Dessins d'Enfants & Machine-Learning

Edward Hirst

City, University of London

edward.hirst@city.ac.uk

arXiv: 2004.05218

15th Dec, 2020

Overview

Dessins d'Enfants: Definition

Belyĭ's theorem

X has an algebraic model over $\overline{\mathbb{Q}}$ iff there exists a (surjective) map $\beta : X \mapsto \mathbb{P}^1$ which is ramified at exactly 3 points

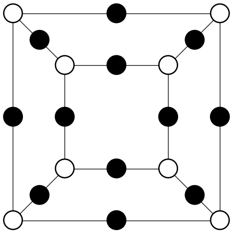
Defining the dessin bipartite graph

$\beta^{-1}(0) \mapsto \circ, \quad \beta^{-1}(1) \mapsto \bullet, \quad \beta^{-1}(0,1) \mapsto -$

Example dessin

$$\beta(z) = \frac{(z^8-14z^4+1)^3}{(-108(z(z^4+1))^4)}$$

Images $\beta(z)$	# Preimages	Ram Index
0	8	3
1	12	2
∞	6	4



Dessins d'Enfants: Motivation & Use

Quivers

Quivers diagrammatically represent gauge theories.

Quiver gauge theories with a toric VMS satisfy:

$$N_0 - N_1 + N_2 = 0 \implies \chi_{g=1}$$

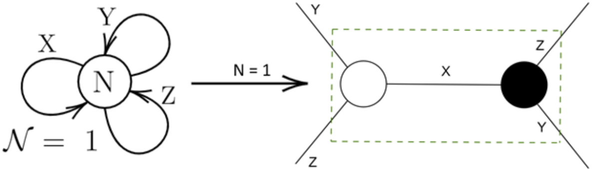
...hence important in string theories where the VMS's are $CY \subset X_{toric}$

Quivers $\mapsto$ Brane Tilings

Quiver & Superpotential $\mapsto$ 'brane-tiling' (i.e. dessin d'enfant!)

Quick example: At the $\mathcal{N} = 1$ level of $\mathcal{N} = 4$ SYM ...

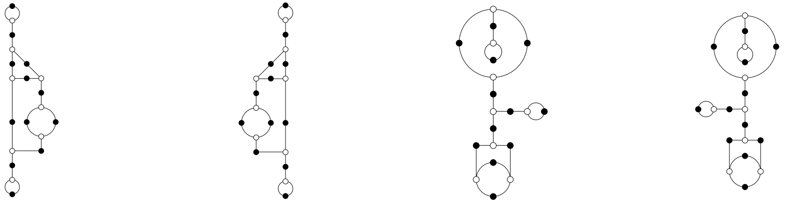
$$W = Tr(XYZ - XZY)$$



Dessins d'Enfants: Motivation & Use

Galois orbits

Absolute Galois group: $Gal(\,\overline{\mathbb{Q}}\,/\,\mathbb{Q}\,)$ ...acts faithfully on dessins
Orbit sizes = degree of field subextension
Example quartic orbit: $\sqrt[4]{a}\,/\,\mathbb{Q} \;\Leftarrow x^4 - a^4 = 0\,(x, a \in \mathbb{Q})$



Dessin Database used

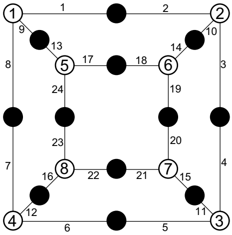
Motivated by 112 genus 0, torsion-free, index 24 subgroups of $PSL(2,\mathbb{Z})$
...under extended quotient action on $\mathscr{H} \times \mathbb{C} \Rightarrow$ elliptically-fibred K3
Modular surface's j-invariant $\Rightarrow$ Belyĭ maps (!) $\Rightarrow$ orbits of dessins
$\therefore$ 191 dessins, in orbits of size $1 \mapsto 4$

ML Dessins: Representation & ML

Data Representation

Tensors: $\begin{pmatrix} 0&1&0&1&1&0&0&0\\ 1&0&1&0&0&1&0&0\\ 0&1&0&1&0&0&1&0\\ 1&0&1&0&0&0&0&1\\ 1&0&0&0&0&1&0&1\\ 0&1&0&0&1&0&1&0\\ 0&0&1&0&0&1&0&1\\ 0&0&0&1&1&0&1&0 \end{pmatrix}$

$\{W : \{\{8,1,9\}, \{2,3,10\}, \{4,5,11\}, \{12,6,7\}, \{18,14,19\},$
$\{22,16,23\}, \{13,24,17\},\{21,20,15\}\}$, B: $\{\{1,2\},\{9,13\},\{10,14\},\{3,4\},$
$\{15,11\},\{24,23\},\{17,18\}, \{8,7\},\{5,6\},\{12,16\}, \{21,22\},\{20,19\}\}\}$



Machine-Learning Galois Orbit

$k = 5$
Train batches of 32 for 20 epochs
Layers: 4×512
Optimiser: *Adam*
Loss: *Sparse-categorical Cross-entropy*

Data Input Type	Accuracy	MCC	Bias
$\mathcal{M}_{8x8}$	0.536 $\pm$ 0.002	0.180 $\pm$ 0.008	0.54 $\pm$ 0.03
$\mathcal{V}_{48}$ (1000:0:0)	0.92 $\pm$ 0.03	0.88 $\pm$ 0.04	0.47 $\pm$ 0.13

Summary

Conclusion

• Simple NNs can classify Belyĭ maps according to the field they're defined over to accuracy > 0.9 (from just respective dessins' combinatoric data).
• Perhaps implies the field information is encoded in the combinatoric data somehow.

Outlook

• Further research aims to examine other dessin databases (associated to other modular groups / CY manifolds).
• Perhaps shift focus to genus 1 dessins with more relevant physical interpretation.
• Trial use of 'Representation learning' to extract the relevant combinatoric information for Galois orbit size.