Boundaries, Vermas, and Factorisation

Based on 2010.09741 with M. Bullimore, S. Crew and 2010.09732 with S.Crew and N. Dorey

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Background and Motivation

$$\{Q, Q^\dagger\} = H$$

Partition Function (Witten Index)

$$\mathrm{Tr}_{\mathscr{H}}(-)^F e^{-\beta\{Q,Q^\dagger\}} x^J$$

$$\mathrm{ch}_V x^J \quad = \quad \mathscr{X}(\mathscr{QM})[x]$$

Characters of quantum algebras acting on BPS States

(Equivariant) indices/ invariants on moduli spaces (of solitons)

Representation Theory

Enumerative Geometry

Outline

- Background on $3d\ \mathcal{N} = 4$
- Moduli spaces of vacua and chiral algebras
- Boundary conditions and modules
- Exceptional Dirichlet and Verma modules
- Half Indices, $S^1 \times H^2$ Partition Functions
- Factorisation and IR Formulae

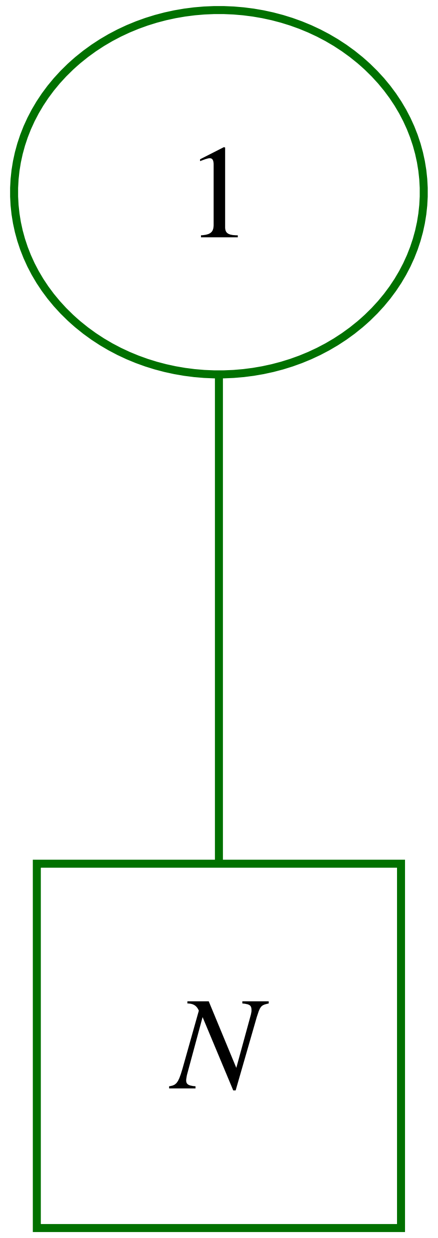
$3d\ \mathcal{N} = 4$ Theories

- 8 supercharge theory $\mathcal{Q}_{\alpha}^{a\dot{a}}$
- R-symmetry group $SU(2)_H \times SU(2)_C \supset R_H \times R_C$
- Fix a gauge group G
- 2 types of multiplets:
  - Hypermultiplets: $(X_i, Y_i),\ i = 1,...,N$ Global symmetry G_H

    $\times$
  - Vector multiplet: (A_μ, σ, φ) Global symmetry G_C
- Deformations: $\vec{m} \in \mathfrak{t}_H,\ \vec{\eta} \in \mathfrak{t}_C$.
- We'll stick to Lagrangian theories which flow in the UV to SCFTs.

Example: SQED[N]

Our favourite theory



SQED[N] - Supersymmetric QED with N hypermultiplets

- A $G = U(1)$ vector multiplet (A_μ, σ, φ)
- N hypermultiplets $(X_i, Y_i),\ i = 1,\ldots, N$ transforming in the fundamental of G
- $G_H = PSU(N)\ ,\ G_C = U(1)$

Moduli Spaces of Vacua

In general, the vacuum moduli space complicated. Work with two distinct ‘branches’.

Higgs Branch $\mathscr{M}_H$

Hypermultiplet scalars (X, Y) get VEVs

Classically:

$$\mu_{\mathbb{R}} = X \cdot X^{\dagger} - Y \cdot Y^{\dagger}$$
$$\mu_{\mathbb{C}} = X \cdot Y$$
$$\mathscr{M}_H = \{\mu_{\mathbb{C}}^{-1}(\eta_{\mathbb{C}}) \cap \mu_{\mathbb{R}}^{-1}(\eta_{\mathbb{R}})\}/G$$

Hyperkähler reduction

Protected via non-renormalisation theorems so the classical computation is exact.

Turning on masses $m_{\mathbb{R}}$ restricts vacua to fixed points of tri-Hamiltonian isometry G_H.

Coulomb branch $\mathscr{M}_C$

Scalars σ, φ and also periodic scalar γ dual to the photon A get VEVs.

Classically:

$$\mathscr{M}_C = (\mathbb{R}^3 \times S^1)^{\text{rk}G}/\text{Weyl}(G)$$

Receives 1-loop quantum corrections. For SQED[N], this is the N-centered Taub-NUT space

SUSY algebra guarantees still Hyperkähler

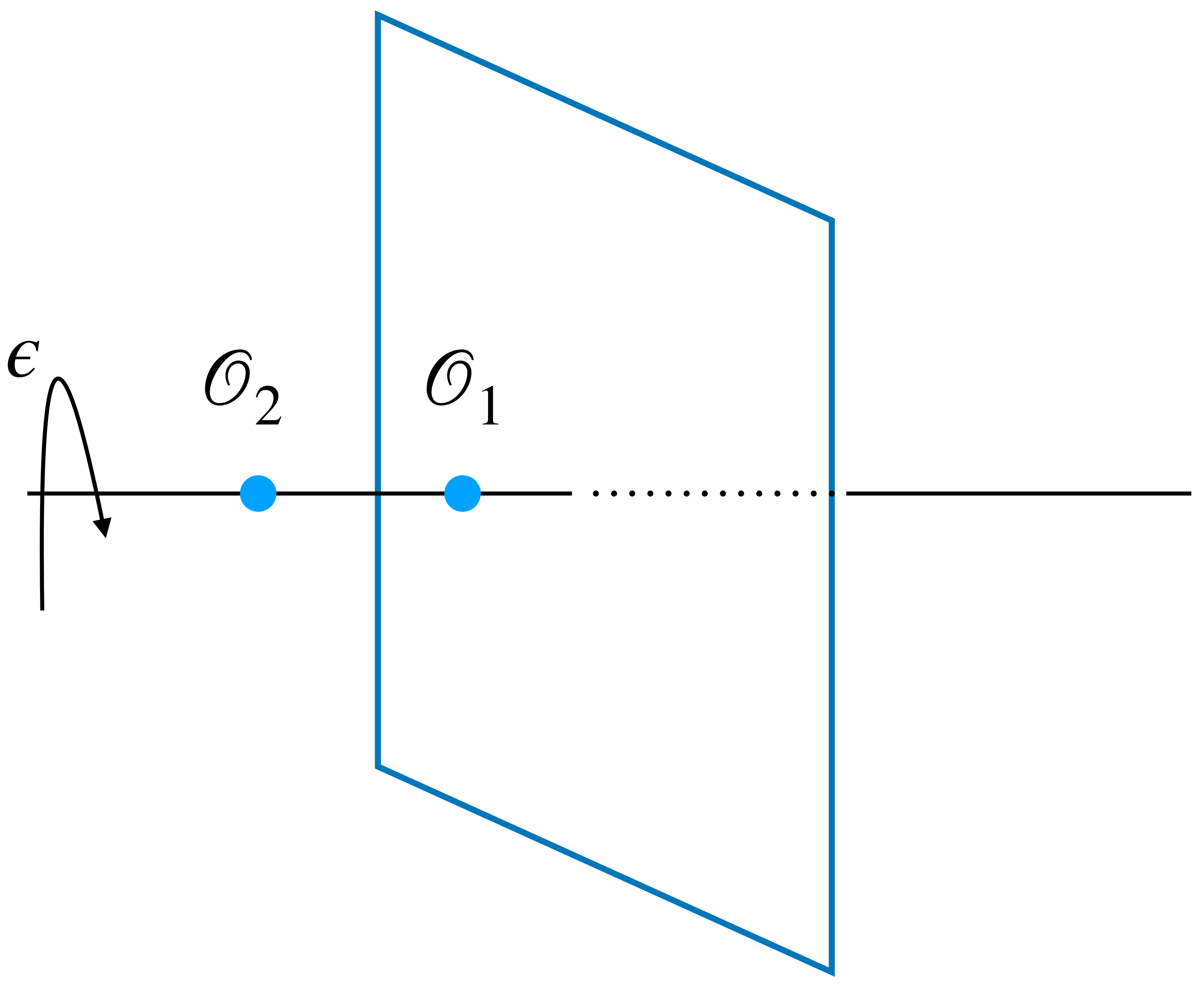
Turning on FI $\eta_{\mathbb{R}}$ restricts vacua to fixed points of tri-Hamiltonian isometry G_C

Algebras of Chiral Operators

- Operators which parametrise vacuum manifolds are annihilated by 2 supercharges. Call them *chiral.*
  - Higgs operators $Q_+^{1\dot{1}}, Q_-^{1\dot{2}}$. Coulomb operators $Q_+^{1\dot{1}}, Q_-^{2\dot{1}}$
  - By SCFT unitary bound arguments, we can just consider the cohomology of $Q_H = Q_+^{1\dot{1}} + Q_-^{1\dot{2}}$ and $Q_C = Q_+^{1\dot{1}} + Q_-^{2\dot{1}}$.
- Translation is Q-exact: $\frac{\partial}{\partial x} \approx \{Q, \tilde{Q}\}$.
- Chiral algebras are just the coordinate rings of the moduli spaces of vacua. $\mathbb{C}[\mathscr{M}_H]$ and $\mathbb{C}[\mathscr{M}_C]$.
- Explicitly: $\mathbb{C}[\mathscr{M}_H] = \mathbb{C}[X^i, Y^i]^G/(\mu_{\mathbb{C}} - \eta_{\mathbb{C}})$. Complex symplectic reduction of free ring.
- Coulomb branch is more difficult. Need to determine ring relations, Poisson bracket etc. Done in [Bullimore, Dimofte, Gaiotto]. Mathematically: [Braverman, Finkelberg, Nakajima].

Omega Deformation and Quantised Algebras

- Ω-deformation: deform such that $Q^2_{H,C} = \epsilon \mathscr{L}_V$.
- V is the Killing vector generating rotations about an axis in $\mathbb{R}^3$
- Localises to a quantum mechanics: a non-linear σ-model to either the Higgs or Coulomb branch [Yagi].
- Operator ordering now matters. We are lead to quantised algebras $\hat{\mathbb{C}}[\mathscr{M}_H]$ and $\hat{\mathbb{C}}[\mathscr{M}_C]$, which are non-commutative. Puts some hats on the operators.



Example: SQED[N]

Higgs Branch $\mathscr{M}_H = T^*P^{N-1}$

Classically: $\mathbb{C}[\mathscr{M}_H] = \mathbb{C}[T^*P^{N-1}]$

Quantised algebra: $\{Y_i, X_j\}_{\text{PB}} \mapsto [\hat{Y}_i, \hat{X}_j] = \epsilon,$
$i, j = 1,\ldots,N$

Gauge invariant generators:

$$\left.\begin{aligned} h_j &= \hat{X}_j\hat{Y}_j - \hat{X}_{j+1}\hat{Y}_{j+1}\,, \\ e_j &= \hat{X}_j\hat{Y}_{j+1} \quad j = 1,\ldots,N-1\,, \\ f_j &= \hat{X}_{j+1}\hat{Y}_j \quad j = 1,\ldots,N-1\,. \end{aligned}\right\} \quad U(\mathfrak{sl}_N)$$

Impose complex moment map: $\sum_{j=1}^{N} {:}\hat{X}_j\hat{Y}_j{:} = \eta_{\mathbb{C}}$

Fixes Casimirs, so $\hat{\mathbb{C}}[\mathscr{M}_H]$ is a central quotient of $U(\mathfrak{sl}_N)$

Coulomb Branch $\mathscr{M}_C$

Unresolved: A_{N-1} singularity

Classically, $\mathscr{M}_C$: $\quad v^+v^- = \varphi^N$

Quantised algebra:

$$[\hat{\varphi}, \hat{v}_\pm] = \pm\,\epsilon\hat{v}_\pm\,,$$

$$\hat{v}_+\hat{v}_- = \prod_{i=1}^{N}\left(\varphi + m_{i,\mathbb{C}} - \frac{\epsilon}{2}\right)\,,$$

$$\hat{v}_-\hat{v}_+ = \prod_{i=1}^{N}\left(\varphi + m_{i,\mathbb{C}} + \frac{\epsilon}{2}\right)\,,$$

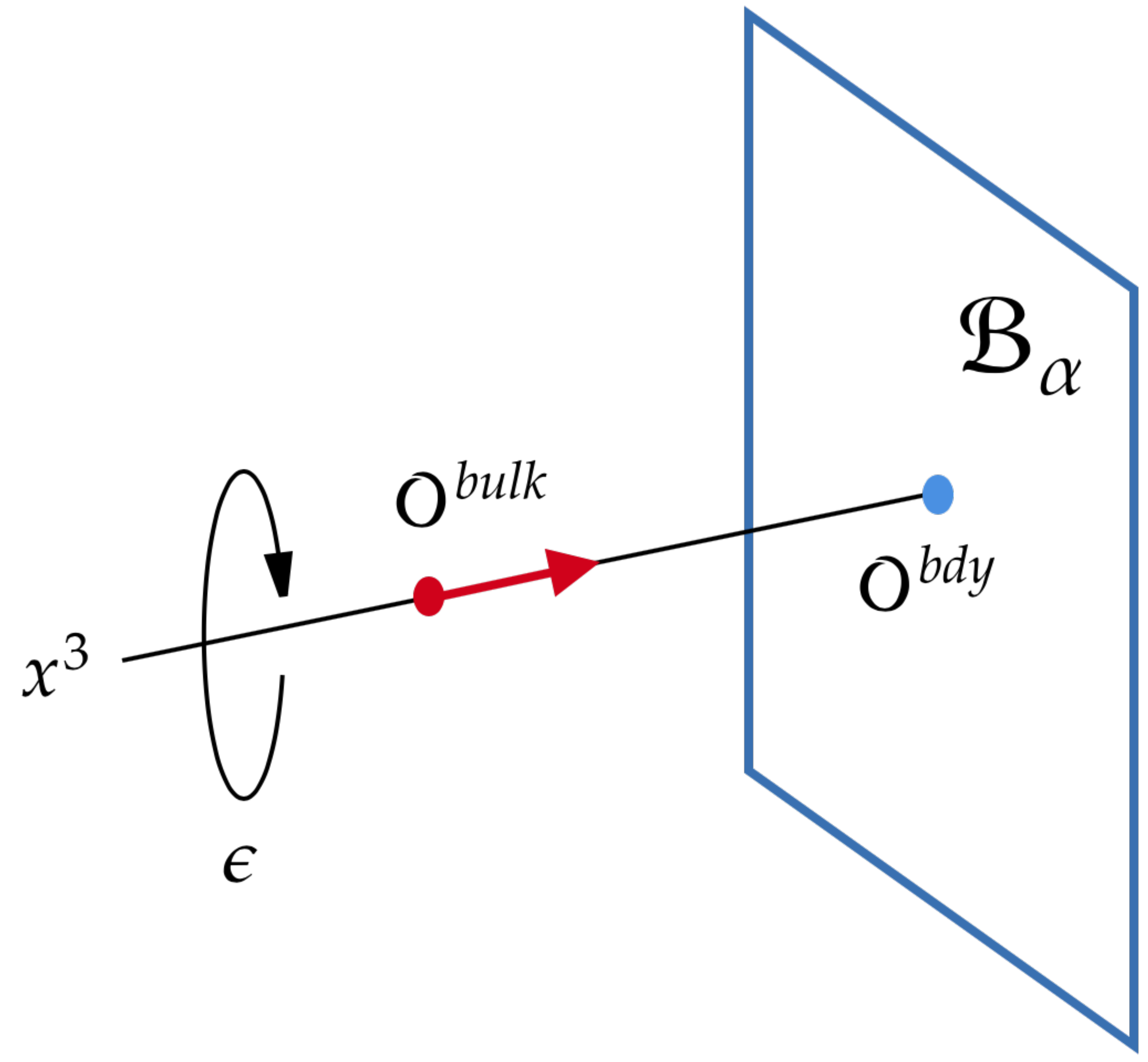
which is a spherical rational Cherednik algebra.

Boundary Conditions and Modules

- Enrich the set-up by introducing $\mathcal{N} = (2,2)$ boundary conditions on $\mathbb{R}^2$. Identify $R_H = R_V$, $R_C = R_A$
- Certain fields/operators are supported by the boundary condition. Both Coulomb and Higgs branch operators.
- Turning on an $\Omega_{H,C}$-deformation (Higgs or Coulomb) - we get a module of $\hat{\mathbb{C}}[\mathcal{M}_H]$ or $\hat{\mathbb{C}}[\mathcal{M}_C]$, by bringing bulk operators to the boundary [Bullimore, Dimofte, Gaiotto, Hilburn].
- There can be boundary 't Hooft anomalies. For a $3d$ $\mathcal{N} = 4$ theory, the only possibilities are the following mixed anomalies:

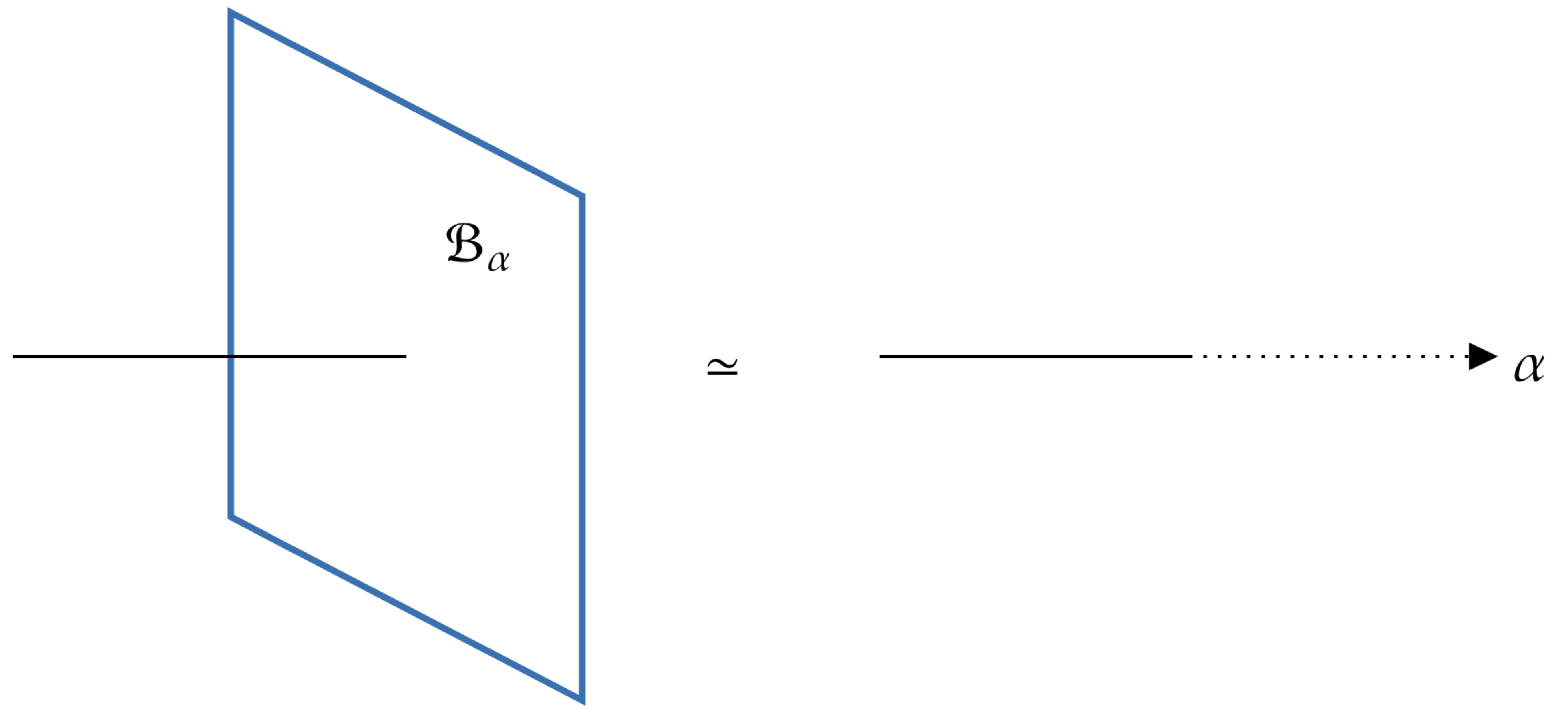
$$T_H - T_C\,, \quad T_H - R_A\,, \quad T_C - R_V\,, \quad R_V - R_A$$

$$k\,, \qquad k_A\,, \qquad k_V\,, \qquad \tilde{k}$$



$= \langle \mathcal{O}^{\text{bulk}} | \mathcal{O}^{\text{bdy}} \rangle$

Exceptional Dirichlet and Verma Modules



- Turning on $m_{\mathbb{R}}, \eta_{\mathbb{R}}$, we have distinct, isolated vacua α.
- Focus on special class of BC, 'thimbles', which mimic a vacuum α at infinity for a choice of chamber of $m_{\mathbb{R}}, \eta_{\mathbb{R}}$. Preserves $T_H \times T_C$ - 'exceptional'.
- (2,2) BPS equations at boundary are Morse flow, with respect to Morse function e.g. $m_{\mathbb{R}} \cdot \mu_{H,\mathbb{R}}$.
- We choose a Lagrangian splitting of the hyper-multiplets X^i, Y^i such that the image of the boundary condition is the attracting Lagrangian $\mathscr{L}_\alpha$ of the fixed point $\alpha \in \mathscr{M}_H$.
- Also impose $A_{||} = 0$ - 'Dirichlet', this supports boundary monopole operators.

Exceptional Dirichlet and Verma Modules

- [Bullimore, Dimofte, Gaiotto, Hilburn] argue on general grounds this yields lowest weight Verma modules $\mathscr{V}_{\alpha}^{H,C}$ for $\hat{\mathbb{C}}[\mathscr{M}_{H,C}]$, for each vacuum. Lowest weight is w.r.t. e.g. $\hat{\mu}_{H,\mathbb{C}}$.

- We find the charge of the lowest weight state is given by the anomalies induced by the boundary condition:

$$\hat{\mu}_{H,\mathbb{C}} \,|\, \mathscr{B}\rangle^{H} = \left(\frac{1}{2}k_A + \frac{1}{\epsilon}\eta_{\mathbb{C}} \cdot k\right) |\, \mathscr{B}\rangle^{H}$$

$$\hat{\mu}_{C,\mathbb{C}} \,|\, \mathscr{B}\rangle^{C} = \left(\frac{1}{2}k_V + \frac{1}{\epsilon}\eta_{\mathbb{C}} \cdot k\right) |\, \mathscr{B}\rangle^{C}$$

Example: SQED[N]

Higgs $\mathscr{M}_H$. Fix a chamber $m_{\mathbb{R},1} < \ldots < m_{\mathbb{R},N}$.

Thimble boundary condition for i^{th} vacuum is:

$$\begin{aligned} \partial_\perp Y_j = 0, \quad & X_j = c\delta_{ij} \quad & j \leq i \\ \partial_\perp X_j = 0, \quad & Y_j = 0 \quad & j > i\,. \end{aligned}$$

The quantisation of the bulk operators acts as:

$$\begin{aligned} \hat{Y}_j = \times Y_j\,, \quad & \hat{X}_j = \epsilon\partial_{Y_j} + c\delta_{ij} \quad & j \leq i \\ \hat{X}_j = \times X_j\,, \quad & \hat{Y}_j = \partial_{X_j} \quad & j > i\,. \end{aligned}$$

The states in the module are polynomials in the scalars assigned Neumann BC. The lowest weight state $|\mathscr{B}_i\rangle$ associated to this module is represented by 1. Identifying generating set of raising operators:

$$\begin{aligned} f_{i,j} = \hat{X}_i\hat{Y}_j \quad & j < i \\ f_{k,j} = \hat{X}_k\hat{Y}_i \quad & k < i \end{aligned}$$

We measure weights with respect to $h_m := \sum_{j=1}^{N} m_j :\hat{\mu}_{H,j,\mathbb{C}}: = \sum_{j=1}^{N} m_j :\hat{X}_j\hat{Y}_j:$. On the vacuum:

$$h_m|\mathscr{B}_i\rangle = \left[\frac{\epsilon}{2}\left(\sum_{j>i}(m_j - m_i) - \sum_{j<i}(m_j - m_i)\right) + \eta_{\mathbb{C}}m_i\right]|\mathscr{B}_i\rangle$$

The coefficients of the fugacities precisely encodes the boundary 't Hooft anomalies induced by the boundary condition.
For example, the $m_i\eta_{\mathbb{C}}$ term precisely encodes the mixed Higgs/Coulomb branch flavour symmetry anomaly.

Half Indices

$$\mathcal{I}_{\mathcal{B}_\alpha} = \mathrm{Tr}(-1)^F q^{J+\frac{R_V+R_A}{4}} t^{\frac{R_V-R_A}{2}} x^{F_H} \xi^{F_C} \,,$$

- Associated to a given boundary condition $\mathcal{B}_\alpha$.
- A count of boundary operators in $Q_+^{1\dot{1}}$ cohomology. Includes Higgs and Coulomb branch chiral operators, but also many others.
- Gradings by Higgs and Coulomb branch flavour symmetries, and also particular combinations of boundary R-symmetries which commute with the supercharge.
- Starts at 1 - the identity operator is uncharged.
- Derived in [Dimofte, Gaiotto, Paquette]

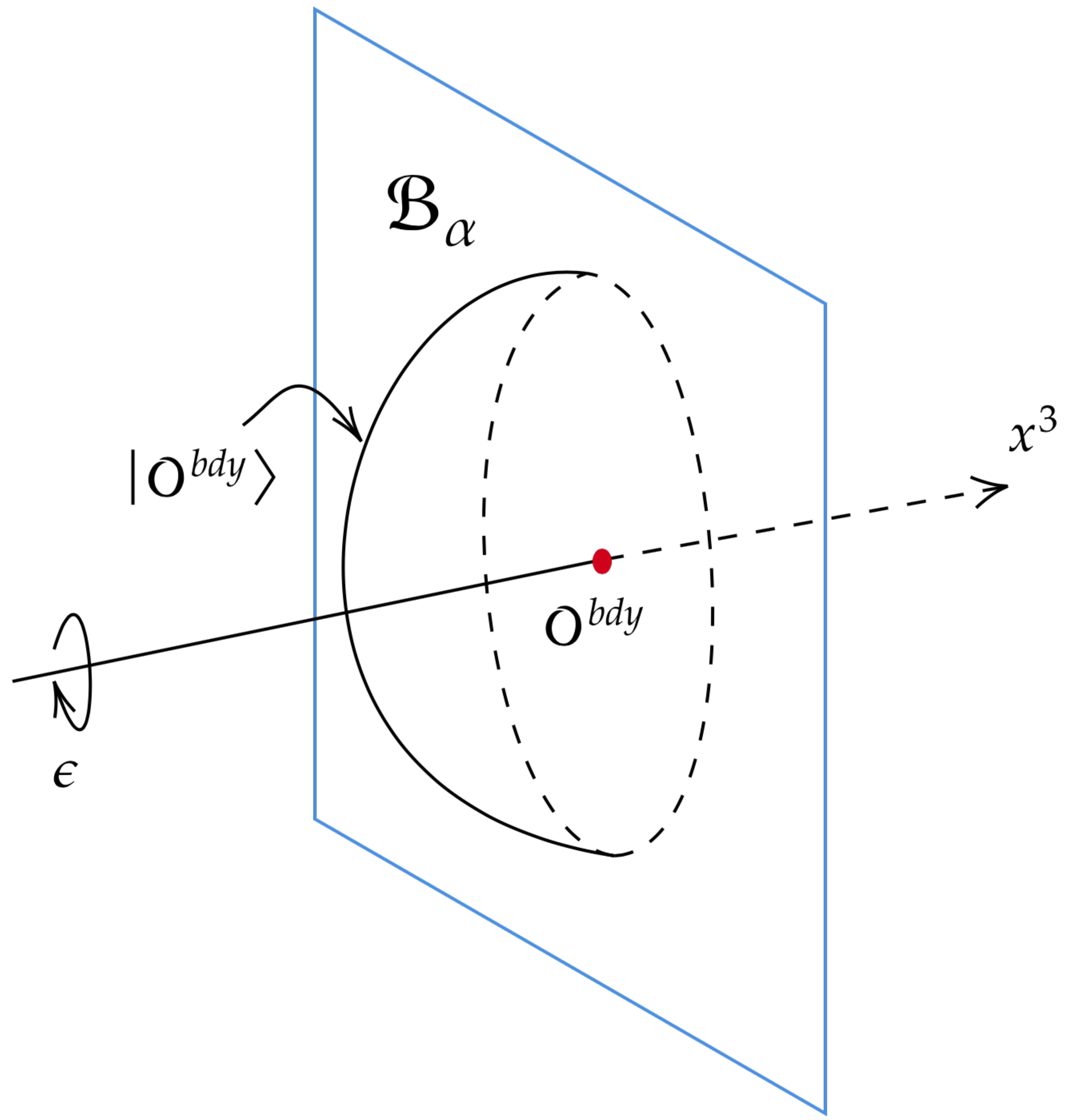
Hemisphere Partition Function

- Alternatively, via state-operator correspondence, we compute the partition function on $S^1 \times H^2$. The boundary condition $\mathscr{B}_\alpha$ for a vacuum α is imposed on the boundary T^2.
- Counts states on a hemisphere, annihilated by $Q_+^{1\dot{1}}$.
- Compute via supersymmetric localisation. For a Dirichlet boundary condition on the vector multiplet: BPS configurations are monopole configurations on the hemisphere.

$$\mathscr{Z}_{\mathscr{B}_\alpha}^{S^1\times H^2} = \lim_{\delta\to\infty} \int \mathscr{D}\Phi e^{-S[\Phi]-\delta Q\cdot V[\Phi]}$$

$$= \sum_{\mathfrak{m}\in\mathbb{Z}^k} e^{-S_{\text{cl}}[\Phi^{(0)}]} Z_{\text{1-loop}}(q, z, x, \xi, \mathfrak{m}),$$

- Related to half index by a ‘Casimir energy’ e^ϕ.



Casimir Energies, Boundary ’t Hooft Anomalies

$$\mathcal{Z}^{S^1\times H^2}_{\mathcal{B}_\alpha} = e^{\phi_{\mathcal{B}_\alpha}} \mathcal{I}_{\mathcal{B}_\alpha} = e^{\phi_{\mathcal{B}_\alpha}} \mathcal{Z}_{1-loop} \mathcal{Z}_{\mathsf{V}}$$

- Casimir Energy is exactly the anomaly polynomial for the boundary ’t Hooft anomaly (actually true for $\mathcal{N}=2$ theories too).

$$\begin{aligned}\phi_{\mathcal{B}} = &\frac{1}{\log q}\left[\log\left(q^{1/4}t^{1/2}\right)\cdot\tilde{k}\cdot\log\left(q^{1/4}t^{-1/2}\right)\right]\\ &+\frac{1}{\log q}\left[\log\left(q^{1/4}t^{-1/2}\right)\cdot k_A\cdot\log x\right]\\ &+\frac{1}{\log q}\left[\log\xi\cdot k_V\cdot\log\left(q^{1/4}t^{1/2}\right)\right]\\ &+\frac{1}{\log q}\left[\log\xi\cdot k\cdot\log x\right]\end{aligned}$$

- For an empty bulk, this is just the statement on $(2,2)$ elliptic genera in [Bobev, Bullimore, Kim].

Higgs and Coulomb Character Limits

$$\mathcal{Z}^{S^1\times H^2}_{\mathcal{B}_\alpha} = e^{\phi_{\mathcal{B}_\alpha}}\mathcal{I}_{\mathcal{B}_\alpha} = e^{\phi_{\mathcal{B}_\alpha}}\mathcal{Z}_{1-loop}\mathcal{Z}_{\text{V}}$$

$$\lim_{t\to q^{-\frac{1}{2}}}$$

Additional commuting supercharge $Q_-^{1\dot{2}}$

$$\mathcal{X}^H_{\mathcal{B}_\alpha}(x) = \text{Tr}_{\mathcal{H}^H_{\mathcal{B}_\alpha}} x^{J_H}$$

$$e^{\phi_{\mathcal{B}_\alpha}} \to x^{\frac{k_A}{2}+k\cdot\frac{\log\xi}{\log q}}$$

$\mathcal{Z}_{1-loop} \to$ Verma character denominator

$$\mathcal{Z}_V \to 1$$

$$\lim_{t\to q^{\frac{1}{2}}}$$

Additional commuting supercharge $Q_-^{2\dot{1}}$

$$\mathcal{X}^C_{\mathcal{B}_\alpha}(x) = \text{Tr}_{\mathcal{H}^C_{\mathcal{B}_\alpha}} \xi^{J_C}$$

$$e^{\phi_{\mathcal{B}_\alpha}} \to \xi^{\frac{k_V}{2}+k\cdot\frac{\log x}{\log q}}$$

$$\mathcal{Z}_{1-loop} \to 1$$

$\mathcal{Z}_V \to$ Verma character denominator

In both cases: the limit of the Casimir energy precisely yields the character of the highest weight states of the respective Vermas! ($\log q \to \epsilon$). This was missing from the half-index.

Example: SQED[N]

For the i^{th} vacuum, and Thimble boundary condition B_i :

k_A, mixed $T_H - R_A$ anomaly

$$\mathcal{X}_i^H = \lim_{t \to q^{-\frac{1}{2}}} \mathcal{Z}_{\mathcal{B}_i} = e^{\frac{\log \xi \log(x_i)}{\log q}} \prod_{j<i} \frac{\left(x_i/x_j\right)^{1/2}}{1 - x_i/x_j} \prod_{j>i} \frac{\left(x_j/x_i\right)^{1/2}}{1 - x_j/x_i}$$

k, mixed $T_H - T_C$ anomaly

Verma character denominators
~ bulk raising operators

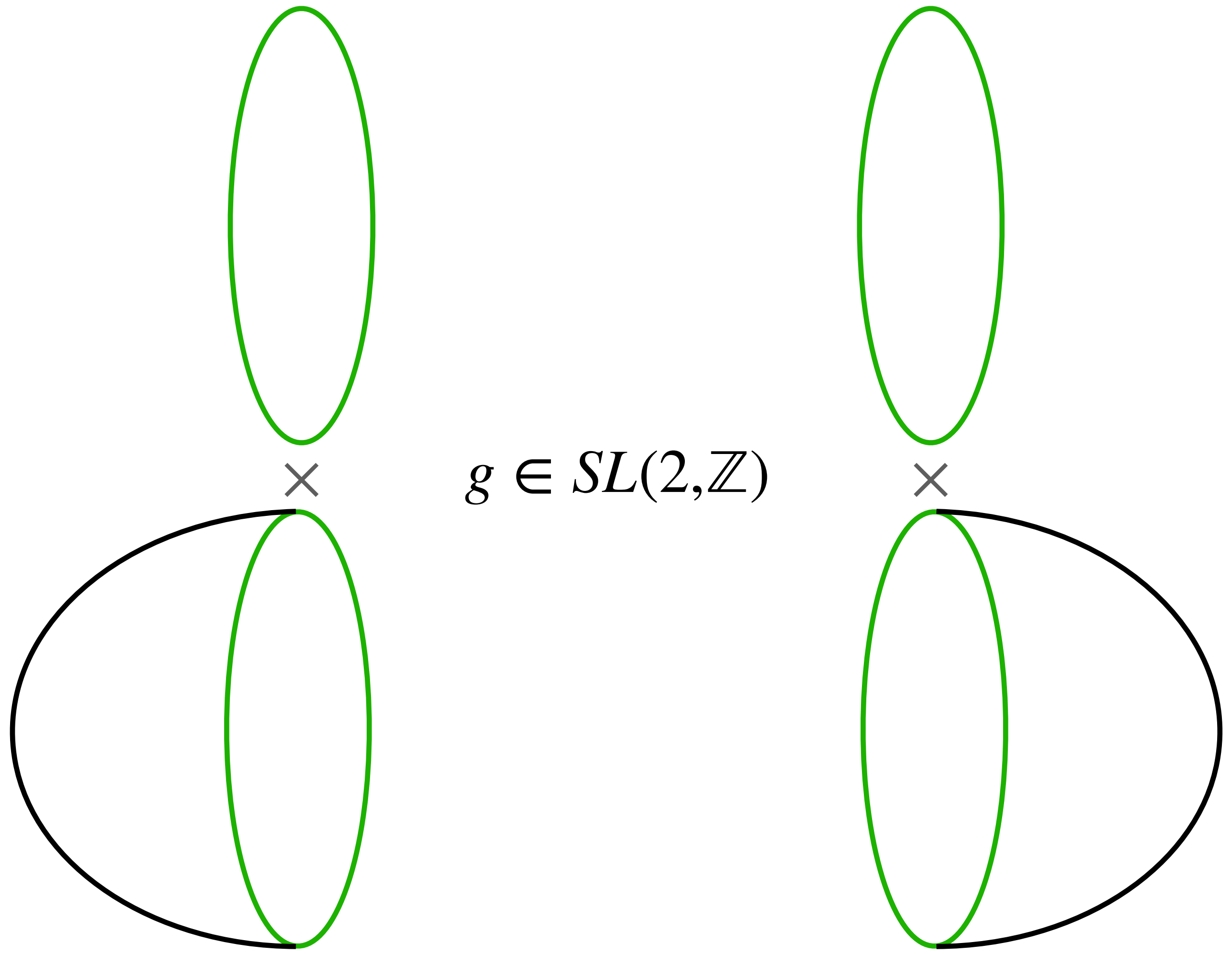
k_V, mixed $T_C - R_V$ anomaly

$$\mathcal{X}_i^C = \lim_{t \to q^{\frac{1}{2}}} \mathcal{Z}_{\mathcal{B}_i} = e^{\frac{\log \xi \log(x_i)}{\log q}} \frac{\xi^{\frac{1}{2}}}{1 - \xi}$$

Holomorphic Factorisation

- Partition functions of $\mathcal{N} \geq 2$ theories on closed 3-manifolds have been shown to factorise, at least partially:

$$\mathcal{M}_3 = \left(S^1 \times H^2\right) \cup_g \left(S^1 \times H^2\right)$$

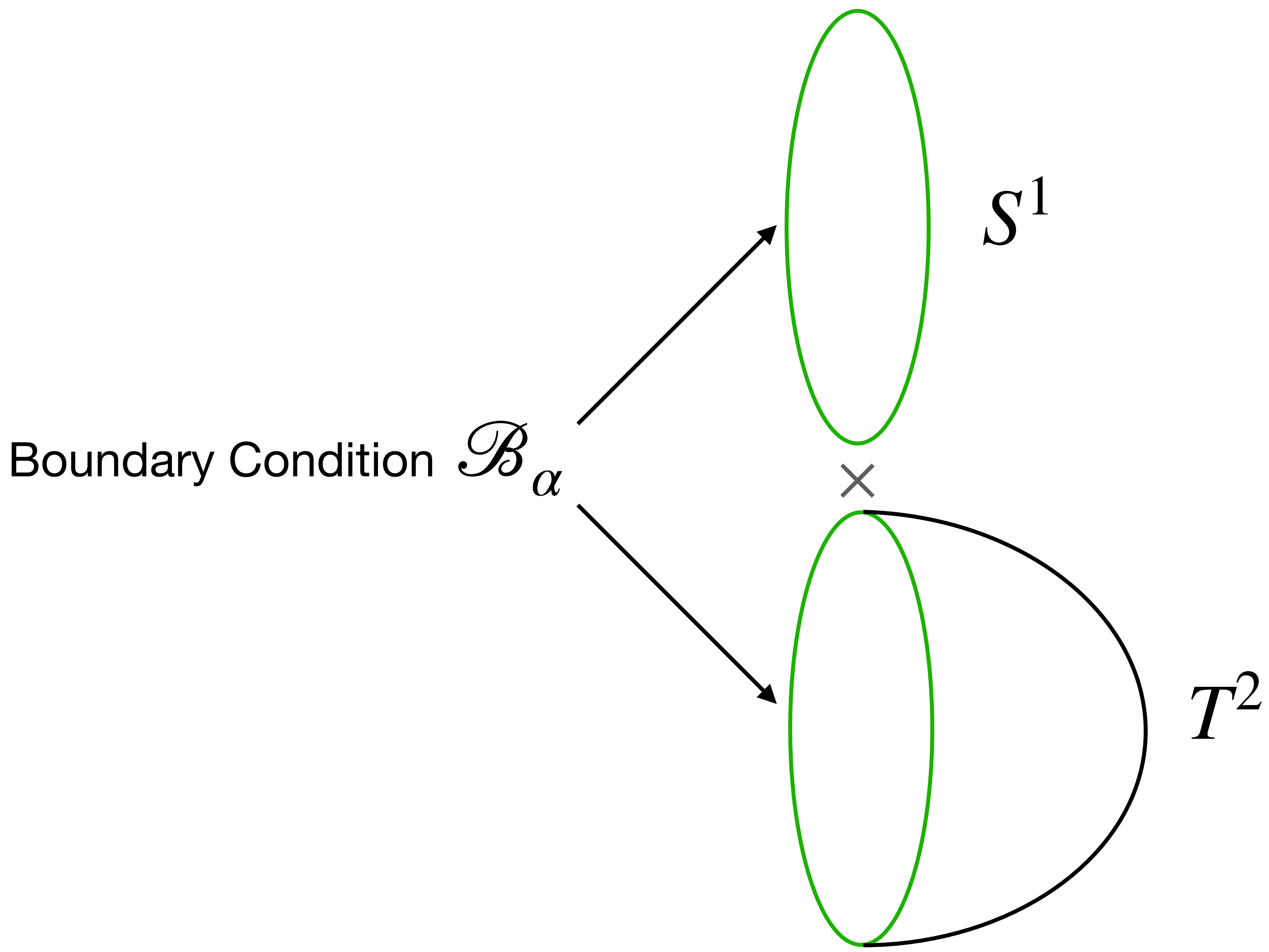


$$\mathcal{Z}_{\mathcal{M}_3} \simeq \sum_\alpha H_\alpha \tilde{H}_\alpha$$

- Holomorphic blocks [Beem, Dimofte, Pasquetti] yields fairly systematic approach, but is an IR calculation. H_α there identified with partition function on infinite cigar obtained by stretching a hemisphere geometry. No exact deformation of $\mathcal{M}_3$ into two copies of cigar geometry exists, and ambiguity in the classical piece.

Holomorphic Factorisation

- Motivated by the fact that UV exceptional Dirichlet/ thimble boundary conditions mimic a vacuum at infinity, for $\mathcal{N} = 4$ theories we propose our basis of hemisphere partition functions associated to vacua as the blocks. We find an exact factorisation (incl. classical pieces)!



$$H_\alpha = \mathscr{Z}_{\mathscr{B}_\alpha}$$

IR Formulae

- Corollary: Various limits of closed 3-manifold partition functions (such as the superconformal index, twisted index and squashed ellipsoid S_b^3 partition function) can be expressed in terms of Verma characters!

[Gaiotto, Okazaki]

$$\mathcal{Z}_{S_b^3} = \sum_\alpha \chi_\alpha^H(x)\chi_\alpha^C(\xi)$$

$$\mathcal{Z}_{\text{SC}}^B = \sum_\alpha \chi_\alpha^H(x)\chi_\alpha^H(x^{-1}) \qquad \mathcal{Z}_{\text{tw}}^B = \sum_\alpha \chi_\alpha^H(x)\chi_\alpha^H(x)$$

$$\mathcal{Z}_{\text{SC}}^A = \sum_\alpha \chi_\alpha^C(\xi)\chi_\alpha^C(\xi^{-1}) \qquad \mathcal{Z}_{\text{tw}}^A = \sum_\alpha \chi_\alpha^C(\xi)\chi_\alpha^C(\xi)$$

Other aspects & future directions

- Non-abelian examples, e.g. 3d ADHM [Crew, Dorey, DZ]
- Leverage factorisation/mathematical understanding of these half-indices to evaluate large N and Cardy limits - these should yield entropy functionals for black hole microstates for theories with AdS duals.
- Enumerative geometry - vortex moduli spaces
- The elliptic stable envelopes [Aganagic, Okounkov]. Mirror symmetry of boundary conditions and modules.