aligned}$$

OR

$$\begin{aligned} t_{S_2} &: \; A_i \to A_i\;,\; i = 1,\,2,\,3\,, \\ t_{S_2} &: \; A_4 \to t\,. \end{aligned}$$

(Nagy Soper)

$$\text{Box}_R \equiv \left(1 - \sum_{i=1}^{4} t_{S_i}\right)\text{Box} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{N_{\text{Box}}}{A_1 A_2 A_3 A_4},$$

$$N_{\text{Box}} = 1 - \frac{A_{24}}{t} - \frac{A_{13}}{s}\,.$$

An one-loop example

- The subtracted integral is now finite in all soft limits.
- Observation: The “soft” counterterms are easier to compute than the original integral (triangle integrals)
- The subtracted integral does not have quadratic poles in epsilon.
- In fact, it does not have single poles in epsilon either….

$$
\begin{aligned}
t_{S_2}\,\mathrm{Box}(s,t,\epsilon) &= t_{S_4}\,\mathrm{Box}(s,t,\epsilon) &= \frac{c_\Gamma}{st\epsilon^2}(-s)^{-\epsilon} \\
t_{S_1}\,\mathrm{Box}(s,t,\epsilon) &= t_{S_3}\,\mathrm{Box}(s,t,\epsilon) &= \frac{c_\Gamma}{st\epsilon^2}(-t)^{-\epsilon}\,.
\end{aligned}
$$

$$
\mathrm{Box}_R = -\frac{1}{st}\left[\pi^2 + \ln^2\left(\frac{t}{s}\right)\right]
$$

An one-loop example

- Let's consider a collinear limit
- Observation: The "soft" counterterms are easier to compute than the original integral (triangle integrals)
- The collinear limit approximation is potentially UV divergent.
- We introduce a UV counterterm to the Collinear counterterm as well *(Nagy, Soper)*.
- In this example, the numerator of the collinear counterterm vanishes.
- ..which explains why our soft-subtractions sufficed to yield a finite result.

$$
\begin{aligned}
t_{C_1}\, A_1 &= A_1\,, \\
t_{C_1}\, A_2 &= A_2\,, \\
t_{C_1}\, A_3 &= (1-x)s\,, \\
t_{C_1}\, A_4 &= xt\,.
\end{aligned}
$$

$$
t_{C_1}\,\mathrm{Box} \equiv \int \frac{d^dk_1}{i\pi^{\frac{d}{2}}} \left(\frac{1}{A_1} - \frac{1}{A_1-\mu^2}\right) \frac{1}{A_2} \left[\frac{1}{stx_1(1-x_1)}\right]
$$

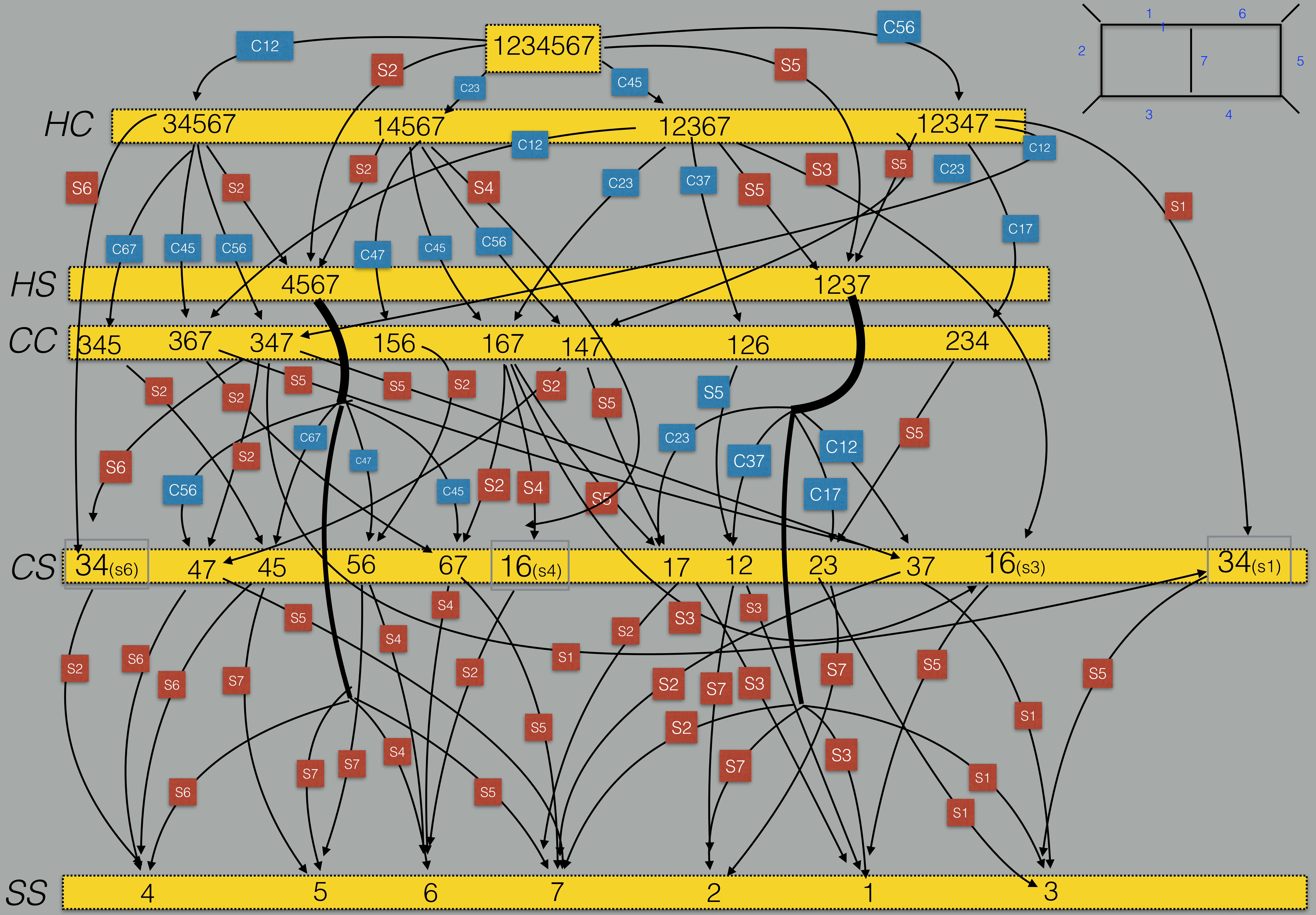
$$
= \int \frac{d^dk_1}{i\pi^{\frac{d}{2}}} \left[\frac{\frac{\mu^2}{\mu^2-A_1}}{A_1A_2stx_1(1-x_1)}\right].
$$

$$
\int \frac{d^dk_1}{i\pi^{\frac{d}{2}}} \left[\frac{N_{\mathrm{Box}}}{A_1A_2A_3A_4} - \frac{\frac{\mu^2}{\mu^2-A_1} N_{\mathrm{Box}}\big|_{k_1=-x_1p_1}}{A_1A_2stx_1(1-x_1)}\right].
$$

$$
\begin{aligned}
N_{\mathrm{Box}}\big|_{k_1=-x_1p_1} &= \left[1 - \frac{A_{13}}{s} - \frac{A_{24}}{t}\right]\Big|_{k_1=-x_1p_1} \\
&= 1-(1-x_1)-x_1 \\
&= 0\,.
\end{aligned}
$$

Does the method work at two-loops?

A complicated web of interconnected singularities….



1234567
HC
34567
14567
12367
12347
HS
4567
1237
CC
345
367
347
156
167
147
126
234
CS
34(s6)
47
45
56
67
16(s4)
17
12
23
37
16(s3)
34(s1)
SS
4
5
6
7
2
1
3
1
2
3
4
5
6
7

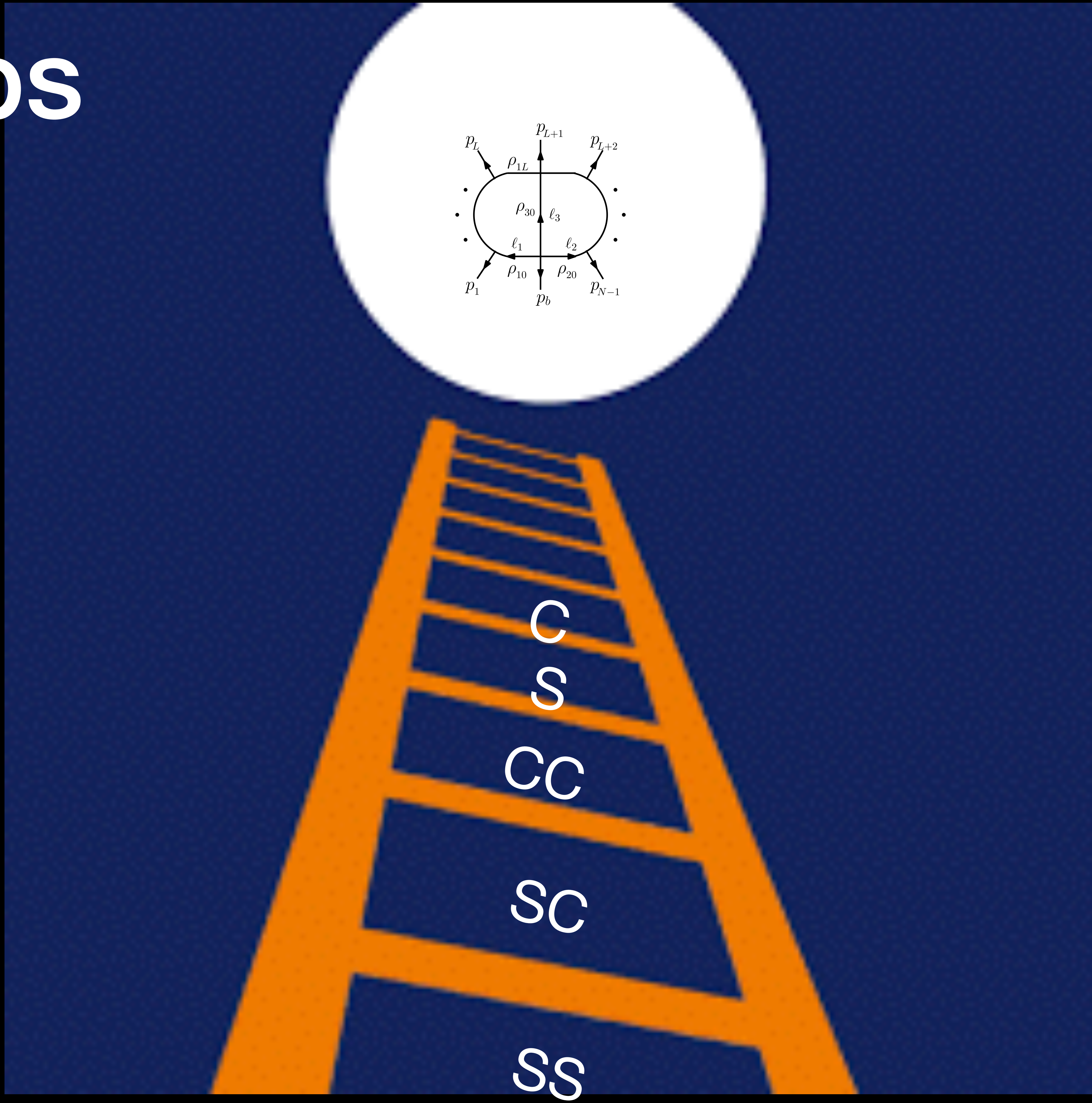
Nested subtractions at 2-loops

- Order of subtractions:
  - double-soft
  - soft-collinear
  - double-collinear
  - single-soft
  - single-collinear

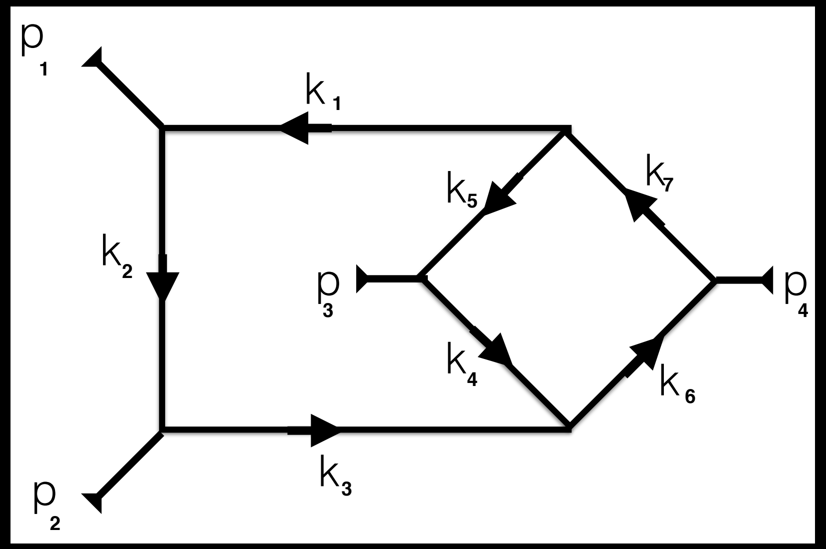
- Approximations in singular regions do not need to be strict limits!

- Good approximations should not introduce ultraviolet divergences

- Good approximations should be easy to integrate exactly.



Example: two-loop cross-box



two-loop limits **single soft** **single collinear**

$$F_{Xbox} = F^{(2)}_{Xbox} + F^{(1s)}_{Xbox} + F^{(1c)}_{Xbox},$$

double -soft ***double -collinear***

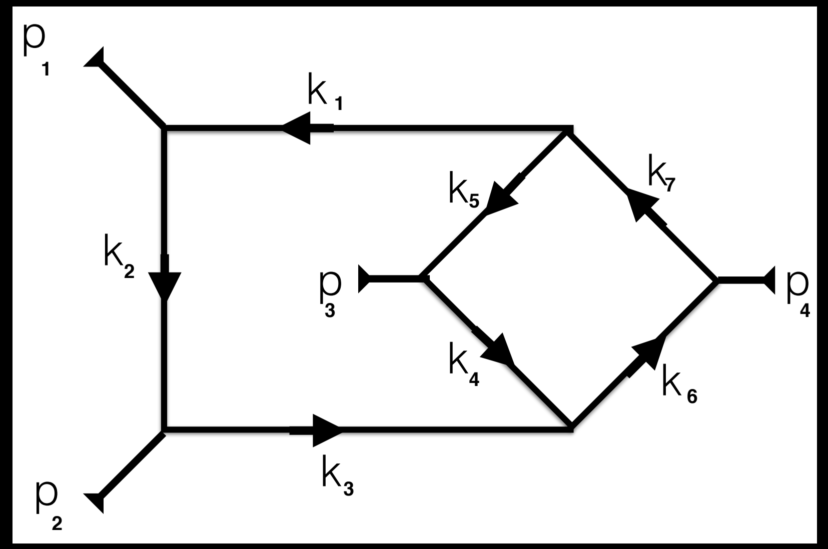
$$F^{(2)}_{Xbox} = \frac{N_5}{A_1 A_2 A_3 A_4 A_5 A_6 A_7},$$

$$\begin{aligned} N_5 = &\left(1 - \frac{A_{13}}{s}\right)^2 + \frac{A_2}{tu}\left(A_2 + s - A_{13}\right) \\ &- \left(1 - \frac{A_1}{s}\right)\left(\frac{A_5}{t} + \frac{A_7}{u}\right) - \left(1 - \frac{A_3}{s}\right)\left(\frac{A_4}{u} + \frac{A_6}{t}\right) + \frac{A_2 A_{4567}}{tu} \\ &- \frac{A_3}{s}\left(\frac{A_7}{t} + \frac{A_5}{u}\right) - \frac{A_1}{s}\left(\frac{A_6}{u} + \frac{A_4}{t}\right) + \frac{(t-u)^2}{s^2}\frac{A_1 A_3}{tu} \end{aligned}$$

$$\begin{aligned} F^{(1c)}_{Xbox} = &- \left[\frac{1}{A_1 A_2} - \frac{1}{B_1 B_2}\right] \frac{1}{s(1-x_1)} \left\{ \left[\frac{N_5}{A_4 A_5 A_6 A_7}\right]_{k_1 = -x_1 p_1} - \left[\frac{N_5}{A_4 A_5 A_6 A_7}\right]_{k_2 = 0} \right\} \\ &- \left[\frac{1}{A_2 A_3} - \frac{1}{B_2 B_3}\right] \frac{1}{s(1-x_3)} \left\{ \left[\frac{N_5}{A_4 A_5 A_6 A_7}\right]_{k_3 = -x_2 p_2} - \left[\frac{N_5}{A_4 A_5 A_6 A_7}\right]_{k_2 = 0} \right\} \\ &- \left[\frac{1}{A_4 A_5} - \frac{1}{B_4 B_5}\right] \left[\frac{N_5}{A_1 A_2 A_3 A_6 A_7}\right]_{k_5 = -x_3 p_3} \\ &- \left[\frac{1}{A_6 A_7} - \frac{1}{B_6 B_7}\right] \left[\frac{N_5}{A_1 A_2 A_3 A_4 A_5}\right]_{k_5 = -x_4 p_4} . \end{aligned}$$

$$F^{(1s)}_{Xbox} = -\frac{1}{A_1 A_2 A_3}\left[\frac{N_5}{A_4 A_5 A_6 A_7}\right]_{k_2 = 0}$$

Example: two-loop cross-box



$$\mathrm{X_{box}}^{\mathrm{fin}} \equiv \int \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{d^d k_5}{i\pi^{\frac{d}{2}}} F_{Xbox} = \mathcal{O}(\epsilon^0).$$

$$s^3 \mathrm{X_{box}}^{\mathrm{fin}} = \frac{f_{\mathrm{X_{box}}}(y)}{y} + \frac{f_{\mathrm{X_{box}}}(1-y)}{1-y},$$

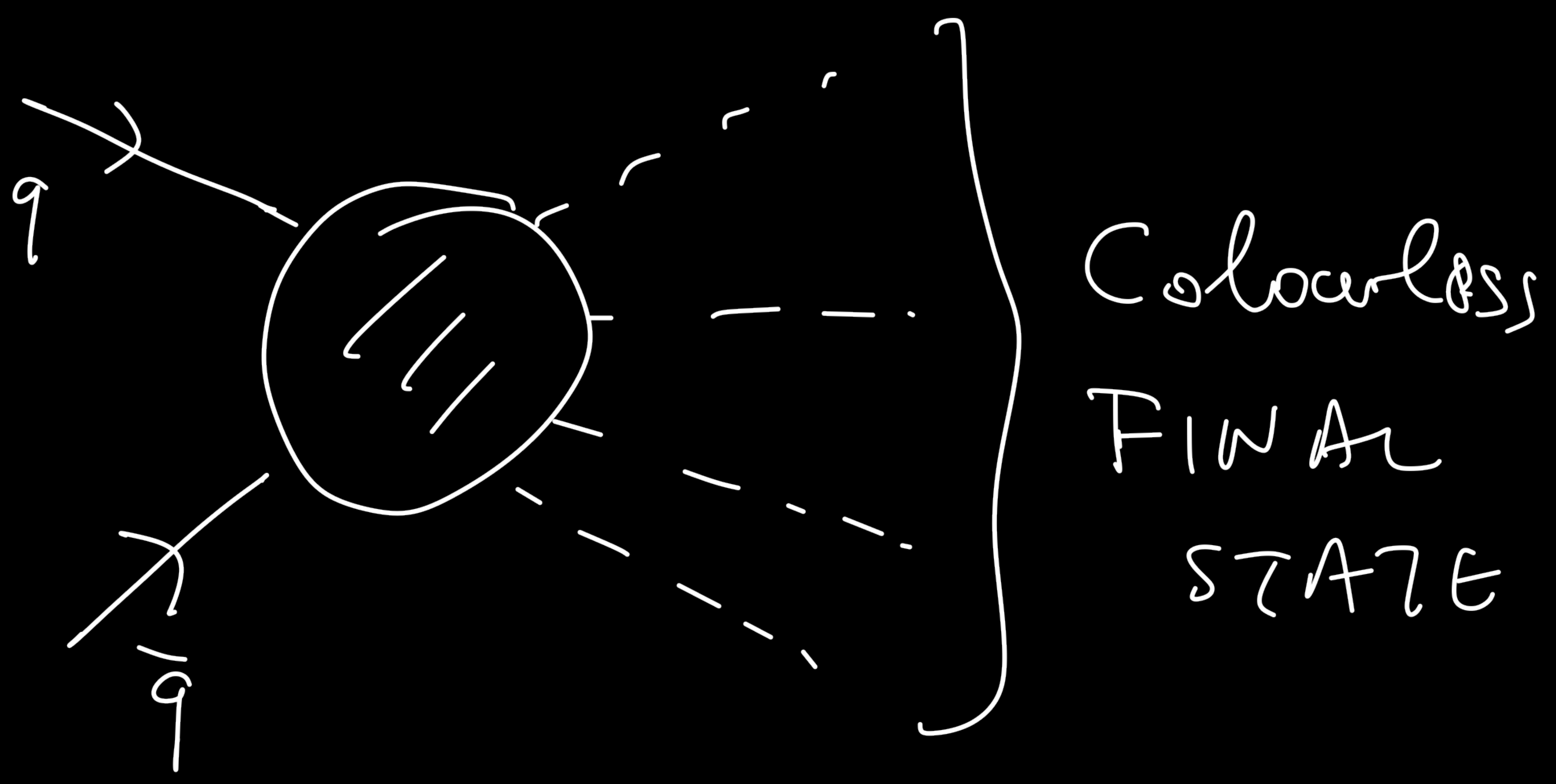
$$f_{\mathrm{X_{box}}}(y) = [G_R(y) + i\pi G_I(y)] \log\left(\frac{\mu^2}{s}\right) + E_R(y) + i\pi E_I(y)$$

$$\begin{aligned}
E_R(y) = & -8\,\pi^2\,\mathrm{Li}_2(y) + 8\,\mathrm{Li}_2(y)\,\log(1-y)^2 - 28\,\log(y)\,\mathrm{Li}_2(y)\,\log(1-y) - 18\,\mathrm{Li}_2(y)\,\log(y)^2 \\
& +44\,\mathrm{Li}_3(y)\,\log(1-y) + 96\,\mathrm{Li}_3(y)\,\log(y) - 188\,\mathrm{Li}_4(y) + \frac{17}{36}\,\pi^4 + \frac{1}{12}\,\log(1-y)^4 \\
& +7\,\log(y)\,\log(1-y)\,\pi^2 - \frac{25}{6}\,\pi^2\,\log(1-y)^2 - \frac{3}{2}\,\log(y)^2\,\pi^2 + \log(y)\,\log(1-y)^3 \\
& +44\,S_{12}(y)\,\log(1-y) - 52\,S_{12}(y)\,\log(y) + 84\,S_{13}(y) + 88\,S_{22}(y) - 44\,\zeta_3\,\log(1-y) \\
& -4\,\log(y)\,\zeta_3 - \frac{1}{4}\,\log(y)^4 + \log(y)^3\,\log(1-y) - \frac{9}{2}\,\log(y)^2\,\log(1-y)^2,
\end{aligned}$$

...

Application to a class of one-loop amplitudes

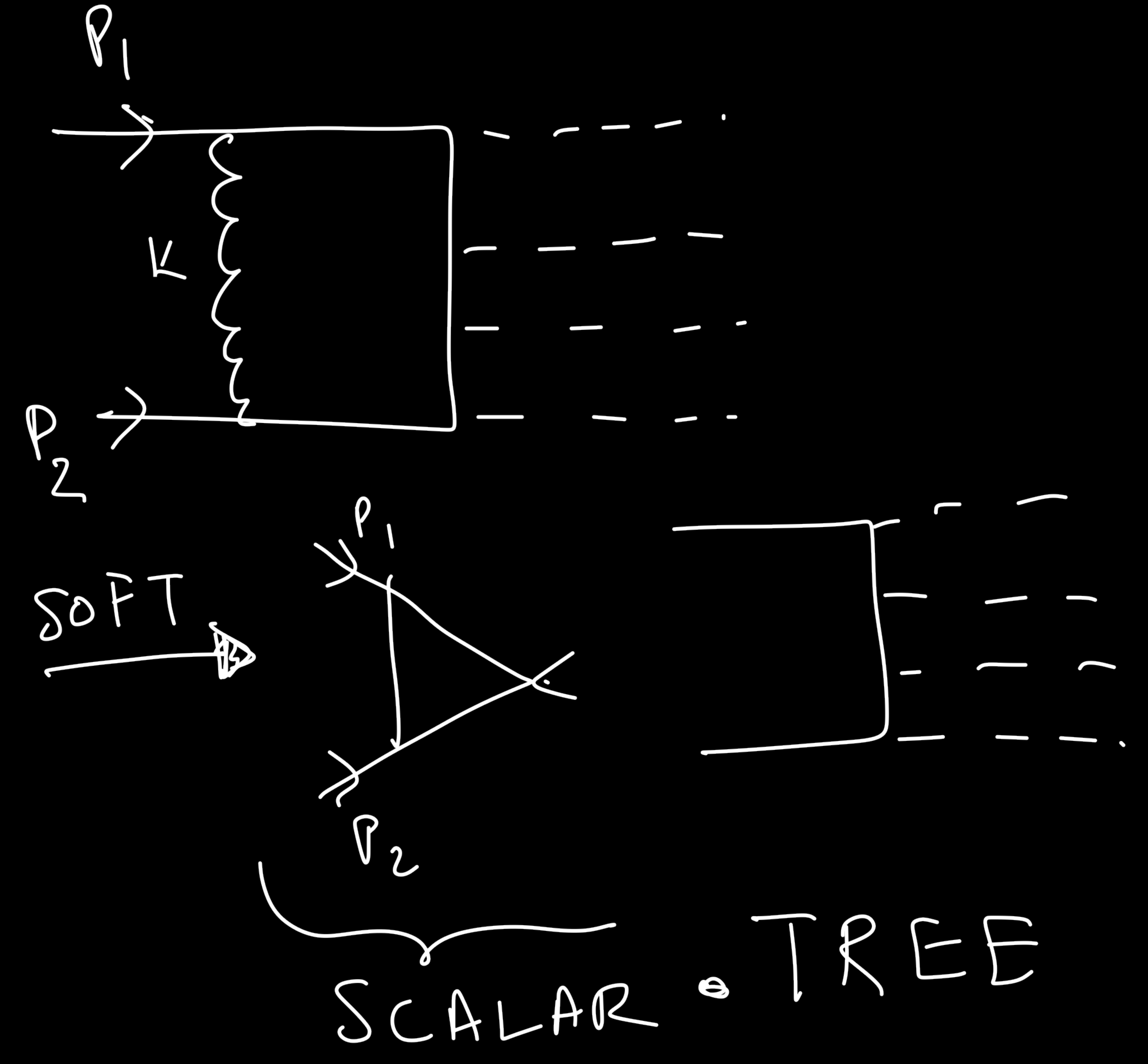
- Consider the process for the production of a heavy colourless final-state from the scattering of a massless quark-antiquark pair.
- This encompasses a large set of processes (multi Z,W, photon production and combinations)
- Easy to verify at one-loop that a simple set of local counterterms exists for all these processes.



Application to amplitudes

- Per tree-diagram, there is one 1-loop diagram with a soft singularity.

- The soft limit is (up to trivial factors), an one-loop scalar integral times a tree-diagram.



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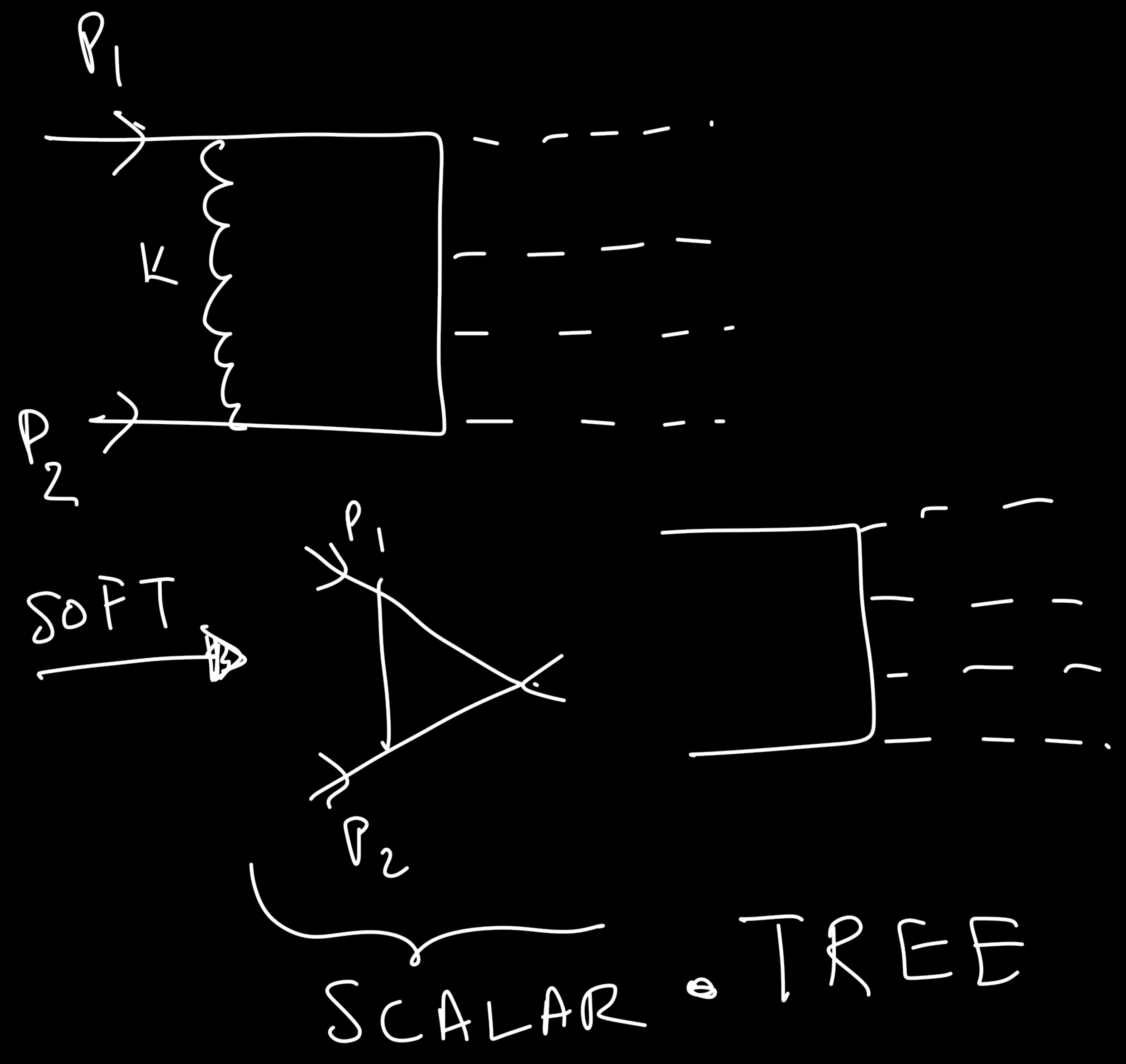
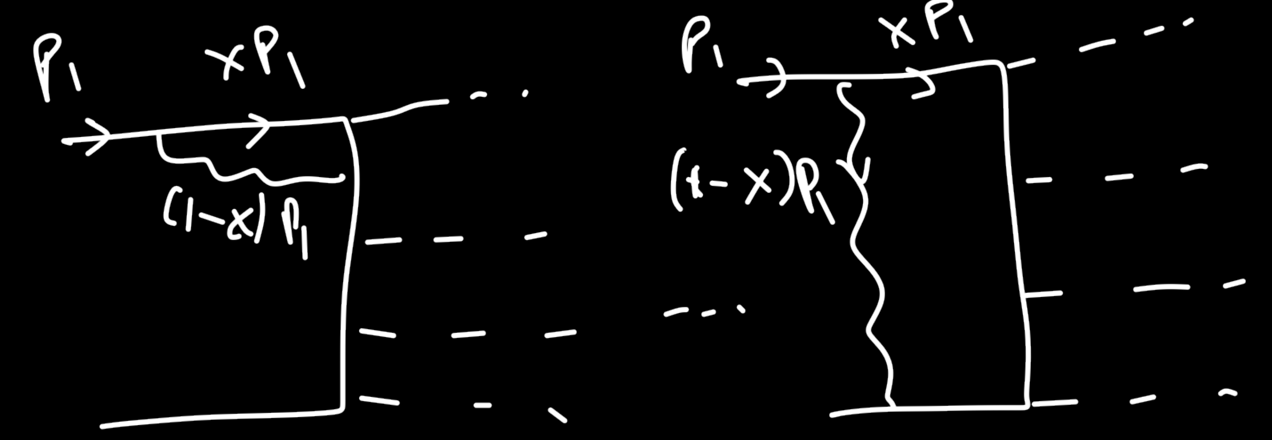
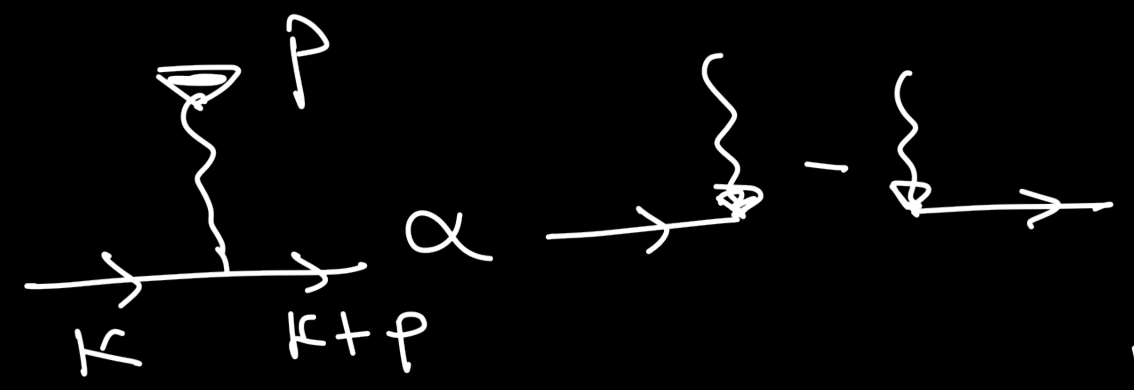


Application to amplitudes

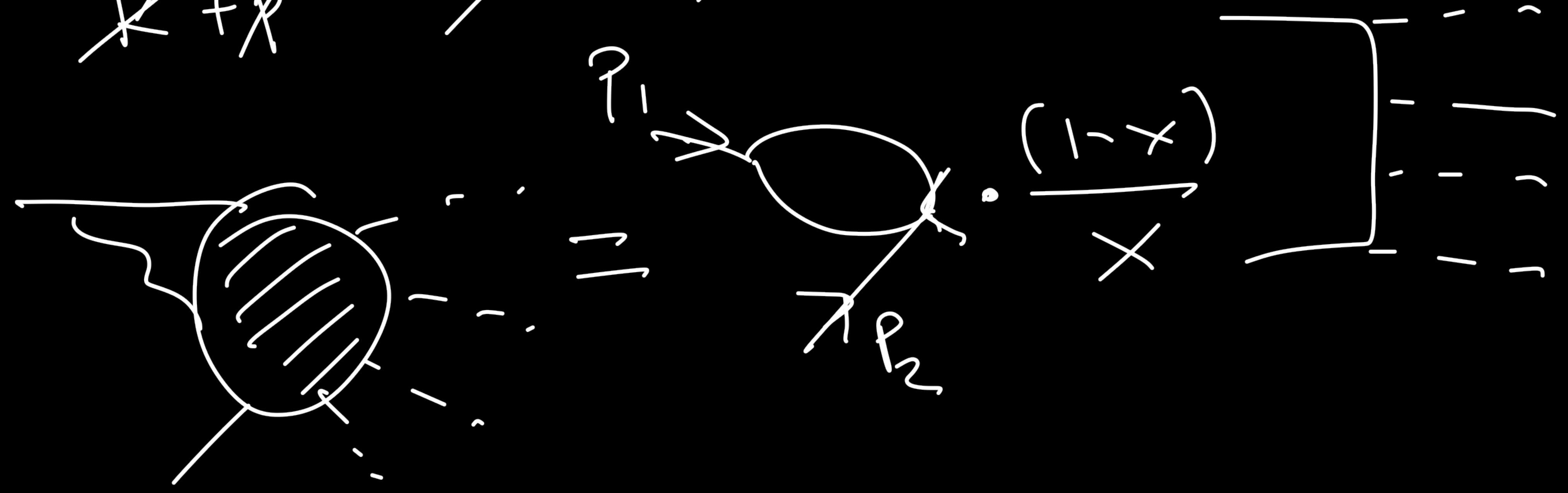
- Many graphs yield collinear divergences.
- Summing over all such graphs, cancellations take place (“Ward”-identity)
- The net-result is factorization of the amplitude in the collinear limit in terms of a splitting-functions and a tree-diagram.



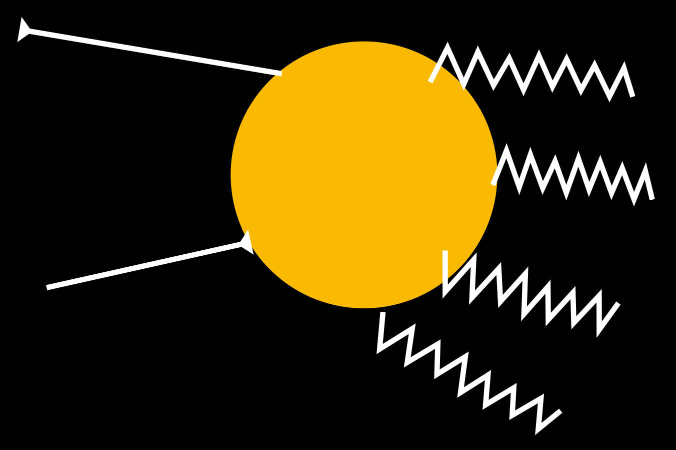
WARD - IDENTITY



$$\frac{1}{\not{k}+\not{p}}\,\not{p}\,\frac{1}{\not{k}} = \frac{1}{\not{k}} - \frac{1}{\not{k}+\not{p}}$$



Does the method work for complicated amplitudes?



$$e^+(p_1) + e^-(p_2) \to \gamma(q_1) + \ldots \gamma(q_n)$$

Example case: QED amplitudes for multi photon production in electron positron annihilation through two-loops

YES!

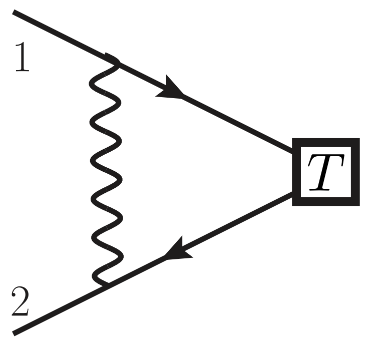
CA, Haindl, Sterman, Yang, Zeng

Simple subtractions! Valid locally! Factorized and universal!

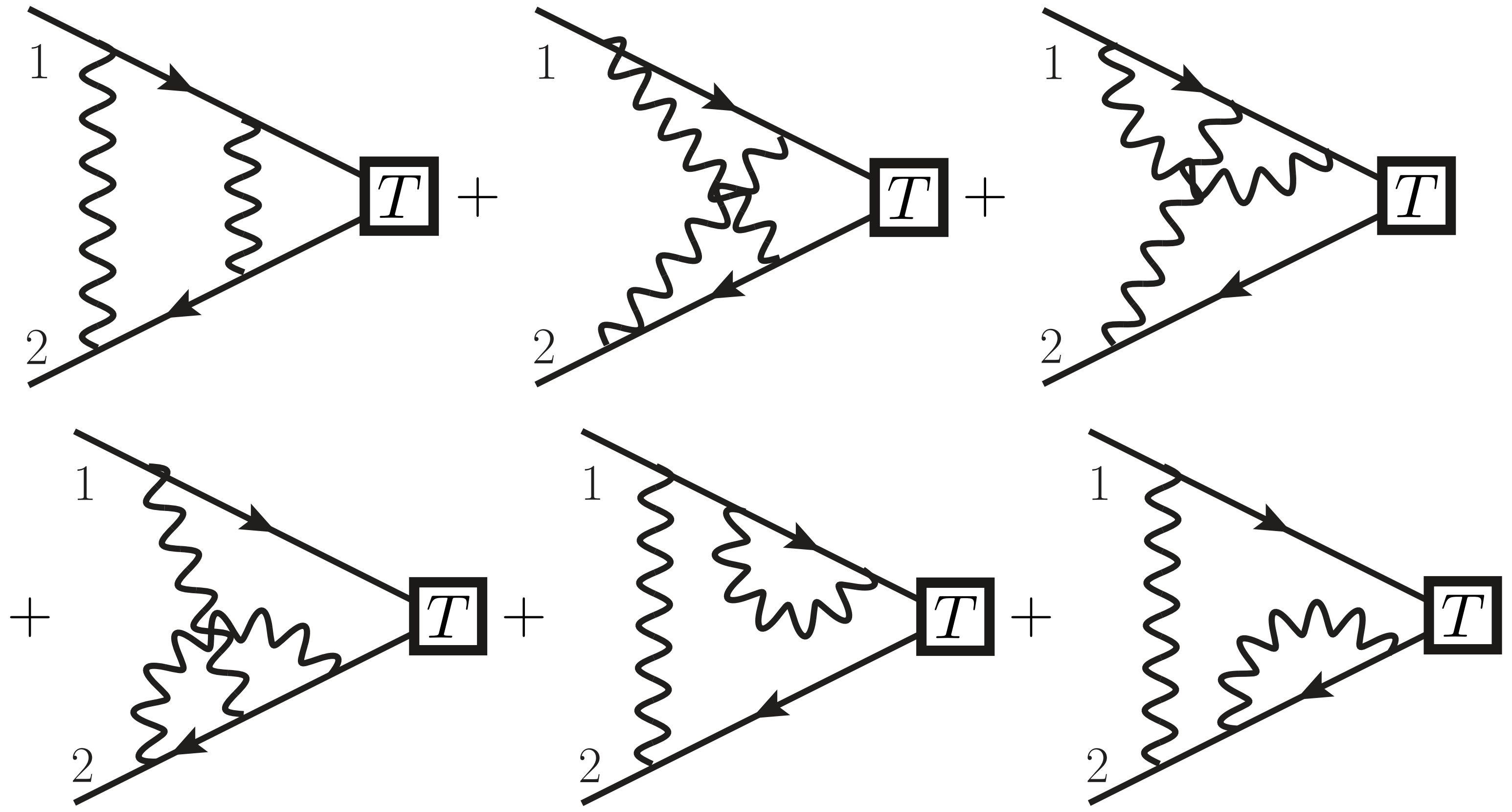
$$\mathcal{M}^{(1)}_{\mathrm{IR-finite}} = \mathcal{M}^{(1)} - \mathcal{F}^{(1)}\left[\mathbf{P}_1 \widetilde{\mathcal{M}}^{(0)} \mathbf{P}_1\right].$$

$$\mathcal{M}^{(2)}_{\text{IR-finite}} = \mathcal{M}^{(2)} - \mathcal{F}^{(2)}\left[\mathbf{P}_1 \widetilde{\mathcal{M}}^{(0)} \mathbf{P}_1\right] - \mathcal{F}^{(1)}\left[\mathbf{P}_1 \widetilde{\mathcal{M}}^{(1)}_{\text{IR-finite}} \mathbf{P}_1\right],$$

$\mathcal{F}^{(1)}\left[\mathbf{P}_1 T \mathbf{P}_1\right] =$



$\mathcal{F}^{(2)}\left[\mathbf{P}_1 T \mathbf{P}_1\right] =$



CA, Haindl, Sterman, Yang, Zeng

Check

$$k^\mu = \frac{33}{17}\delta^{\omega_k^+}\, p_1^\mu - \frac{48}{89}\delta^{\omega_k^-}\, p_2^\mu + \delta^{\omega_k^T}\left(\frac{21}{23}\, n_x^\mu + \frac{21}{41}\, n_y^\mu\right),$$
$$l^\mu = \frac{47}{23}\delta^{\omega_l^+}\, p_1^\mu - \frac{7}{61}\delta^{\omega_l^-}\, p_2^\mu + \delta^{\omega_l^T}\left(-\frac{37}{73}\, n_x^\mu - \frac{39}{67}\, n_y^\mu\right).$$

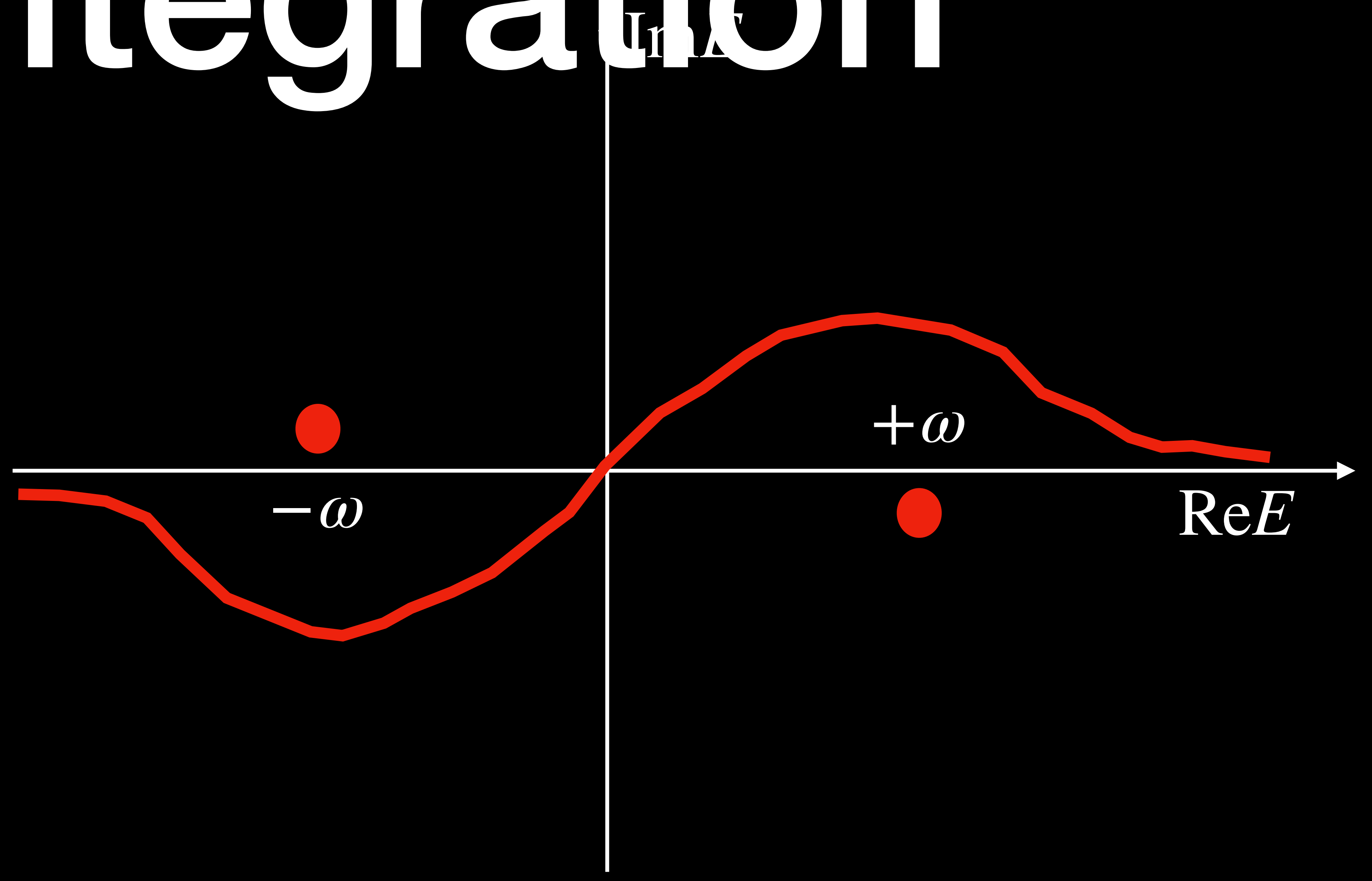
Limit	ω_k^+	ω_k^-	ω_k^T	ω_l^+	ω_l^-	ω_l^T	amplitude scaling
k soft	1	1	1	0	0	0	δ^1
$k \parallel p_1$	0	2	1	0	0	0	δ^1
k, l soft	1	1	1	1	1	1	δ^2
k soft, $l \parallel p_1$	2	2	2	0	2	1	δ^3
$k \parallel p_1$, $l \parallel p_2$	0	2	1	2	0	1	δ^2
$k, l \parallel p_1$	0	2	1	0	2	1	δ^2
$k, l \to \infty$	-1	-1	-1	-1	-1	-1	δ
$k \to \infty$	-1	-1	-1	0	0	0	δ

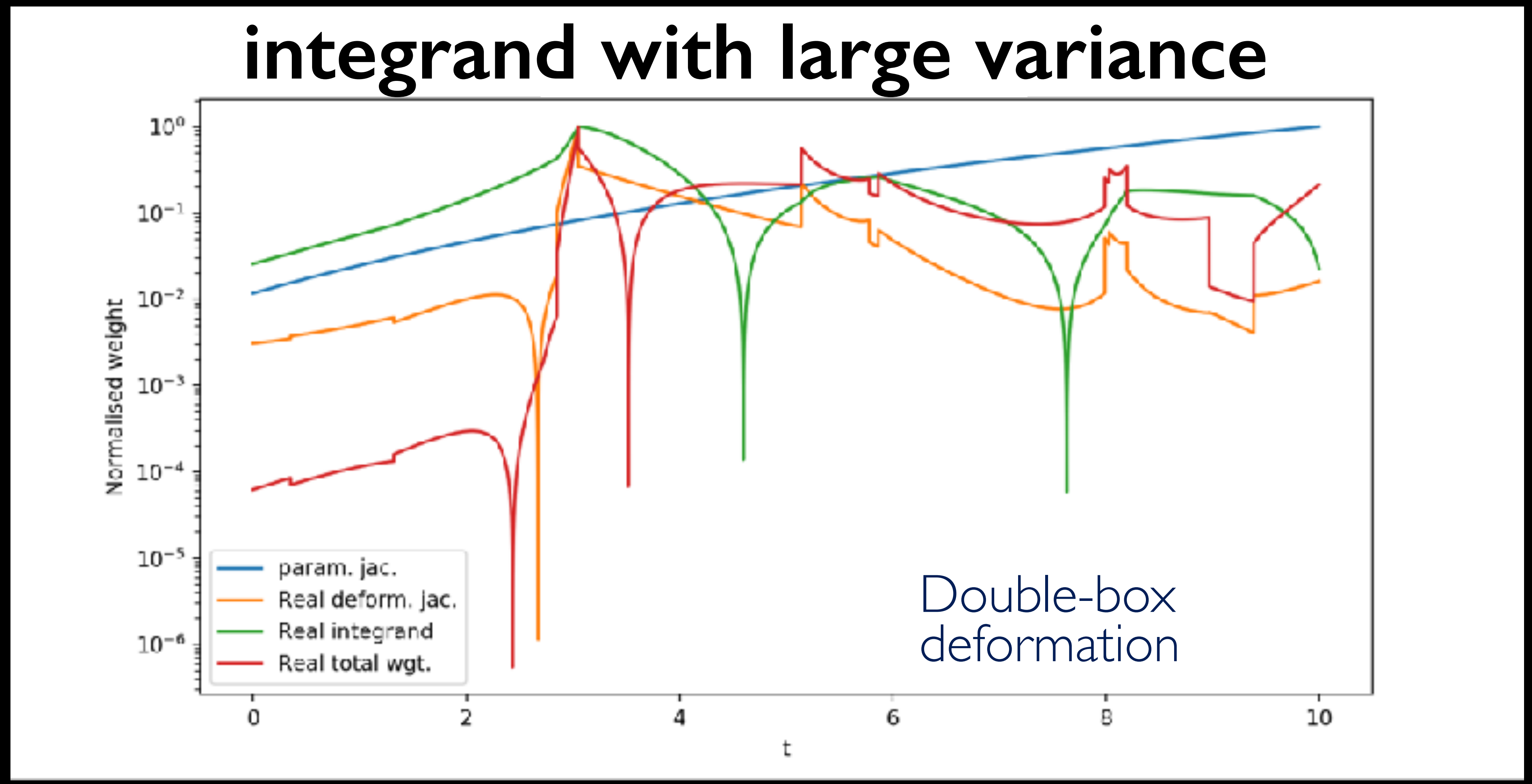
Amplitude scaling in all singular limits

CA, Haindl, Sterman, Yang, Zeng

Numerical integration

- Can such subtractions be used for evaluating loop amplitudes numerically?
- They are an important ingredient! They remove “pinch” singularities.
- Other singularities which can be avoided with appropriate contour-deformations are equally important.





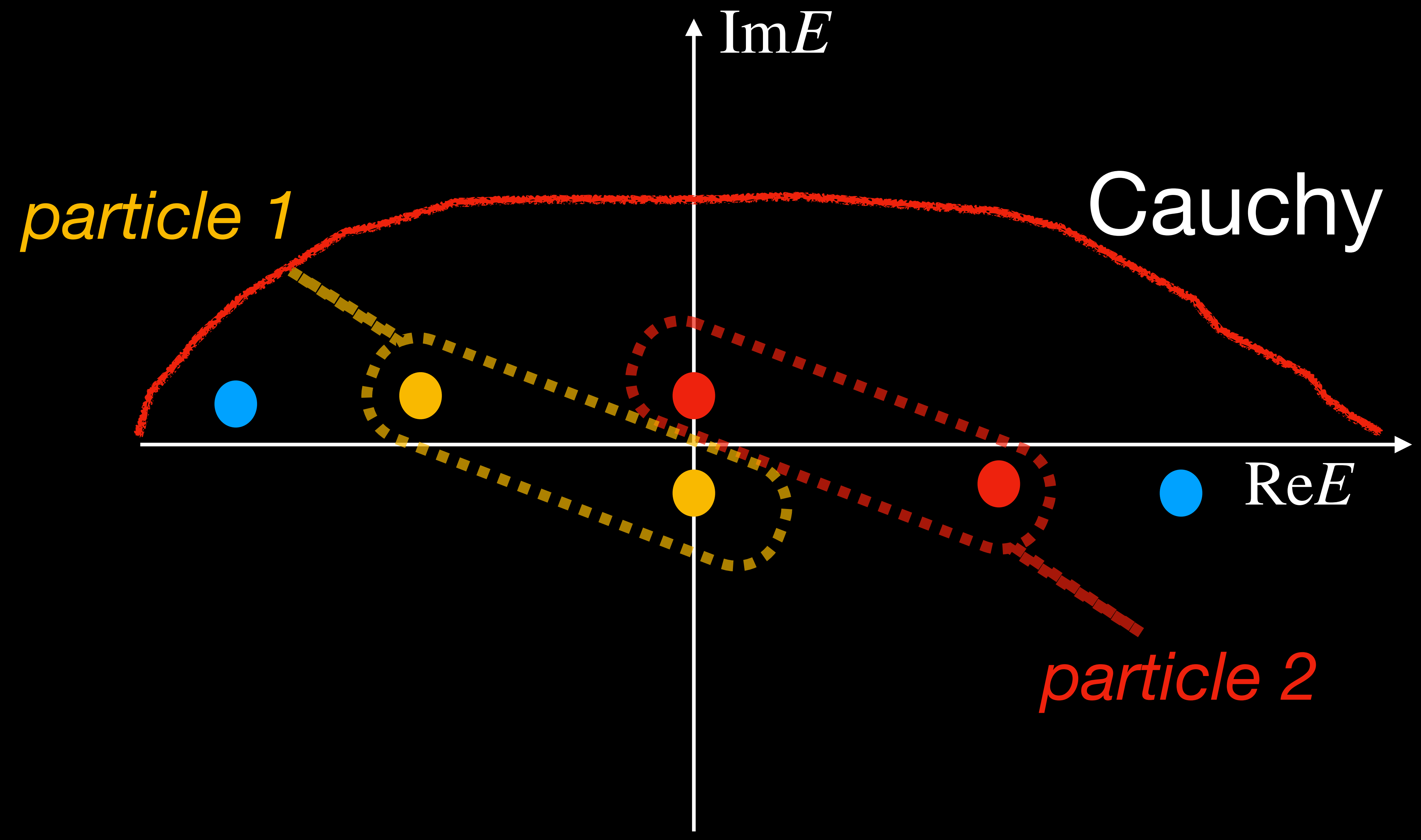
Numerical integration

- A breakthrough in numerical integration has been achieved recently

- First integrate over the energy component of all loop momenta using Cauch [Loop-Tree duality]

- This reduces the number of integrations.

Catani,Gleisberg,Krauss,Rodrigo, Winter; Bierenbaum, Catani, Draggiotis, Rodrigo; Capatti,Hirschi, Kermanschah, Ruijl; Aguilera-Verdugo, Driencourt-Mangin, Plenter, Ramırez-Uribe, Rodrigo, Sborlini,Torres Bobadilla, Tracz; Runkel, Szőr, Vesga, Weinzierl;…

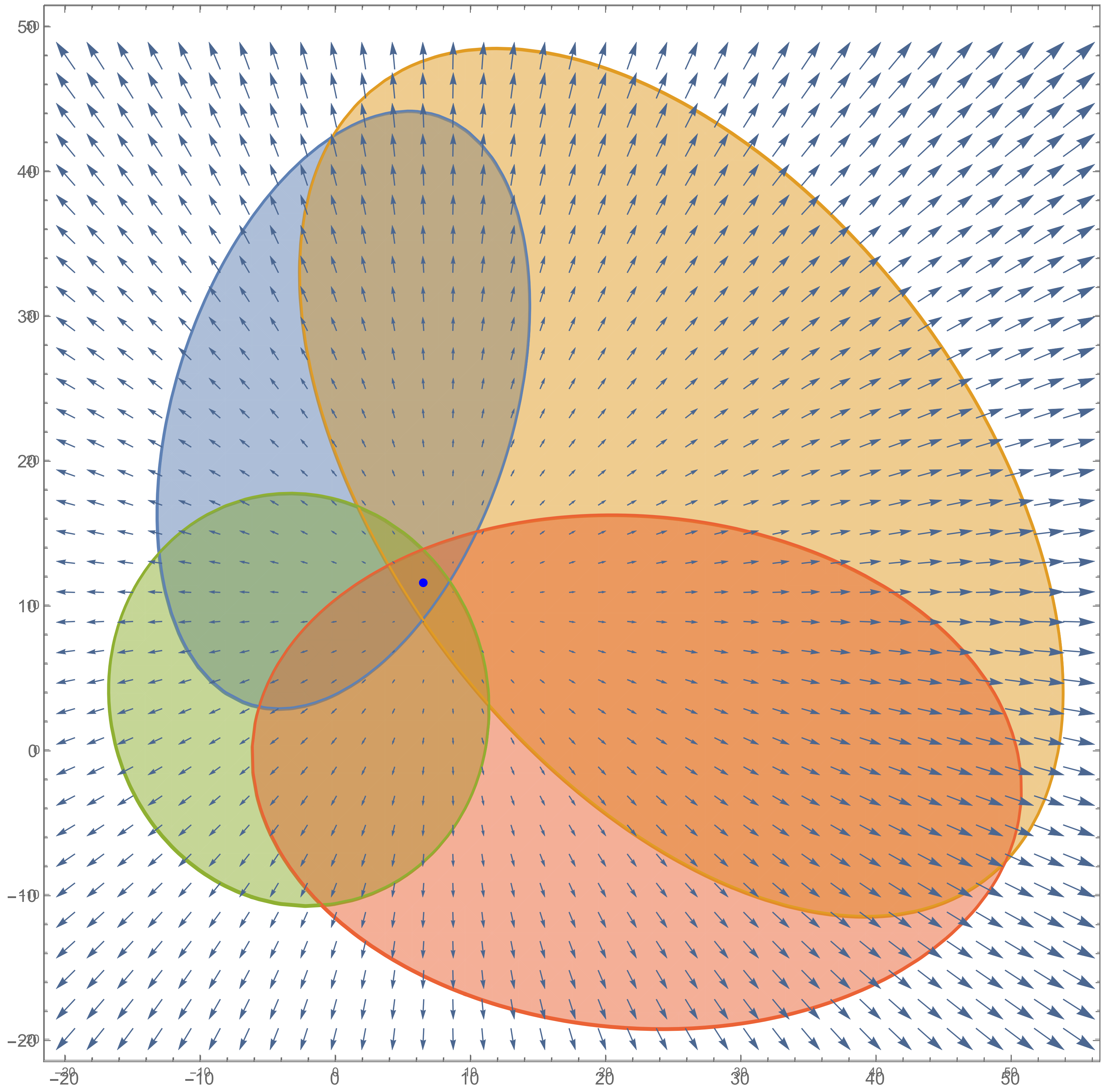


$$\int d^4k \to \int \frac{d^3\vec{k}}{|\vec{k}|} = \int d^4k\delta(k^2)\Theta(k^0)$$

Capatti, Hirschi, Kermanschah, Ruijl

Numerical integration

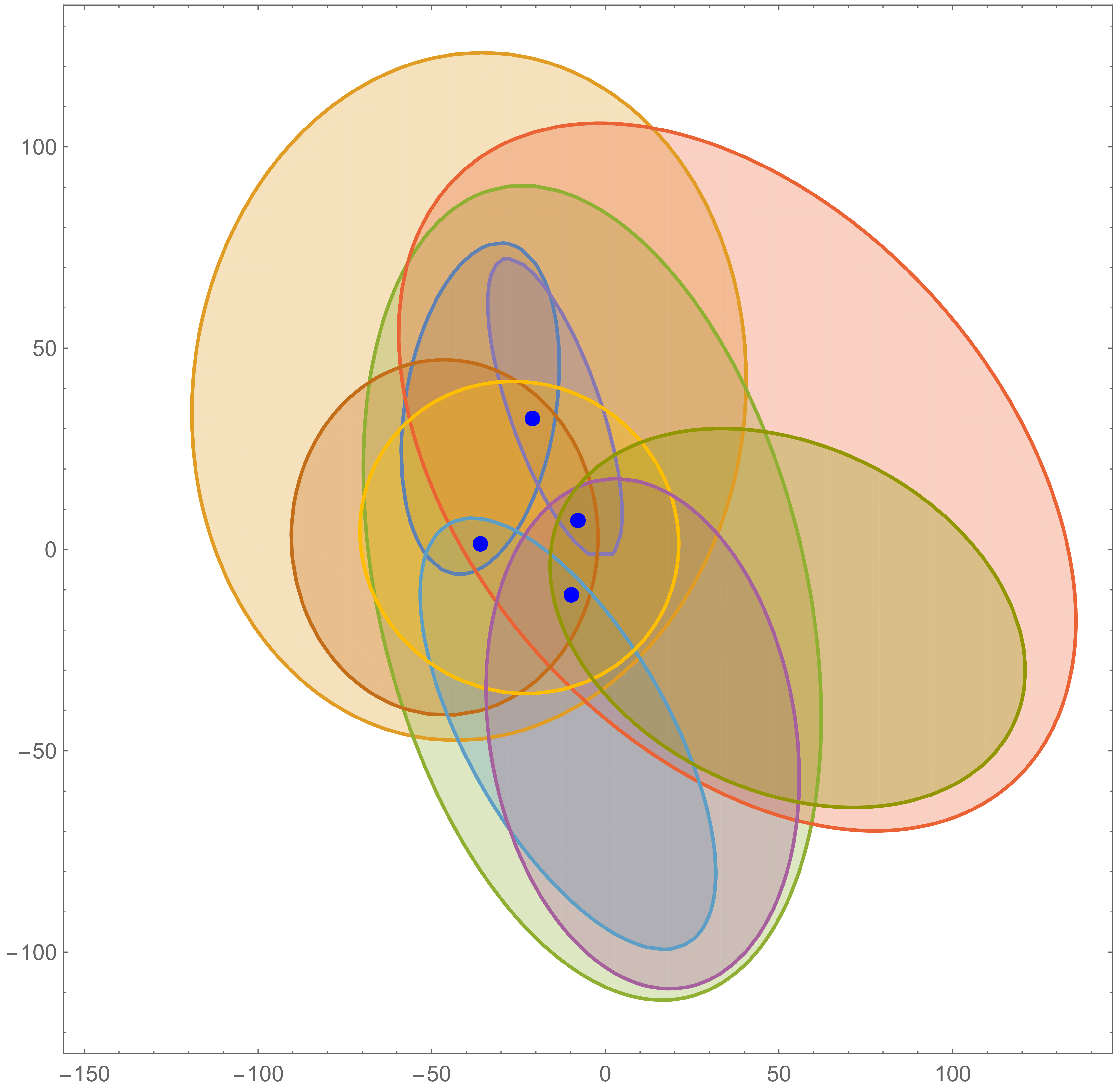
- A breakthrough in numerical integration has been achieved recently
- First integrate over the energy component of all loop momenta using Cauch [Loop-Tree duality]
- This reduces the number of integrations.
- Then devise an algorithm to move the contour of remaining integrations away from non-pinched singularities.

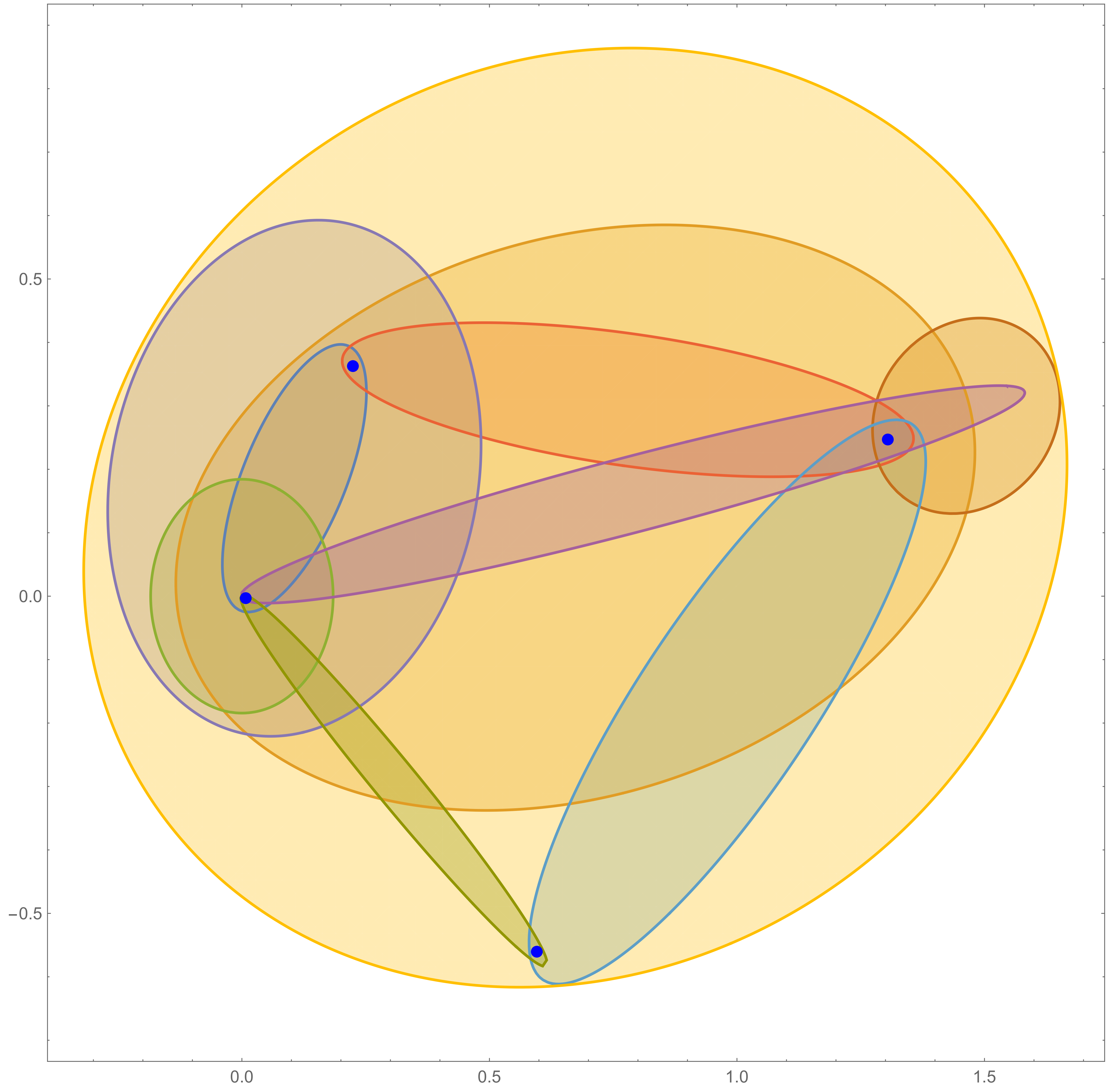


A **radial field** centered in the inside of all ellipsoids!

Capatti, Hirschi, Kermanschah, Pelloni, Ruijl

More complicated cases require multiple centres





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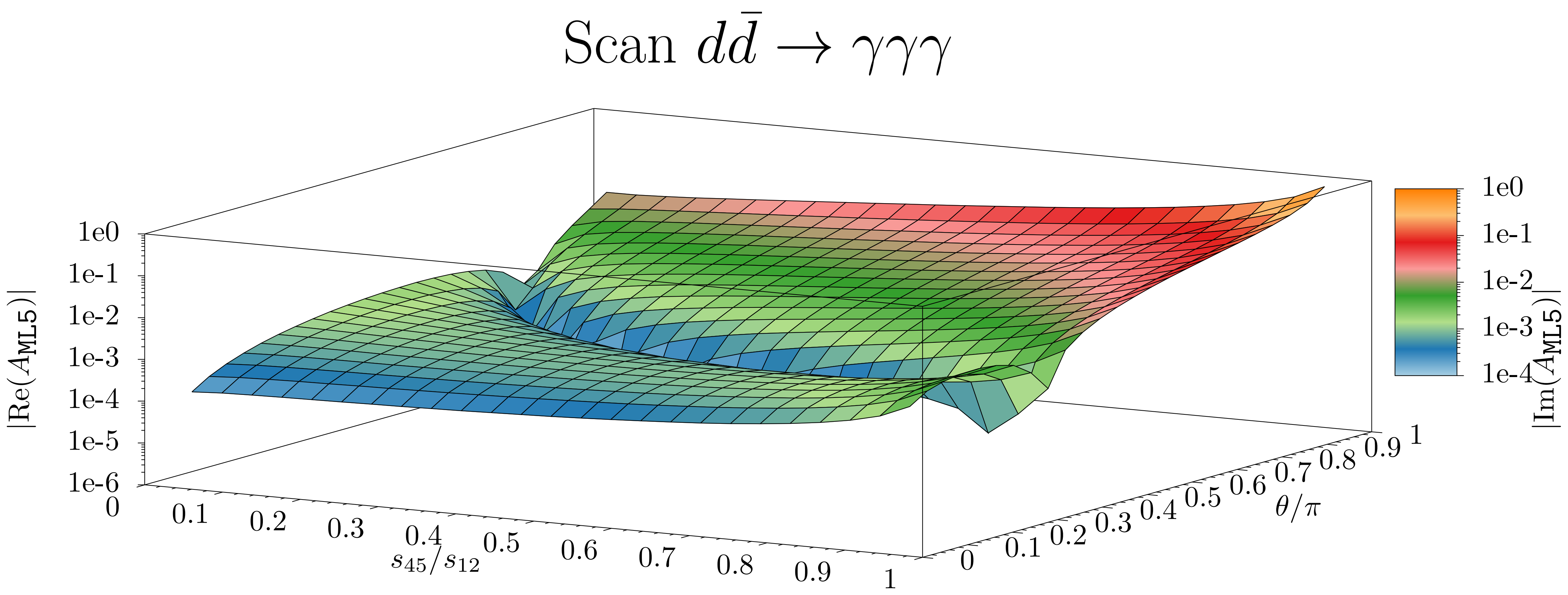
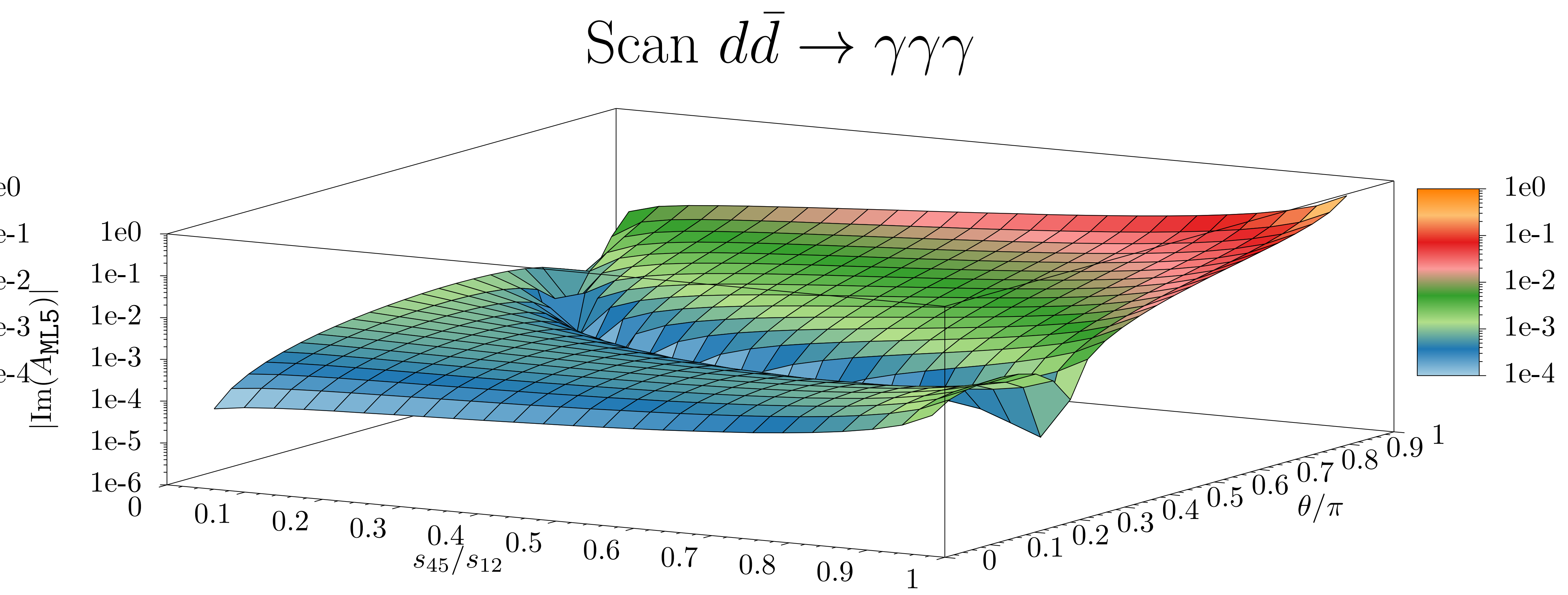
Numerical integration at one-loop



(a) Real part of the amplitude



(b) Imaginary part of the amplitude.

A few per mille numerical precision easily reached

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Numerical integration at many-loops

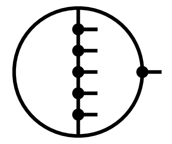
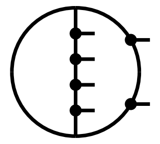
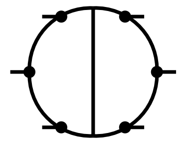
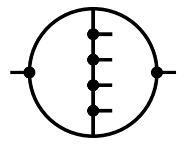
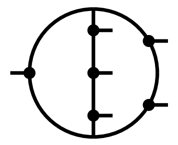
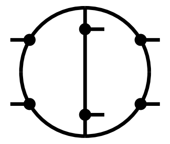
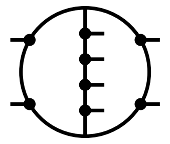
Topology	Kin.	N_C	N_E	N_S	L_{max}	N_p [10^9]	t/p [μs]	Phase	Exp.	Reference	Numerical LTD	Δ [σ]	Δ [%]	Δ [%]|·|
	K1	20	20	15	[19, 14]	3	100	Re	-09	n/a	4.58688 +/- 0.05132		1.119	1.059
								Im		n/a	5.04144 +/- 0.05075		1.007	
2L6P.a	K1*	20	17	24	[12, 13, 13, 13, 13]	3	116	Re	-09	n/a	-1.04316 +/- 0.35247		33.79	10.99
								Im		n/a	-4.42468 +/- 0.35421		8.005	
	K1	23	23	15	[22, 19]	3	91	Re	-09	n/a	1.17336 +/- 0.00888		0.757	0.303
								Im		n/a	3.99809 +/- 0.00896		0.224	
2L6P.b	K1*	23	20	20	[18, 17, 18]	3	103	Re	-09	n/a	5.35217 +/- 0.00153		0.029	0.033
								Im		n/a	3.81579 +/- 0.00150		0.039	
	K1	24	22	16	[20, 21]	3	89	Re	-09	n/a	4.90974 +/- 0.01407		0.286	0.375
								Im		n/a	-2.13974 +/- 0.01434		0.670	
2L6P.c	K1*	24	20	22	[17, 17, 17, 17]	3	108	Re	-08	n/a	1.05934 +/- 0.15850		14.96	14.87
								Im		n/a	1.03698 +/- 0.15312		14.77	
	K1	24	20	26	[16, 7, 14, 14, 4]	3	136	Re	-08	n/a	1.90487 +/- 0.05753		3.020	2.017
								Im		n/a	-3.55267 +/- 0.05746		1.617	
2L6P.d	K1*	24	17	30	[13, 12, 12, 12, 2]	3	144	Re	-08	n/a	-2.97419 +/- 0.00961		0.323	0.367
								Im		n/a	-2.18847 +/- 0.00957		0.437	
	K1	26	21	34	[16, 9, 9, 14, 15, 9, 7]	3	163	Re	-07	n/a	2.87833 +/- 0.00951		0.330	0.386
								Im		n/a	1.99937 +/- 0.00961		0.481	
2L6P.e	K1*	26	18	43	[13, 12, 7, 7, 12, 12, 12, 12, 7, 5]	3	172	Re	-07	n/a	1.67332 +/- 0.00578		0.346	0.482
								Im		n/a	-0.21788 +/- 0.00571		2.620	
	K1	27	27	22	[24, 21, 24]	3	121	Re	-08	n/a	-0.95486 +/- 0.00890		0.932	0.368
								Im		n/a	3.28530 +/- 0.00889		0.271	
2L6P.f	K1*	27	24	34	[19, 20, 20, 20, 20]	3	152	Re	-08	n/a	2.55104 +/- 0.00208		0.082	0.097
								Im		n/a	-1.63019 +/- 0.00205		0.126	
	K1	39	46	40	[37, 42, 41, 40]	3	237	Re	-12	n/a	-5.15438 +/- 0.03310		0.642	0.544
2L8P								Im		n/a	6.78546 +/- 0.03243		0.478	

Table 7: Results for two-loop topologies for scattering kinematics ($2 \to N$) for massless and massive propagators (indicated by a *). When there is no reference result, Δ[%] and Δ[%]|·| refer to the Monte-Carlo accuracy relative to the central value. See the main text for details.

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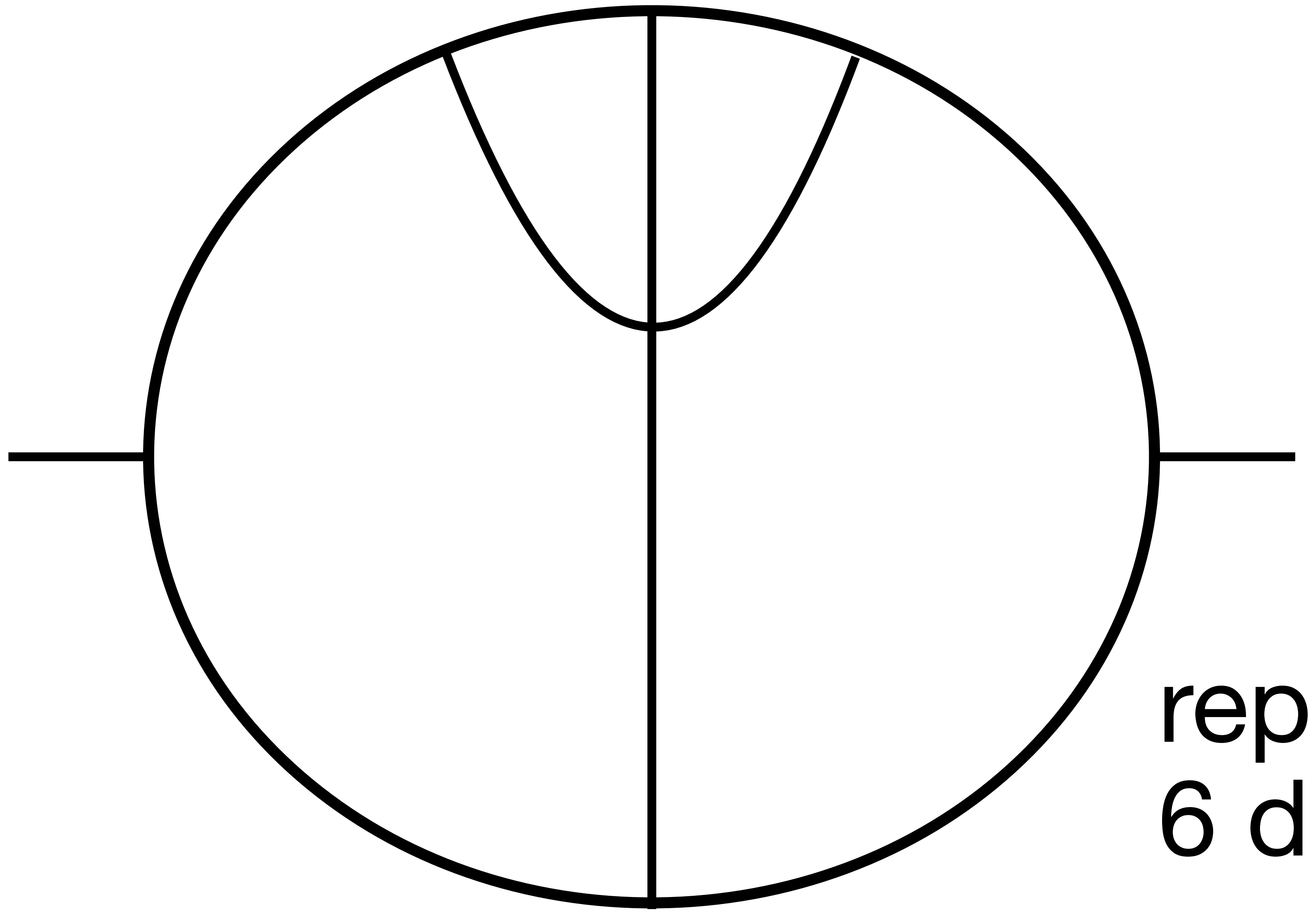
Back to UNITARITY: LTD makes the dimensionalities of loop and phase-space integrals to match! Implementing local unitarity is natural in this framework.

- "N3LO" $\{\phi^3, \phi^4\}$ cross-section from one individual 4-loop topology :

CUTKOSKY + LTD CUTS [] = $5\zeta(5)$

reproduced numerically to 6 digits.

IR cancellations can be made to happen for **any theory**, at **any loop count** and **separately for each individual topology**!

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Conclusions

- We have witnessed rapid progress in perturbative QCD, matching the precision of the LHC experiments. So far!
- Can we keep up? A need to keep reinventing our field and understanding perturbation theory at deeper levels.
- Infrared factorization and Unitarity have been crucial historically. These properties can be exploited further.
- A dream which can become true: Perform perturbative QCD computations at very high orders efficiently, automatically and in D=4 exactly!
- Bring theoretical predictions to the frontline of the precision physics at the LHC (and gravitational waves and cosmology….)