From tree-level perturbation theory to the S-matrix bootstrap in two dimensions

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Based on a ongoing work with Patrick Dorey

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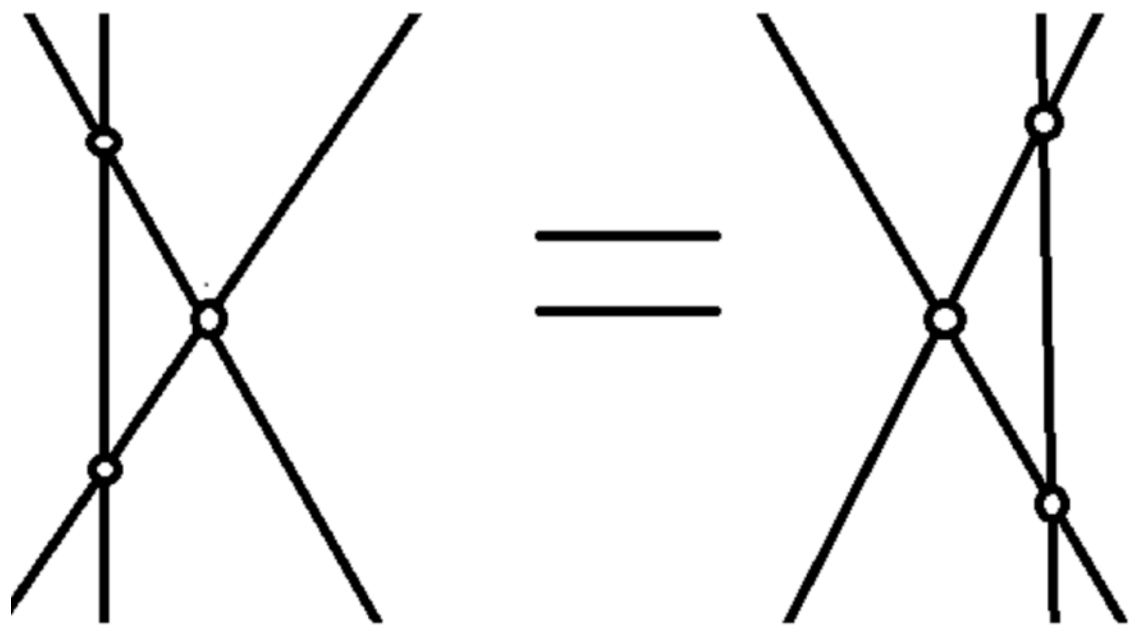
INTRODUCTION

Integrable theories in two dimensions are characterised by the presence of higher spin conserved charges that constrain the scattering processes to be diagonal and factorised.

$$S_{3\to3} = S_{2\to2}S_{2\to2}S_{2\to2}$$



Axiomatic procedure

The exact S-matrix for a variety of 2d integrable quantum field theories has been found in the last 30 years using bootstrap.

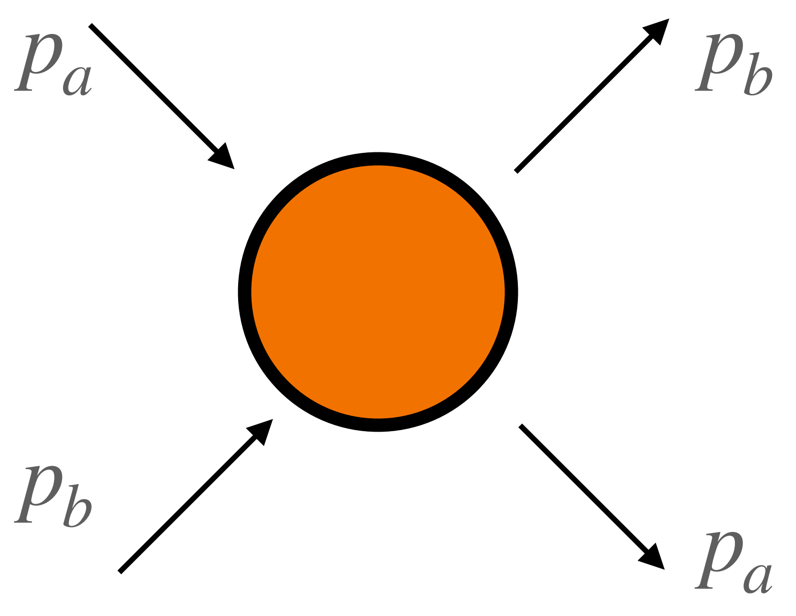
Perturbation theory

Integrability manifests itself in a surprising cancellation of Feynman diagrams contributing to non-elastic scattering (incoming state $\neq$ outgoing state). This mechanism is not completely understood also at the tree-level.

We focus on a class of integrable theories in (1+1) dimensions described by a scalar Lagrangian of the form

$$L = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} m_a^2 \phi_a^2 - \frac{1}{3!} C_{abc}\phi_a\phi_b\phi_c - \frac{1}{4!} C_{abcd}\phi_a\phi_b\phi_c\phi_d - \dots$$

Analyticity structure of 4 point S-matrix in two dimensions

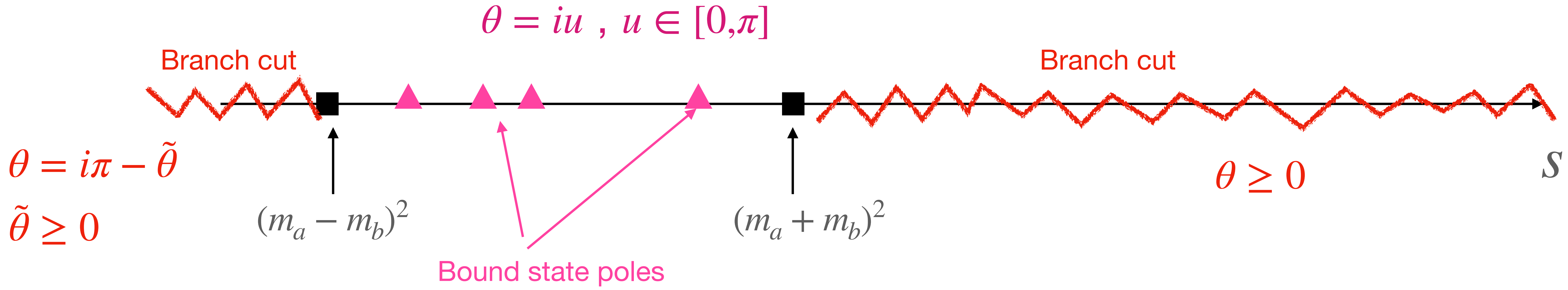


$$p_j = m_j \, (\cosh \theta_j, \sinh \theta_j)$$

$$s = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2 m_a m_b \cosh \theta \qquad \geq (m_a + m_b)^2$$

$\theta = \theta_a - \theta_b$

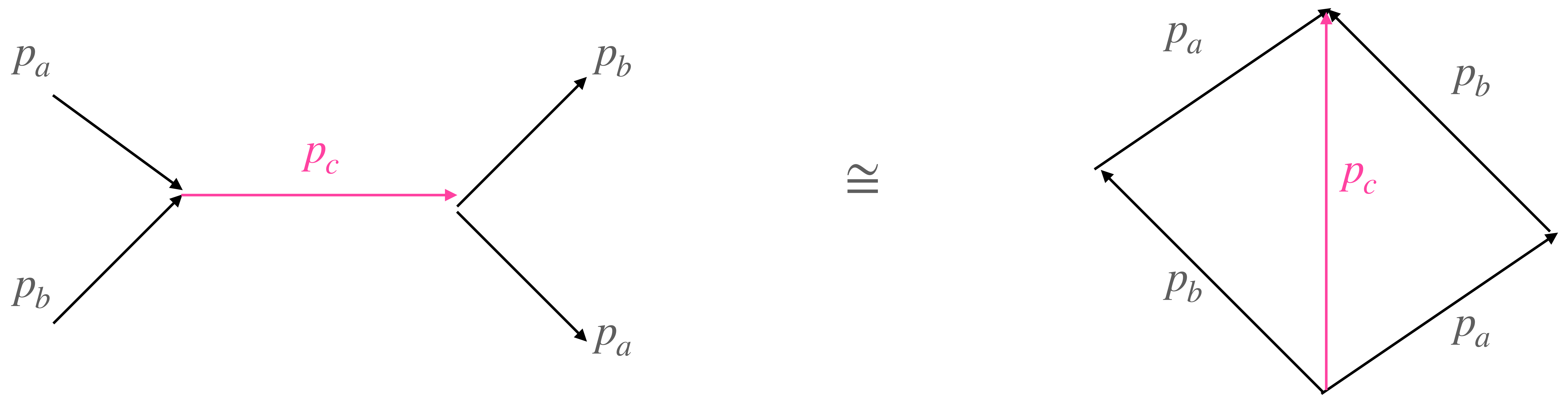
Condition on physical momenta



On the bound state region momenta are complex numbers having absolute value equal to their masses

$$p_a = m_a \, (\cos u_a, i \sin u_a) \cong m_a e^{i u_a} \qquad\qquad p_b = m_b \, (\cos u_b, i \sin u_b) \cong m_b e^{i u_b}$$

If c is a possible bound state propagating particle (i.e. $C_{ab\bar{c}} \neq 0$) on the pole position $s = m_c^2$
Feynman diagrams have a geometrical meaning



On-shell dual description of Feynman diagrams

While on the LHS the lengths and the directions of the momenta do not have a physical interpretation, on the RHS they are meaningful. Arrow lengths correspond to the masses of propagating particles and their directions correspond to the associated rapidities.

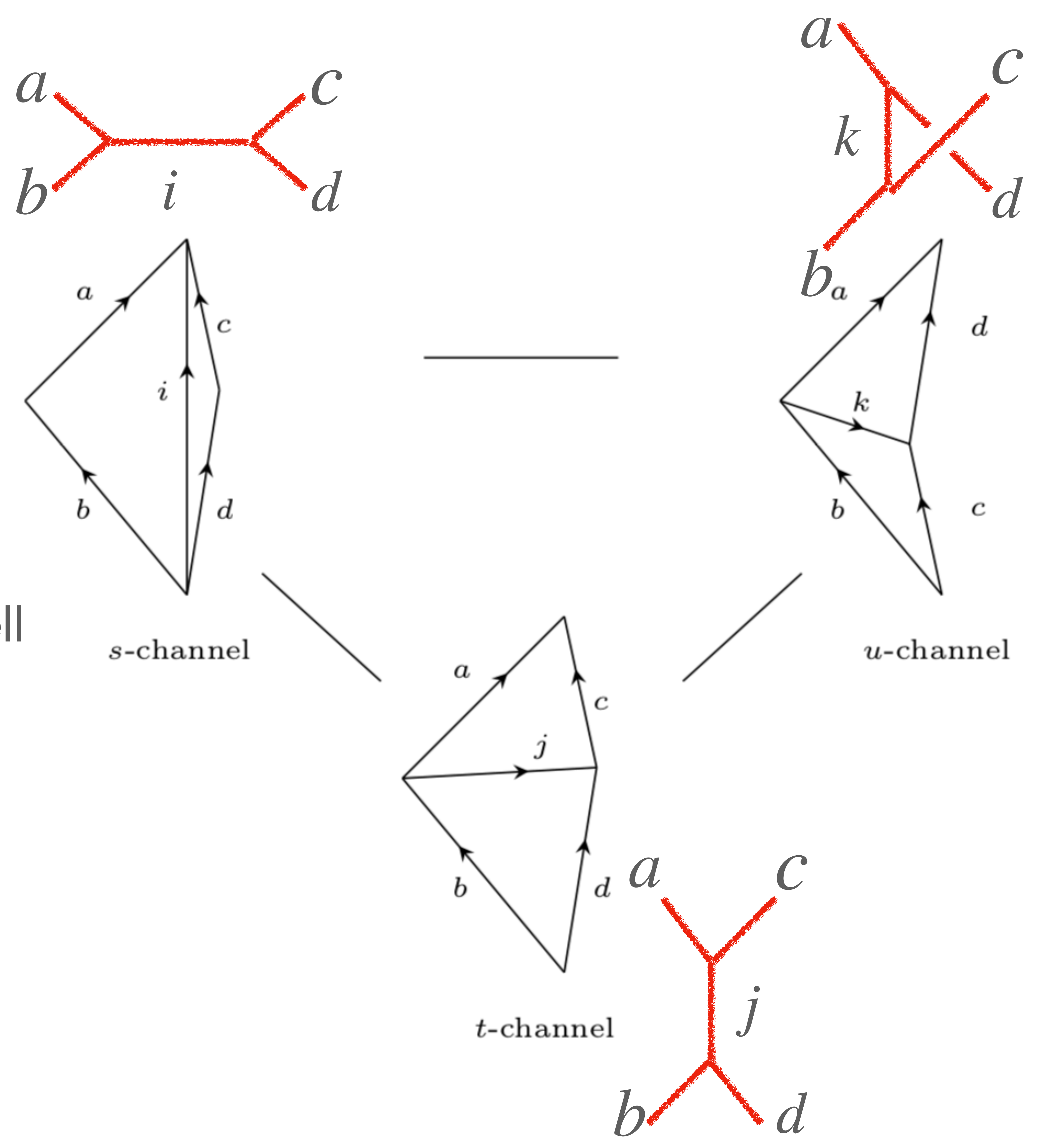
Each non-zero vertex can be represented by a triangle having as sides the masses of the fusing particles

A FEW WORDS ABOUT CANCELLATION OF 4-POINT NON-DIAGONAL PROCESSES

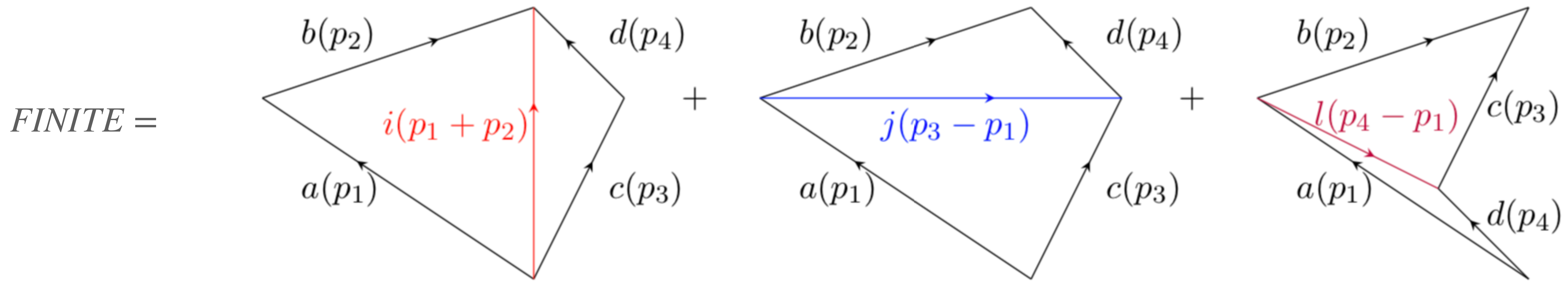
Poles in non-diagonal processes of the form $a + b \to c + d$ cancel through a **flipping rule**.

If for a choice of external kinematic an internal propagator goes on-shell generating a pole, then there is an other propagator in an other channel going on-shell for the same choice of the kinematic.

In this way there are always copies (or triplets) of singular Feynman diagrams that cancel each other.



FINDING THE RESIDUES OF 4-POINT NON-ALLOWED PROCESSES



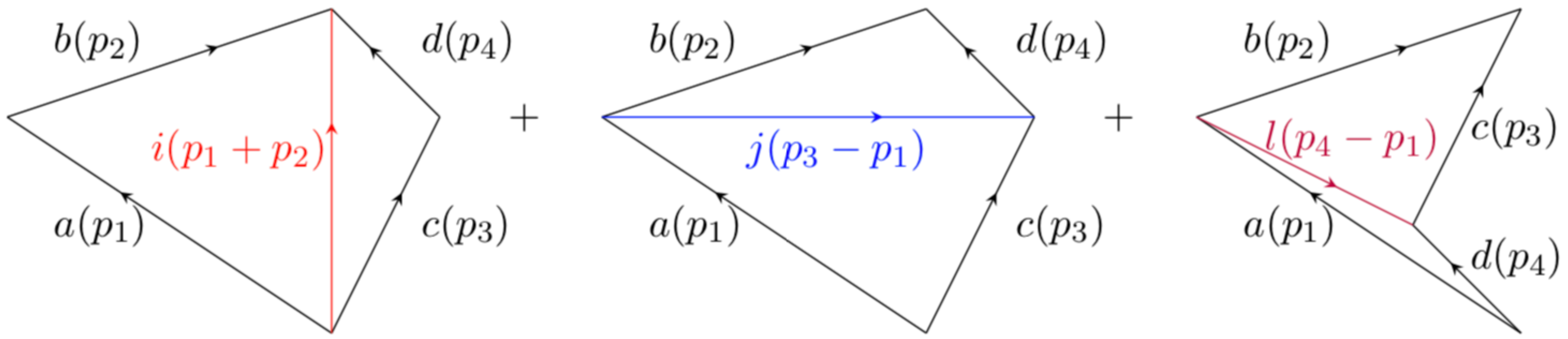
In 2 dimensions the Mandelstam variables t and u can be expressed as functions of s

$$\left.\frac{dt}{ds}\right|_{s=m_i^2} = -\frac{\Delta_{acj}\Delta_{jbd}}{\Delta_{abi}\Delta_{icd}} \qquad \left.\frac{du}{ds}\right|_{s=m_i^2} = \frac{\Delta_{adl}\Delta_{lbc}}{\Delta_{abi}\Delta_{icd}}$$

$$t = m_j^2 - \frac{\Delta_{acj}\Delta_{jbd}}{\Delta_{abi}\Delta_{icd}}(s - m_i^2) + \dots \qquad u = m_l^2 - \frac{\Delta_{adl}\Delta_{lbc}}{\Delta_{abi}\Delta_{icd}}(s - m_i^2) + \dots$$

FINDING THE RESIDUES OF 4-POINT NON-ALLOWED PROCESSES

FINITE =



$$\sim \frac{1}{s-m_i^2}\left[\frac{C_{ab\bar{i}}C_{i\bar{c}\bar{d}}}{\Delta_{abi}\Delta_{icd}} - \frac{C_{a\bar{c}j}C_{\bar{j}b\bar{d}}}{\Delta_{acj}\Delta_{jbd}} + \frac{C_{a\bar{d}l}C_{\bar{l}b\bar{c}}}{\Delta_{adl}\Delta_{lbc}}\right]$$

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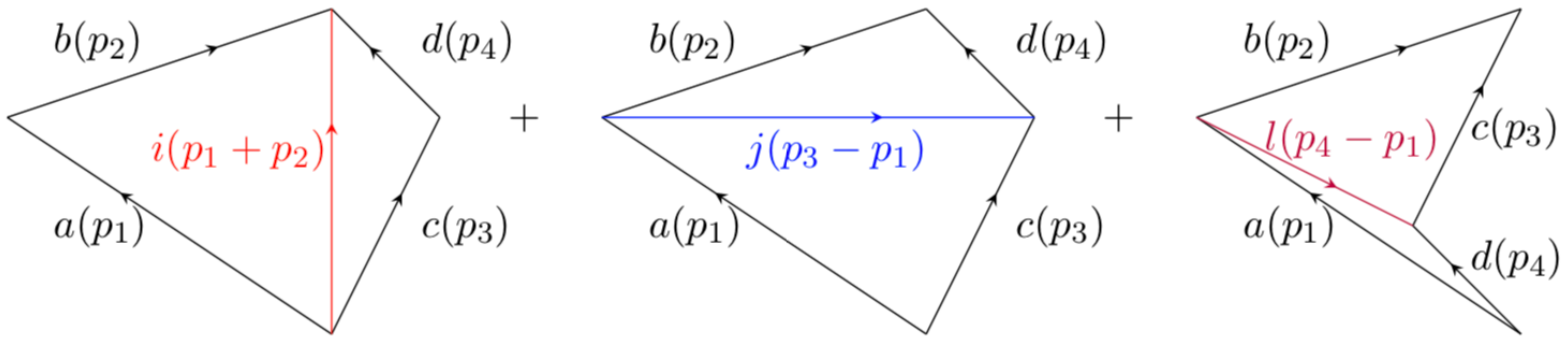

$$\sim \frac{1}{s-m_i^2}\left[\frac{C_{ab\bar{i}}C_{i\bar{c}\bar{d}}}{\Delta_{abi}\Delta_{icd}} - \frac{C_{a\bar{c}j}C_{\bar{j}b\bar{d}}}{\Delta_{acj}\Delta_{jbd}} + \frac{C_{a\bar{d}l}C_{\bar{l}b\bar{c}}}{\Delta_{adl}\Delta_{lbc}}\right]$$

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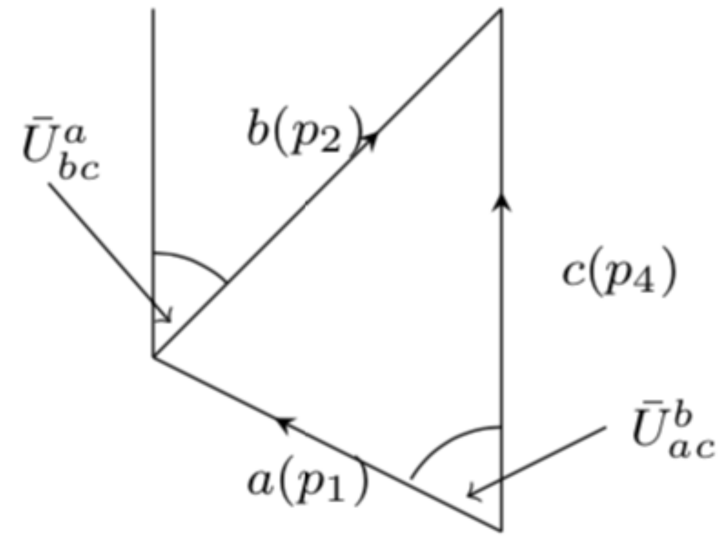
$$C_{abc} = f_{abc}\Delta_{abc} \qquad f_{ab\bar{i}}f_{i\bar{c}\bar{d}} - f_{a\bar{c}j}f_{\bar{j}b\bar{d}} + f_{a\bar{d}l}f_{\bar{l}b\bar{c}} = 0$$

TREE-LEVEL BOOTSTRAP RELATIONS FROM CANCELLATION OF 5-POINT EVENTS

If we set a, b, c on-shell

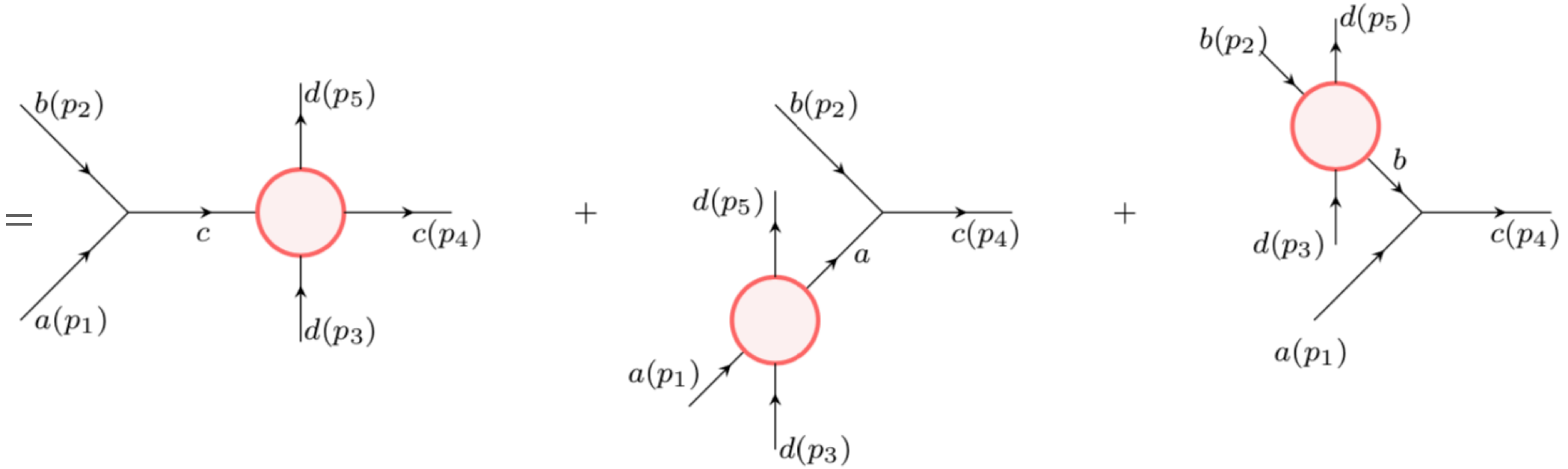


$$a(p_1) + b(p_2) + d(p_3) \to c(p_4) + d(p_5)$$

$$\theta_{14} = i\bar{U}^{b}_{ac}$$

$$\theta_{42} = i\bar{U}^{c}_{bc}$$

FINITE =



$$\sim \frac{C_{ab\bar{c}}}{\theta_5 - \theta_3}\left[S^{tree}_{dc}(\theta_{34}) - S^{tree}_{da}(\theta_{31}) - S^{tree}_{db}(\theta_{32})\right]$$

Defining $\theta_{34} \equiv \theta$ the imposition of not having poles in the 5-point process above becomes

$$S^{tree}_{dc}(\theta) = S^{tree}_{da}(\theta + \theta_{41}) + S^{tree}_{db}(\theta + \theta_{42}) = S^{tree}_{da}(\theta - i\bar{U}^{b}_{ac}) + S^{tree}_{db}(\theta + i\bar{U}^{a}_{bc})$$

What about affine Toda theories?

Let $\mathscr{G}$ be a semisimple Lie algebra and $\mathscr{H}$ its Cartan sub-algebra spanned by $\{h_a\}_{a=1}^r$

We can define a set of simple roots $\{\alpha_a\}_{a=1}^r$ obtained by diagonalising the basis $\{h_a\}_{a=1}^r$

$$[h_a, h_b] = 0 \qquad [h_a, e_\alpha] = \alpha_a e_\alpha \qquad [e_\alpha, e_\beta] = N_{\alpha,\beta}\, e_{\alpha+\beta}$$

$$\mathscr{L} = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{m^2}{g^2}\sum_{i=0}^{r} n_i e^{g\alpha_i^a \phi_a}$$

$n_0 = 1$

n_i : set of integer numbers characteristic for each of the affine Dynkin diagram such that

$$\alpha_0 = -\sum_{i=1}^{r} n_i \alpha_i$$

Expanding the potential around $\phi = 0$ and diagonalising the mass term $M_{ab}^2 = m^2 \sum_{i=0}^{r} n_i \alpha_i^a \alpha_i^b$

we can start applying perturbation theory

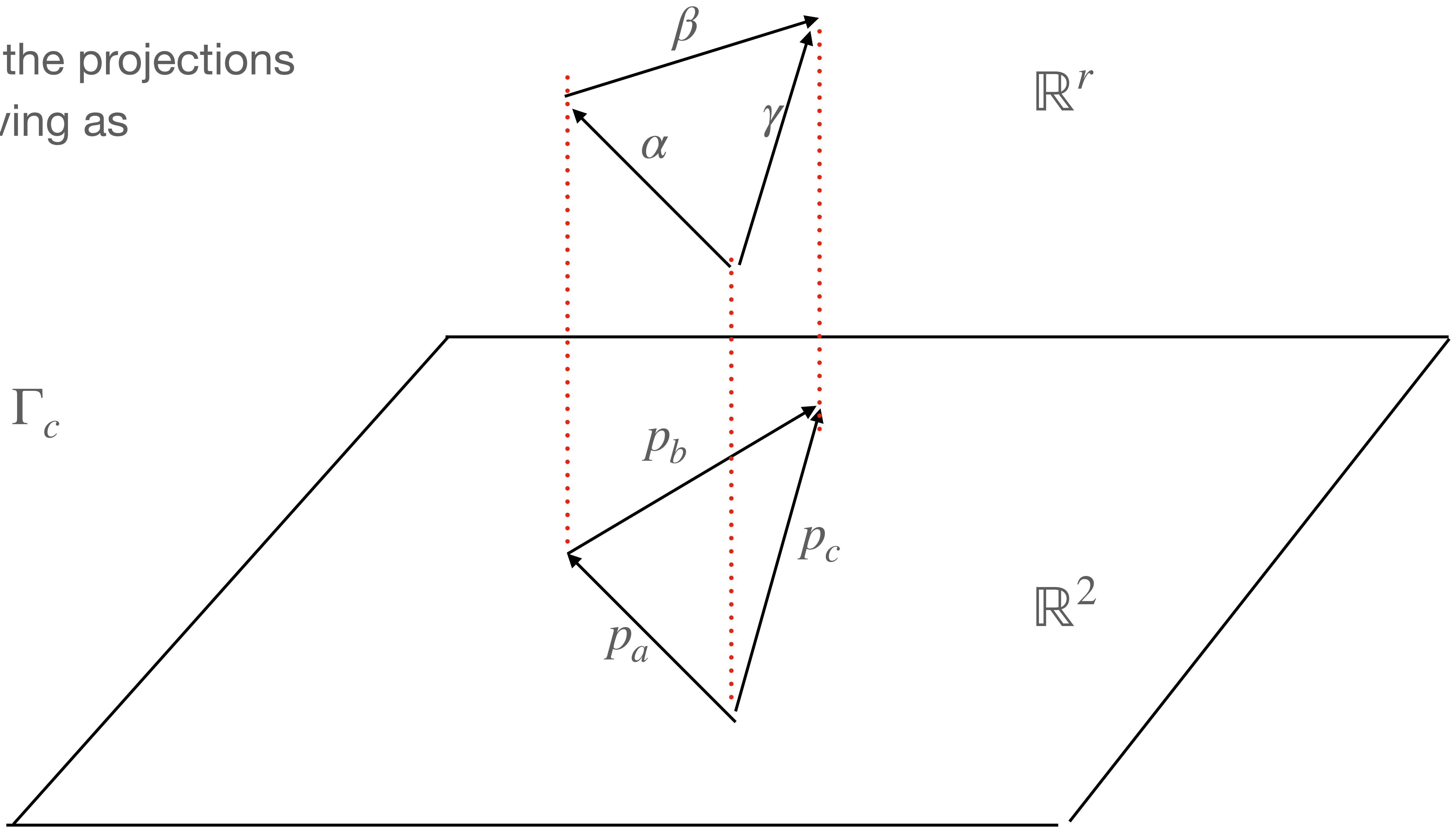
Coxeter element $w = \prod_{i=1}^{r} w_i$

Weyl reflection respect to the plane perpendicular to the simple root α_i

w defines r orbits having h elements each one:

$$\Gamma_a: \quad \gamma_a \to w\gamma_a \to w^2\gamma_a \to \ldots \to w^{h-1}\gamma_a \to \gamma_a \qquad a = 1,\ldots,r$$

In affine Toda theories we discover that the on-shell momenta are the projections of the roots on the plane defined by the two eigenvectors of w having as eigenvalues $exp(\pm\frac{2\pi i}{h})$



Property 1: $C_{ab\bar{c}} \neq 0$ iff $\exists\ \alpha + \beta = \gamma$ with $\alpha \in \Gamma_a,\ \beta \in \Gamma_b,\ \gamma \in \Gamma_c$

Property 2: If property 1 is satisfied then we have

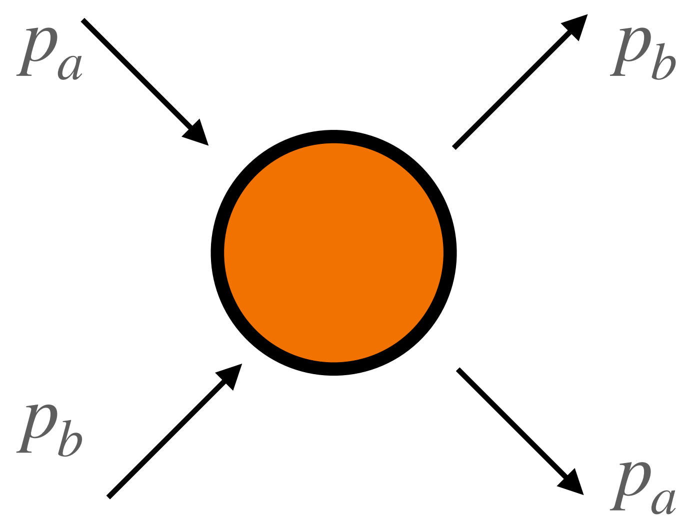
$$C_{ab\bar{c}} = \frac{4g}{\sqrt{h}} \Delta_{abc} N_{\alpha,\beta} sign\left[\sin(u_a - u_b)\right]$$

$$f_{ab\bar{c}} \equiv \frac{4g}{\sqrt{h}} N_{\alpha,\beta} sign\left[\sin(u_a - u_b)\right]$$

A UNIVERSAL FORMULA FOR ALLOWED SCATTERING

$$M_{ab} = -\,i\sum_i \frac{|C_{abi}|^2}{s-m_i^2} - i\sum_j \frac{|C_{a\bar{b}j}|^2}{t-m_j^2} - \frac{g^2}{h}\frac{m_a^2 m_b^2}{m^2}$$

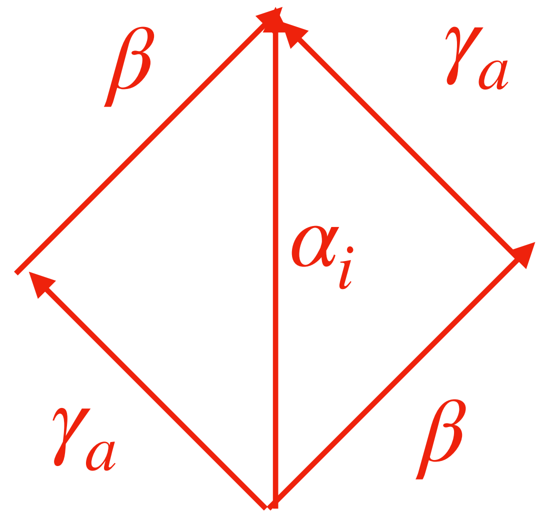
s-channels t-channels u-channels + 4-point coupling

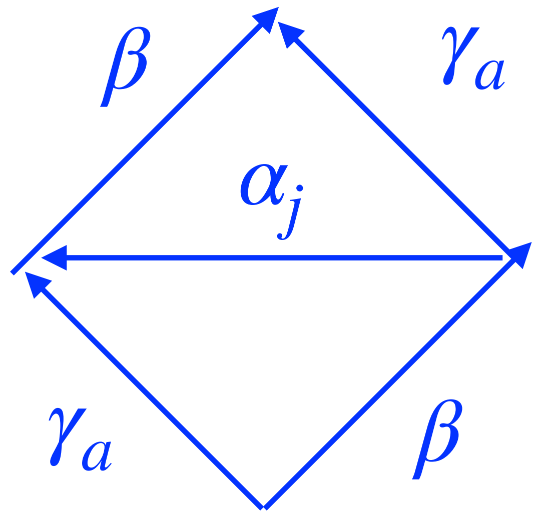


$$s = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2m_a m_b \cosh\theta$$

$$t = (p_a - p_b)^2 = m_a^2 + m_b^2 - 2m_a m_b \cosh\theta$$

$$u = (p_a - p_a)^2 = 0$$





It is possible absorbing this constant inside the sum by promoting $\sinh iu_{\gamma_a,\beta} \to \sinh\theta$

$$M_{ab} = \frac{ig^2}{h} m_a m_b \sum_{\substack{\beta\in\Gamma_b \\ \beta\neq\pm\gamma_a}} \left(|N_{\gamma_a,\beta}|^2 - |N_{\gamma_a,-\beta}|^2\right)\frac{\sinh^2 iu_{\gamma_a\beta}}{\cosh\theta - \cos u_{\gamma_a\beta}} - i\frac{g^2}{h}\frac{m_a^2 m_b^2}{m^2}$$

$\left(|N_{\gamma_a,\beta}|^2 - |N_{\gamma_a,-\beta}|^2\right) \to -(\gamma_a,\beta)$

$$M_{ab} = -\frac{ig^2}{h} m_a m_b \sum_{\beta\in\Gamma_b} (\gamma_a,\beta)\frac{\sinh^2\theta}{\cosh\theta - \cos u_{\gamma_a\beta}}$$

$$S_{ab}^{tree}(\theta) = \frac{1}{4m_a m_b \sinh\theta} M_{ab}(\theta)$$

CONCLUSIONS AND OUTLOOK

A. We have proved absence of particle production at tree-level in all the untwisted affine Toda theories connecting them with properties of the underlying Lie algebra

B. Tree-level bootstrap relations can be derived from the root system showing how different S-matrix elements are connected

C. Still remains to be understood the higher loop structure of the building blocks. Possible ways: optical theorem and dispersion relation

D. It would be interesting investigating the space of integrable theories by the only imposition of absence of particle production at the tree-level

Thank you

SAGEX
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