Double Copy and Homotopy Algebras

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Based on joint work (2007.13803; 2012.?????)
with L Borsten, B Jurčo, T Macrelli, C Saemann, M Wolf
(all errors/oversimplifications are mine alone)

Summary

- Kinematic Jacobi identity, double copy:

$$\text{YM ampl.} = \frac{c_i^{a_1\dots a_n} n_i^{\mu_1\dots\mu_n}}{d_i} \qquad \text{GR ampl.} = \frac{n_i^{\mu_1\dots\mu_n} n_i^{\nu_1\dots\nu_n}}{d_i}$$

- Cubic action manifesting double copy, with BRST:

$$S_{\text{YM}} = c_{ab} C_{ij} A^{ai} \Box A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$

$$S_{\text{GR}} = C_{ij} C_{i'j'} A^{ii'} \Box A^{jj'} + F_{ijk} F_{i'j'k'} A^{ii'} A^{jj'} A^{kk'}$$

$$(Q_{\text{YM}} A)_{ai} = c_{ab} Q_{ij} A^{bj} + f_{abc} Q_{ijk} A^{bj} A^{ck}$$

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- Field theory = cyclic L_∞-algebra
- Factorise field theories as L_∞-algebras:

$$\mathfrak{YM} = \mathfrak{colour} \otimes \mathfrak{kinematic} \otimes_\tau \mathfrak{Scalar}$$

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Kinematic Jacobi identity

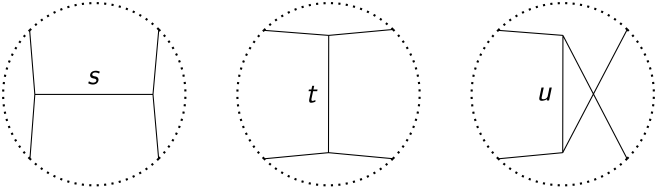
$$\text{YM } n\text{-pt } l\text{-loop amplitude} = \epsilon^{(1)}_{a_1\mu_1} \cdots \epsilon^{(n)}_{a_n\mu_n} \int_{\text{loops}} \sum_{i\in\Gamma_{n,l}} \frac{c_i^{a_1\ldots a_n} n_i^{\mu_1\ldots\mu_n}}{|\text{Aut}(i)| d_i}$$

- c_i: **colour factor** (contraction of group structure constants $f^{a_1a_2a_3}$)
- n_i: **kinematic numerator** (polynomial of momenta p^μ)
- d_i: **denominator** ($\prod_{e\in\text{internal edges}} p_e^2$)
- $\epsilon^{(1)}, \ldots, \epsilon^{(n)}$: polarisation vectors
- $\Gamma_{n,l}$: set of trivalent graphs with n external legs, l loops

c_i and n_i have same (anti)symmetry properties

Kinematic Jacobi identity[1,2]

. . . such that n_i behaves like c_i



$$c_s + c_t + c_u = 0 \qquad \text{(colour Jacobi)}$$

$$n_s + n_t + n_u = 0 \qquad \text{(kinematic Jacobi)}$$

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[1] Bern–Carrasco–Johansson 0805.3993

[2] Proof of tree-level kinematic Jacobi: Stieberger 0907.2211, Bjerrum-Bohr–Damgaard–Vanhove 0907.1425

Double copy[3]

$$\text{YM amplitude} = \epsilon^{(1)}_{a_1\mu_1}\cdots\epsilon^{(n)}_{a_n\mu_n}\int_{\text{loops}}\sum_{i\in\Gamma_{n,l}}\frac{c_i^{a_1\dots a_n}n_i^{\mu_1\dots\mu_n}}{|\operatorname{Aut}(i)|d_i}$$

$$\text{Gravity amplitude} = \epsilon^{(1)}_{\mu_1\nu_1}\cdots\epsilon^{(n)}_{\mu_n\nu_n}\int_{\text{loops}}\sum_{i\in\Gamma_{n,l}}\frac{n_i^{\mu_1\dots\mu_n}n_i^{\nu_1\dots\nu_n}}{|\operatorname{Aut}(i)|d_i}$$

$$\epsilon_{\mu\nu} = \text{graviton} + \text{Kalb–Ramond 2-form} + \text{dilaton}$$

known tree kinematic Jacobi	$\Longrightarrow$	known tree double copy
loop kinematic Jacobi not known	$\Longrightarrow$	loop double copy what we want to show

[3] Bern–Carrasco–Johansson 0805.3993, 1004.0476; Bern–Dennen–Huang–Kiermaier 1004.0693

Tolotti–Weinzierl action[4]

Is there an action such that the Feynman diagrams are the trivalent graphs?

$$\mathcal{L} = A \cdot \Box A + \sum_{n=3}^{\infty} \sum_{i \in \Gamma_n} \frac{c_i^{a_1 \cdots a_n} N_i^{\mu_1 \cdots \mu_n}}{d_i} A_{a_1 \mu_1} \cdots A_{a_n \mu_n}$$

- at $n = 4$, the A^4 term is written as $\Box A^4 / \Box$, such that the s-, t-, and u-channels are separated
- at $n \geq 5$, the $\mathcal{O}(A^n)$ term $= 0$ due to colour Jacobi identities
- Order n terms correct wrong partition of the n-point amplitude given by using only order $< n$ vertices

[4] Bern–Dennen–Huang–Kiermaier 1004.0693; Tolotti–Weinzierl 1306.2975

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BRST and all that

1. BRST symmetry

$$Q: \qquad A \mapsto \mathrm{D}c \qquad \underbrace{c}_{\text{ghost}} \mapsto cc \qquad \overbrace{\bar{c}}^{\text{antighost}} \mapsto \underbrace{b}_{\text{Nakanishi–Lautrup}} \mapsto 0$$

2. Gauge condition

$$G[A] = \partial \cdot A + \cdots$$

3. Gauge-fixed action

$$\begin{aligned} \mathcal{L} &= F^2 + Q\big((G[A] + \xi b)\bar{c}\big) \\ &= \underbrace{F^2}_{\text{physical}} + \underbrace{(G[A] + \xi b)b}_{\text{gauge-fixing}} + \underbrace{(\partial \cdot \mathrm{D}c)\bar{c} + \cdots}_{\text{ghost}} \end{aligned}$$

Kinematic Jacobis for unphysically polarised gluons

Kinematic Jacobi identity *fails* for unphysically polarised gluons ($\epsilon_i \cdot p_i \neq 0$) in the usual gauges.
We need new terms like

$$\mathcal{L}' = \frac{(\partial \cdot A)^2 A^2}{\Box} + \cdots$$

to ensure kinematic Jacobi identity without using transversality.
These terms can be generated by a careful choice of gauge-fixing condition.
These change amplitudes of unphysical gluons, but not physical (transverse) gluons.

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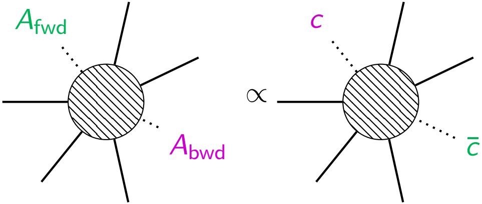
Ward identities

Ward identities relate on-mass-shell (anti)ghost amplitudes with on-mass-shell amplitudes of unphysical gluon modes.



As a byproduct of the modified gauge condition, we get extra ghost terms.

$$\mathcal{L}' = \frac{(\partial \cdot A)^2 A^2}{\Box} + \frac{\bar{c}\partial c \cdot A \partial \cdot A}{\Box} + \cdots$$

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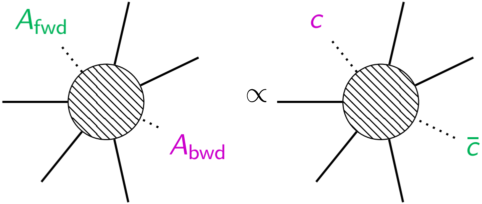

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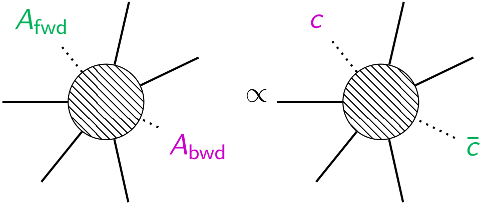

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Strictification[5]

Put in auxiliary fields to make the action local and cubic

$$S = c_{ab} C_{ij} A^{ai} \Box A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$

where

- a, b, c: adjoint colour index
- i, j, k: DeWitt index over all particle species (gluon, ghost, antighost, Nakanishi–Lautrup, strictification auxiliary fields)
- c_{ab}, f_{abc}: structure constants of colour Lie algebra (with invariant bilinear form)
- C_{ij}, F_{ijk}: "structure constants" of "kinematic algebra"
  - F_{ijk}: trilinear differential operator
  - C_{ij}: bilinear form

[5] Cf. prior work on "kinematic structure constants": Monteiro–O'Connell 1105.2565, 1311.1151; Bjerrum-Bohr–Damgaard–M–O'C 1203.0944; Chen–Johansson–Teng–Wang 1906.10683

Strictified action with ghosts

A *cubic action* such that, for a given trivalent tree topology Γ_i,

$$\sum \text{all Feynman diagrams for topology } \Gamma_i$$
$$= \epsilon^{(1)}_{a_1\mu_1} \cdots \epsilon^{(n)}_{a_n\mu_n} \frac{c_i^{a_1\ldots a_n} n_i^{\mu_1\ldots\mu_n}}{|\operatorname{Aut}(i)| d_i}$$
$$(\text{mod terms} \propto \textstyle\sum_i p_i \text{ or } \propto p_i^2 \text{ for external } p_i)$$

- We *don't* assume transversality ($\epsilon_i \cdot p_i = 0$) of the external gluons
- External ghosts, longitudinal gluons also satisfy colour Jacobi identity

Double-copied action

$$\mathcal{L}(\mathfrak{YM}) = c_{ab} C_{ij} A^{ai} \Box A^{bj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$
$$\mathcal{L}(\mathfrak{GR}) = C_{ij} C_{i'j'} A^{ii'} \Box A^{jj'} + F_{ijk} F_{i'j'k'} A^{ii'} A^{jj'} A^{kk'}$$

By construction (tree-level double copy), double-copied action gives correct tree-level amplitudes of $\mathcal{N} = 0$ supergravity.

Does it give correct loop amplitudes?

$\Leftrightarrow$ Is it correctly quantised (i.e. has BRST)?

Double copy of BRST

$$\mathcal{L}_{\text{YM,strict}} = c_{ab} C_{ij} A^{ai} \square A^{bj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$
$$(QA)_{ai} = c_{ab} Q_{ij} A^{bj} + f_{abc} Q_{ijk} A^{bj} A^{ck}$$

$\Downarrow$ double copy

$$\mathcal{L}_{\text{GR,strict}} = C_{ij} C_{i'j'} A^{ii'} \square A^{jj'} + F_{ijk} F_{i'j'k'} A^{ii'} A^{jj'} A^{kk'}$$
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Suffices if (C_{ij}, F_{ijk}) behaves like (c_{ab}, f_{abc}):

$$c_{ab} = c_{(ab)} \qquad f_{abc} = f_{[abc]} \qquad f_{[ab|d} c^{dd'} f_{d'|c]e} = 0$$
$$C_{ij} = C_{(ij)} \qquad F_{ijk} = F_{[ijk]} \qquad F_{[ij|l} C^{ll'} F_{l'|k]m} = 0$$

We don't quite have this because we only have tree-level kinematic Jacobi identity (possible violation up to $\mathcal{O}(\square)$). A field redefinition argument gets around this issue.

Thus, the double-copied action correctly describes $\mathcal{N} = 0$ SUGRA.

Double copy of BRST

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Homotopy algebras

- Abstract nonsense gadget
  - In: definition of a kind of algebra (quadratic operad)
  - Out: definition of a kind of algebra generalising the original definition ("algebraic identity up to homotopy")

In	Out
Lie algebra	L_∞-algebra
associative[6] algebra	A_∞-algebra
comm. assoc. algebra	comm. A_∞-algebra ("C_∞-")

- Can be given an invariant inner product (**cyclic structure**)
- Functorial

$$\text{assoc.} \xrightarrow{\text{antisym}} \text{Lie} \qquad A_\infty \xrightarrow{\text{antisym}} L_\infty$$
$$\text{Lie} \otimes \text{comm. assoc.} = \text{Lie} \qquad L_\infty \otimes C_\infty = L_\infty$$

[6] not necessarily unital

Field theory as homotopy algebra

$$\mathcal{L} = \underbrace{c_{ij}}_{\mu_1} \phi^i \phi^j + \underbrace{c_{ijk}}_{\mu_2} \phi^i \phi^j \phi^k + \underbrace{c_{ijkl}}_{\mu_3} \phi^i \phi^j \phi^k \phi^l + \cdots$$

$$\deg \phi^i = 1 - \mathrm{gh}\, \phi^i$$

Cyclic L_∞-algebra of cyclic degree -3:

- k-ary operator μ_k for each $(k+1)$-ary term in the Lagrangian — only the totally (graded-anti)symmetric parts appear in the action
- inner product degree compensates for

$$\text{field [1] + antifield [2]} = 3$$

YM as colour ⊗ kinematic ⊗ scalar

$$\mathfrak{YM} = \mathfrak{colour} \otimes \mathsf{Maxw} = \mathfrak{colour} \otimes \mathfrak{kinematic} \otimes_\tau \mathfrak{Scalar}$$

$$\mathcal{L}_{\mathrm{YM,strict}} = \mathcal{L}(\mathfrak{YM}) = c_{ab} C_{ij} A^{ai} \Box A^{bj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$
$$\mathcal{L}(\mathfrak{Scalar}) = \phi \Box \phi + \phi^3$$

where

$$\mathfrak{kinematic} = \underbrace{\mathbb{R}^{1,d-1}}_{\text{gluon}} \oplus \overbrace{\mathbb{R}}^{\text{ghost}} \oplus \underbrace{\mathbb{R}}_{\text{antighost}} \oplus \overbrace{\mathbb{R}}^{\text{NL}} \oplus \underbrace{\cdots}_{\text{strictification auxiliaries}}$$

- $\mathfrak{kinematic}$ acts on $\mathfrak{Scalar}$ (as differential operators)
- analogous to $\rtimes$

Double copy using homotopy algebras

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The end