Finite Fields for Di-Photon Amplitudes

Francesco Sarandrea

University of Durham - Università di Torino

YTF 20
16/12/2020

Motivation: Why Do We Care?

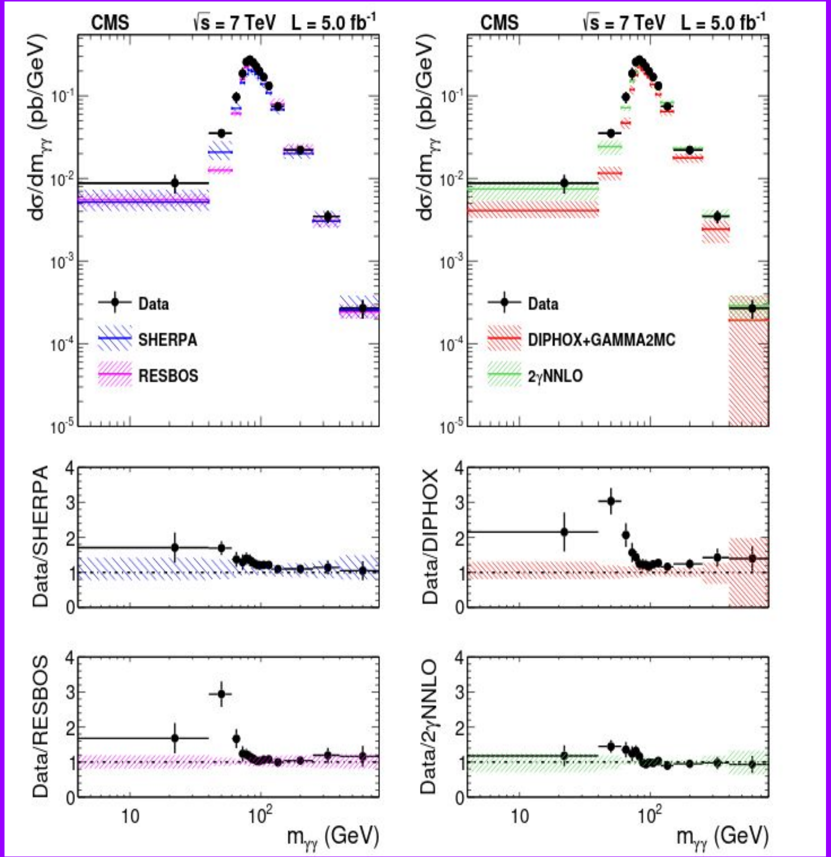
Di-photon production is one of the most important processes at LHC[1]

Final state signature is very clean:

- $H \rightarrow \gamma\gamma$ channel is crucial in the study of Higgs' properties
- Can be used to probe New Physics

LO: $q\bar{q} \rightarrow \gamma\gamma$

NNLO: $gg \rightarrow \gamma\gamma$ is finite, gauge invariant and enhanced at high luminosity



[1]Maltoni, Fabio, Manoj K. Mandal, and Xiaoran Zhao. "Top-quark effects in diphoton production through gluon fusion at NLO in QCD." arXiv preprint arXiv:1812.08703 (2018).

Chatrchyan, Serguei, et al. "Measurement of differential cross sections for the production of a pair of isolated photons in pp collisions at √s= 7 TeV." *The European Physical Journal C* 74.11 (2014): 3129.

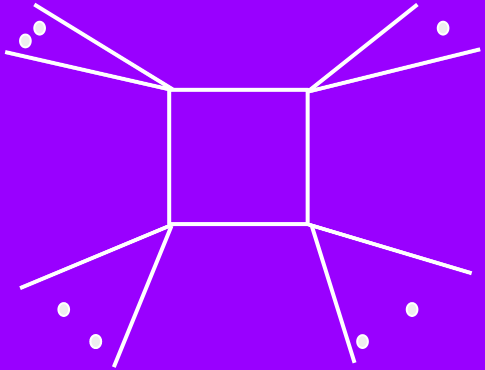
5-point amplitudes

Methods:

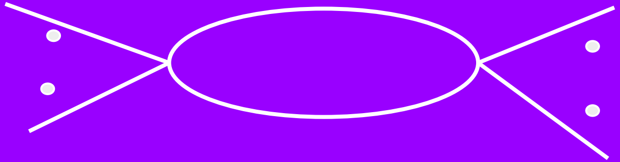
- Feynman Diagrams
- Colour Ordering
- OPP Integrand Reduction
- Finite Fields Reconstruction

$$A_n^{\text{1-loop}} = \sum_{i \in C} d_i \, I_4^{\,i} + \sum_{j \in D} c_j \, I_3^{\,j} + \sum_{k \in \varepsilon} b_k \, I_2^{\,k} + R_n$$

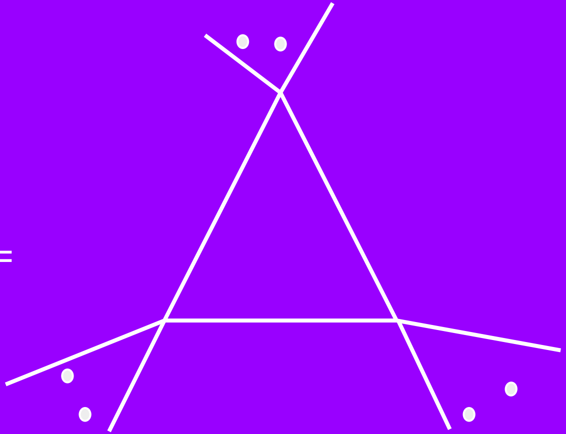
$I_4^{\,i} =$



$I_2^{\,k} =$



$I_3^{\,j} =$



Integrand Reduction

$$A_n^{\text{1-loop}} = \mu^{2\varepsilon} \int \frac{d^d k}{(2\pi)^d} \; I_n$$

$$I_n = \frac{N(k)}{\prod_i D_i}$$

$$N(k) = \text{RSP} + \text{ISP}$$

In dimensional regularisation:

$d = 4-2\varepsilon \qquad k^\mu = \bar{k} + k_\perp \qquad k^2 = \bar{k}^2 + k_\perp{}^2 = m^2 - \mu^2$

$$A_n^{1\text{-loop}} = \sum_{i \in C} d_i I_4^i + \sum_{j \in D} c_j I_3^j + \sum_{k \in \varepsilon} b_k I_2^k + R_n$$

$$N(k) = \sum_{i<j<k<l} [d_{ijkl} + \tilde{d}_{ijkl}(k) + \dot{d}_{ijkl}(\mu^2)] \prod_{\alpha \neq i,j,k,l} D_\alpha +$$

$$\sum_{i<j<k} [c_{ijk} + \tilde{c}_{ijk}(k) + \dot{c}_{ijk}(\mu^2)] \prod_{\alpha \neq i,j,k} D_\alpha +$$

$$\sum_{i<j} [b_{ij} + \tilde{b}_{ij}(k) + \dot{b}_{ij}(\mu^2)] \prod_{\alpha \neq i,j} D_\alpha +$$

$$\sum_{i} [a_i + \tilde{a}_i(k) + \dot{a}_i(\mu^2)] \prod_{\alpha \neq i} D_\alpha +$$

$$\tilde{p}(k) \prod_{\alpha} D_\alpha$$

$$A_n^{\text{1-loop}} = \sum_{i \in C} d_i I_4^i + \sum_{j \in D} c_j I_3^j + \sum_{k \in \varepsilon} b_k I_2^k + R_n$$

after integration:

$$N(k) = \sum_{i<j<k<l} [d_{ijkl} + \underbrace{\tilde{d}_{ijkl}(k)}_{=0} + \dot{d}_{ijkl}(\mu^2)] \prod_{\alpha \neq i,j,k,l} D_\alpha +$$

$$\sum_{i<j<k} [c_{ijk} + \underbrace{\tilde{c}_{ijk}(k)}_{=0} + \dot{c}_{ijk}(\mu^2)] \prod_{\alpha \neq i,j,k} D_\alpha +$$

$$\sum_{i<j} [b_{ij} + \underbrace{\tilde{b}_{ij}(k)}_{=0} + \dot{b}_{ij}(\mu^2)] \prod_{\alpha \neq i,j} D_\alpha +$$

$$\sum_i [a_i + \underbrace{\tilde{a}_i(k)}_{=0} + \dot{a}_i(\mu^2)] \prod_{\alpha \neq i} D_\alpha +$$

$$\underbrace{\tilde{p}(k)}_{=0} \prod_\alpha D_\alpha$$

$$A_n^{\text{1-loop}} = \sum_{i \in C} d_i\, I_4^{\,i} + \sum_{j \in D} c_j\, I_3^{\,j} + \sum_{k \in \varepsilon} b_k\, I_2^{\,k} + R_n$$

$$N(k) = \sum_{i<j<k<l} \left[d_{ijkl} + \tilde{d}_{ijkl}(k) + \dot{d}_{ijkl}(\mu^2) \right] \prod_{\alpha \neq i,j,k,l} D_\alpha \, +$$

$$\sum_{i<j<k} \left[c_{ijk} + \tilde{c}_{ijk}(k) + \dot{c}_{ijk}(\mu^2) \right] \prod_{\alpha \neq i,j,k} D_\alpha \, +$$

$$\sum_{i<j} \left[b_{ij} + \tilde{b}_{ij}(k) + \dot{b}_{ij}(\mu^2) \right] \prod_{\alpha \neq i,j} D_\alpha \, +$$

$$\sum_{i} \left[a_i + \tilde{a}_i(k) + \dot{a}_i(\mu^2) \right] \prod_{\alpha \neq i} D_\alpha \, +$$

$$\tilde{p}(k) \prod_{\alpha} D_\alpha$$

$$A_n^{\text{1-loop}} = \sum_{i \in C} d_i \, I_4^{\,i} + \sum_{j \in D} c_j \, I_3^{\,j} + \sum_{k \in \varepsilon} b_k \, I_2^{\,k} + R_n$$

four-particle cut:

$$N(k) = \sum_{i<j<k<l} \left[d_{ijkl} + \tilde{d}_{ijkl}(k) + \dot{d}_{ijkl}(\mu^2)\right] \prod_{\alpha \neq i,j,k,l} D_\alpha +$$

$$\cancel{\sum_{i<j<k} \left[c_{ijk} + \tilde{c}_{ijk}(k) + \dot{c}_{ijk}(\mu^2)\right] \prod_{\alpha \neq i,j,k} D_\alpha} + \quad =0$$

$$\cancel{\sum_{i<j} \left[b_{ij} + \tilde{b}_{ij}(k) + \dot{b}_{ij}(\mu^2)\right] \prod_{\alpha \neq i,j} D_\alpha} + \quad =0$$

$$\cancel{\sum_{i} \left[a_{i} + \tilde{a}_{i}(k) + \dot{a}_{i}(\mu^2)\right] \prod_{\alpha \neq i} D_\alpha} + \quad =0$$

$$\cancel{\tilde{p}(k) \prod_{\alpha} D_\alpha} \quad =0$$

$$A_n^{\text{1-loop}} = \sum_{i \in C} d_i\, I_4^i + \sum_{j \in D} c_j\, I_3^j + \sum_{k \in \varepsilon} b_k\, I_2^k + R_n$$

$$- \sum_{i<j<k<l} [d_{ijkl} + \tilde{d}_{ijkl}(k) + \dot{d}_{ijkl}(\mu^2)] \prod_{\alpha \neq i,j,k,l} D_\alpha + N(k) =$$

$$\sum_{i<j<k} [c_{ijk} + \tilde{c}_{ijk}(k) + \dot{c}_{ijk}(\mu^2)] \prod_{\alpha \neq i,j,k} D_\alpha +$$

$$\sum_{i<j} [b_{ij} + \tilde{b}_{ij}(k) + \dot{b}_{ij}(\mu^2)] \prod_{\alpha \neq i,j} D_\alpha +$$

$$\sum_{i} [a_i + \tilde{a}_i(k) + \dot{a}_i(\mu^2)] \prod_{\alpha \neq i} D_\alpha +$$

$$\tilde{p}(k) \prod_{\alpha} D_\alpha$$

$$A_n^{\text{1-loop}} = \sum_{i \in C} d_i\, I_4^{\,i} + \sum_{j \in D} c_j\, I_3^{\,j} + \sum_{k \in \varepsilon} b_k\, I_2^{\,k} + R_n$$

$$-\sum_{i<j<k<l} \left[d_{ijkl} + \tilde{d}_{ijkl}(k) + \dot{d}_{ijkl}(\mu^2) \right] \prod_{\alpha \neq i,j,k,l} D_\alpha + N(k) =$$

three-particle cut:

$$\sum_{i<j<k} \left[c_{ijk} + \tilde{c}_{ijk}(k) + \dot{c}_{ijk}(\mu^2) \right] \prod_{\alpha \neq i,j,k} D_\alpha +$$

$$\cancelto{=0}{\sum_{i<j} \left[b_{ij} + \tilde{b}_{ij}(k) + \dot{b}_{ij}(\mu^2) \right] \prod_{\alpha \neq i,j} D_\alpha} +$$

$$\cancelto{=0}{\sum_{i} \left[a_i + \tilde{a}_i(k) + \dot{a}_i(\mu^2) \right] \prod_{\alpha \neq i} D_\alpha} +$$

$$\cancelto{=0}{\tilde{p}(k) \prod_{\alpha} D_\alpha}$$

Momentum Twistor Variables

The kinematics can be represented by momentum twistors $Z_i(\lambda_i, \mu_i)$ for each momentum[3].

λ_i= standard holomorphic spinors

$$\tilde{\lambda}_i = \frac{\langle i, i+1\rangle \mu_{i-1} + \langle i+1, i\rangle \mu_i + \langle i-1, i\rangle \mu_{i+1}}{\langle i, i+1\rangle \langle i-1, i\rangle}$$

Advantages:

- all identities like the Schouten identity, energy-momentum conservation, etc. are satisfied automatically
- the expressions are **rational** in the momentum twistor variables at every step of the calculations

[3]Badger, Simon, Hjalte Frellesvig, and Yang Zhang. "A two-loop five-gluon helicity amplitude in QCD." *Journal of High Energy Physics* 2013.12 (2013): 45.

Five-Particle Example:

$$Z = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1/x_1 & -1/x_1 -1/x_2 & -1/x_1 -1/x_2 -1/x_3 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & x_4 & 1 \\ 0 & 0 & 1 & 1 & x_5/x_4 \end{pmatrix}$$

$s_{12} = x_1$

$s_{23} = x_2 x_4$

$s_{34} = (1/x_2)\,(x_1(x_3(x_4-1) + x_2 x_4) + x_2 x_3 (x_4 - x_5))$

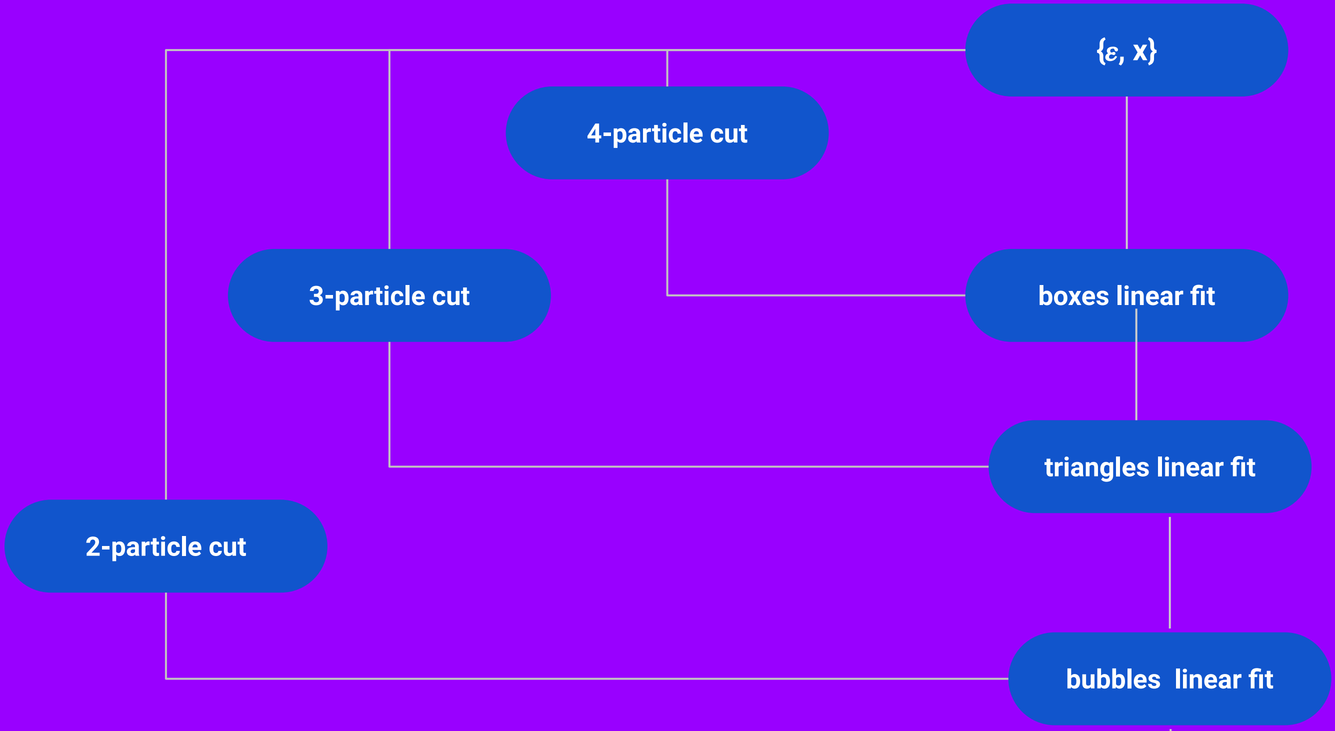
$s_{45} = x_2(x_4 - x_5)$

$s_{15} = -x_3(x_5 - 1)$

$tr_5 = x_1(x_3(x_4(x_5-2) + 1) + x_2 x_4 (x_5 - 1)) + x_2 x_3 (x_5 - x_4)$

Finite Fields Reconstruction

FiniteFlow[2] is a framework for defining and executing numerical algorithms over finite fields and reconstructing multivariate rational functions



[2]Peraro, Tiziano. "Finite Flow: multivariate functional reconstruction using finite fields and dataflow graphs." Journal of High Energy Physics 2019.7 (2019): 31.

Wang's Reconstruction Algorithm

Output: pair of integers (a,b), b>0, such that $\frac{a}{b}$ = c mod(m)
Condition: $|a|, |b| < \sqrt{(m/2)}$

Algorithm:

1. set: $v = (m,0)$, $w = (c,1)$
2. WHILE: $w[1] > \sqrt{(m/2)}$
3. q= FLOOR[$v[1]/w[1]$], $z = v - qw$
4. set: $v \longrightarrow w$, $w \longrightarrow z$

Example: $\frac{a}{b} = 5 \bmod 9$

1. $v = (9,0)$, $w = (5,1)$;
2. $5 > \sqrt{(9/2)}$ TRUE;
3. q= FLOOR[9/5]=1, $z = (4,-1)$;
4. $v \longrightarrow (5,1)$, $w \longrightarrow (4,-1)$;

Example: $\frac{a}{b} = 5 \bmod 9$

1. $v = (5,1)$, $w = (4,-1)$;
2. $4 > \sqrt{(9/2)}$ TRUE;
3. q= FLOOR[5/4]=1, $z = (1,2)$;
4. $v \longrightarrow (4,-1)$, $w \longrightarrow (1,2)$;

Example: $\boxed{\dfrac{a}{b} = 5 \bmod 9}$

1. $v = (4,-1)$, $w = (1,2)$;
2. $1 > \sqrt{(9/2)}$ FALSE;

Hence: a= 1, b=2

$$\boxed{\frac{1}{2} = 5 \bmod(9)}$$

Condition: $|a|, |b| < \sqrt{(m/2)}$ ← **puts a limit of 2^{64}**

Chinese Remainder Theorem: we can deduce a number $a \in Z_n$ from its images $a_i \in Z_{ni}$ if the integers n_i have no common factors.

Given a sequence of primes $\{p_1, p_2, \ldots\}$, from the image of a rational number over several prime fields $Z_{p1}, Z_{p2}, \ldots$ one can deduce the image of the same number over $Z_{p1\,p2}$ ...

5-point results

$$A_{3g2\gamma}(+++++) = 64\, x_3\, x_1^{\,2} x_5^{\,2}$$

$$A_{3g2\gamma}(+++-+) = \frac{64\, x_1^{\,2}\, x_3\, (x_2 - x_4 + x_5 + x_3 x_5 + x_2\, x_3\, x_5)}{(1 + x_3)\,(1 + (1 + x_2)\, x_3)}$$

All the $3g2\gamma$ amplitudes are reconstructed in no more than ~10 minutes on less than 10 cores.
They tested against numerical results and implemented into NJet.

6 ~~5~~-point amplitudes

Methods:

- 6 g Feynman Diagrams
- Colour Ordering

Supersymmetric Decomposition

- OPP Integrand Reduction

BCFW shifts

- Finite Fields Reconstruction
- Permutation Sum

4g2γ amplitudes can be obtained by the sum of **6g** fermion-loop colour-ordered contributions with permuted legs

Simpler 4-point example:

$$M^{(1)\,f}_{gg\to gg} = N_f \times [\, Tr(1,2,3,4) \times A^f(1,2,3,4) + Tr(1,3,4,2) \times A^f(1,3,4,2) + Tr(1,4,2,3) \times A^f(1,4,2,3) +$$

$$Tr(1,3,2,4) \times A^f(1,3,2,4) + Tr(1,4,3,2) \times A^f(1,4,3,2) + Tr(1,2,4,3) \times A^f(1,2,4,3)\,]$$

$$Tr(1,2,3,4) = (T^1)_a{}^b (T^2)_b{}^c (T^3)_c{}^d (T^4)_d{}^a \longrightarrow (T^1)_a{}^b (T^2)_b{}^c \delta_c{}^d \delta_d{}^a = Tr(1,2)$$

$$Tr(1,3,4,2) = (T^1)_a{}^b (T^3)_b{}^c (T^4)_c{}^d (T^2)_d{}^a \longrightarrow (T^1)_a{}^b \delta_b{}^c \delta_c{}^d (T^2)_d{}^a = Tr(1,2)$$

$$\boxed{\begin{array}{l} M^{(1)}_{gg\to\gamma\gamma} = C \times Tr(1,2) \times A_{2g2\gamma}(1,2,3,4) \\ A_{2g2\gamma}(1,2,3,4) = A^f_{4g}(1,2,3,4) + A^f_{4g}(1,3,4,2) + A^f_{4g}(1,4,2,3) + A^f_{4g}(1,3,2,4) + A^f_{4g}(1,4,3,2) + A^f_{4g}(1,2,4,3) \end{array}}$$

Supersymmetric Decomposition

String theory suggests a natural decomposition of QCD amplitudes into supersymmetric and non-supersymmetric parts:

$$A_n^{gluon} = A_n^{N=4} - 4\, A_n^{N=1\ chiral} + A_n^{scalar}$$

$$\boxed{A_n^{f} = A_n^{N=1\ chiral} - A_n^{N=0\ scalar}}$$

contains the non
cut-constructible part

BCFW shifts

$|1'] = |1] + \mathbf{z}\, |2]$

$\mathbf{z} \in C$

$|2'\rangle = |2\rangle - \mathbf{z}|1\rangle$

- $\sum_i p'_i(z)=0$
- $p'^{2}_i(z)=0$
- $p'_1(z) = p_1 + \mathbf{z}\, |1\rangle [2|$
  $p'_2(z) = p_2 - \mathbf{z}\, |1\rangle [2|$

$$s_{13} = -x_1(1+ x_5 - x_8)$$

$$s'_{13}(z) = -x_1(1+ x_5 - x_8) + \mathbf{z}$$

Given A(z) analytic in z:

$$A(z=0) = -\sum_{z^*} \mathrm{Res}_{z=z^*}\left(\frac{A(z)}{z}\right) - \mathrm{Res}_{z=\infty}\left(\frac{A(z)}{z}\right)$$

Possible Poles:

$$[1,j] \longrightarrow [1,j] + z[2,j]$$

$$\langle 2,j\rangle \longrightarrow \langle 2,j\rangle - z\,\langle 1,j\rangle$$

$$s_{1j} \longrightarrow s_{1j} + z\,\langle 1,j\rangle\,[2,j]$$

$$\langle i|\,1\,|\,j\,] \longrightarrow \langle i|\,1\,|\,j] - z\,\langle i\,1\rangle\,[2\,j\,]$$

$$\Delta^{2m}{}_{3}(\,s_{1j}\,,\,s_{klm}) \longrightarrow s_{1j} - s_{klm} + z\,\langle 1\,j\rangle[2\,j]$$

$$\Delta^{3m}{}_{3}(\,s_{1j}\,,\,s_{km}) \longrightarrow \Delta^{3m}{}_{3}(\,s_{1j}\,,\,s_{km}) + 2\,z\,[(s_{1k}+s_{1m}) + (s_{jk}+s_{jm}+s_{km})\langle 1\,j\rangle[2j] + (s_{jk}+s_{jm})\langle 1\,m\rangle[2m]$$

$$+\,z^{2}(\langle 1\,k\rangle^{2}[2\,k]^{2} + \langle 1\,m\rangle^{2}[2\,m]^{2} + \langle 1\,k\rangle[2\,k]\langle 1\,m\rangle[2\,m]\,)$$

second order pole, can’t be used for residues

$$A(z=0) - \sum_{z^*} \mathrm{Res}_{z=z^*}\left(\frac{A(z)}{z}\right) = \textbf{remainder}$$

residue at infinity + any $1/z^2$ contributions

6-point results

The all-plus, single-minus and MHV $4g2\gamma$ amplitudes are reconstructed in no more than ~2 days on less than 10 cores.

Bottlenecks: the reconstruction of 6g, N=0 (−−+−++) and (−+−+−+) amplitudes is too slow (~10^2 days) due to the high polynomial degree of the expressions. Triangle coefficients probably need to be reconstructed using a different method.

Conclusions

- Finite fields reconstruction can be used to efficiently reconstruct one-loop analytic amplitudes

- For 3g2 γ, analytic results can be obtained extremely fast in a very neat and compact form

- For 4g2γ, additional steps are needed to manage the largest expressions, most of the complexity resides in the 3-mass triangle coefficients and some associated parts in the bubbles in the NMHV cases