COVARIANT MULTIPOLE EXPANSION OF LOCAL CURRENTS FOR MASSIVE STATES OF ANY SPIN

S. Cotogno, C. Lorcé, P. Lowdon and Manuel Morales*

(Phys.Rev.D 101, 2020 5, 056016)

Young Theorists' Forum, Durham – 16 December 2020

*Presenter

OUTLINE

- Why should we study interactions involving arbitrary spin particles?
- How can we decompose the amplitudes of these interactions?
- What relevant information can we get from these amplitudes?

WHY SHOULD WE STUDY INTERACTIONS INVOLVING ARBITRARY SPIN PARTICLES?

Picture: CMS

ARBITRARY SPIN STATES

- Elementary particles have **spin 0, 1/2 and 1**.
- Composite systems can have higher spin [P. Data Group]:
    - Fermions with **spin up to 15/2**.
    - Bosons with **spin up to 6**.
- Particles of higher spin are theorised in physics beyond the Standard Model (graviton and gravitino with **spin 2 and 3/2**, respectively).

PARAMETRISATION OF CURRENTS

- We calculate the matrix elements of rank k operators:

$$< p', \lambda' \,|\, \hat{O}^{\mu_1 \cdots \mu_k}(0) \,|\, p, \lambda > = \bar{\eta}(p', \lambda') O^{\mu_1 \cdots \mu_k} \eta(p, \lambda),$$

where p and λ (p' and λ') are the initial (final) momentum and helicity of a **massive on-shell particle of arbitrary spin** j . $\bar{\eta}$ and η are generalised polarisation tensors (GPTs) that satisfy subsidiary conditions to account for $2j+1$ degrees of freedom.

- We impose Hermicity

$$< p', \lambda' \,|\, \hat{O}^{\mu_1 \cdots \mu_k}(0) \,|\, p, \lambda > = < p, \lambda \,|\, \hat{O}^{\mu_1 \cdots \mu_k}(0) \,|\, p', \lambda' >^* ,$$

and T and P discrete symmetries.

PARAMETRISATION OF CURRENTS

- The general amplitude can be written as

$$< p', \lambda' | \hat{O}^{\mu_1 \cdots \mu_k}(0) | p, \lambda > = \bar{\eta}(p', \lambda') O^{\mu_1 \cdots \mu_k}(P, \Delta) \eta(p, \lambda) \,.$$

- When $j = n$ is integer, we have

$$\langle p', \lambda' | \hat{O}^{\mu_1 \cdots \mu_k}(0) | p, \lambda \rangle = (-1)^n \, \varepsilon^*_{\alpha'_1 \cdots \alpha'_n}(p', \lambda') \, O^{\mu_1 \cdots \mu_k, \alpha'_1 \cdots \alpha'_n \alpha_1 \cdots \alpha_n}(P, \Delta) \, \varepsilon_{\alpha_1 \cdots \alpha_n}(p, \lambda) \,.$$

$P = (p' + p)/2$

$\Delta = p' - p$

- When $j = n + 1/2$ is half-integer, we have

$$\langle p', \lambda' | \hat{O}^{\mu_1 \cdots \mu_k}(0) | p, \lambda \rangle = (-1)^n \, \bar{u}_{\alpha'_1 \cdots \alpha'_n}(p', \lambda') \, O^{\mu_1 \cdots \mu_k, \alpha'_1 \cdots \alpha'_n \alpha_1 \cdots \alpha_n}(P, \Delta) \, u_{\alpha_1 \cdots \alpha_n}(p, \lambda) \,.$$

PARAMETRIZATION OF CURRENTS (GENERALISED POLARISATION TENSORS)

- The integer GTPs are given by

$$\varepsilon_{\alpha_1\cdots\alpha_n}(p,\lambda)=\sum_{k=0}^{m/2}\frac{\sum_{\mathscr{P}}\left[\prod_{l=1}^{k}\varepsilon_{\alpha_{\mathscr{P}(l)}}(p,-1)\right]\left[\prod_{l=k+1}^{m-k}\varepsilon_{\alpha_{\mathscr{P}(l)}}(p,0)\right]\left[\prod_{l=m-k+1}^{n}\varepsilon_{\alpha_{\mathscr{P}(l)}}(p,+1)\right]}{2^{k-m/2}\,k!\,(m-2k)!\,(n-m+k)!\sqrt{C_{2n}^{m}}},$$

where $\mathscr{P}$ stands for a permutation of $\{1,\ldots,n\}$ and $m=n-\lambda$.

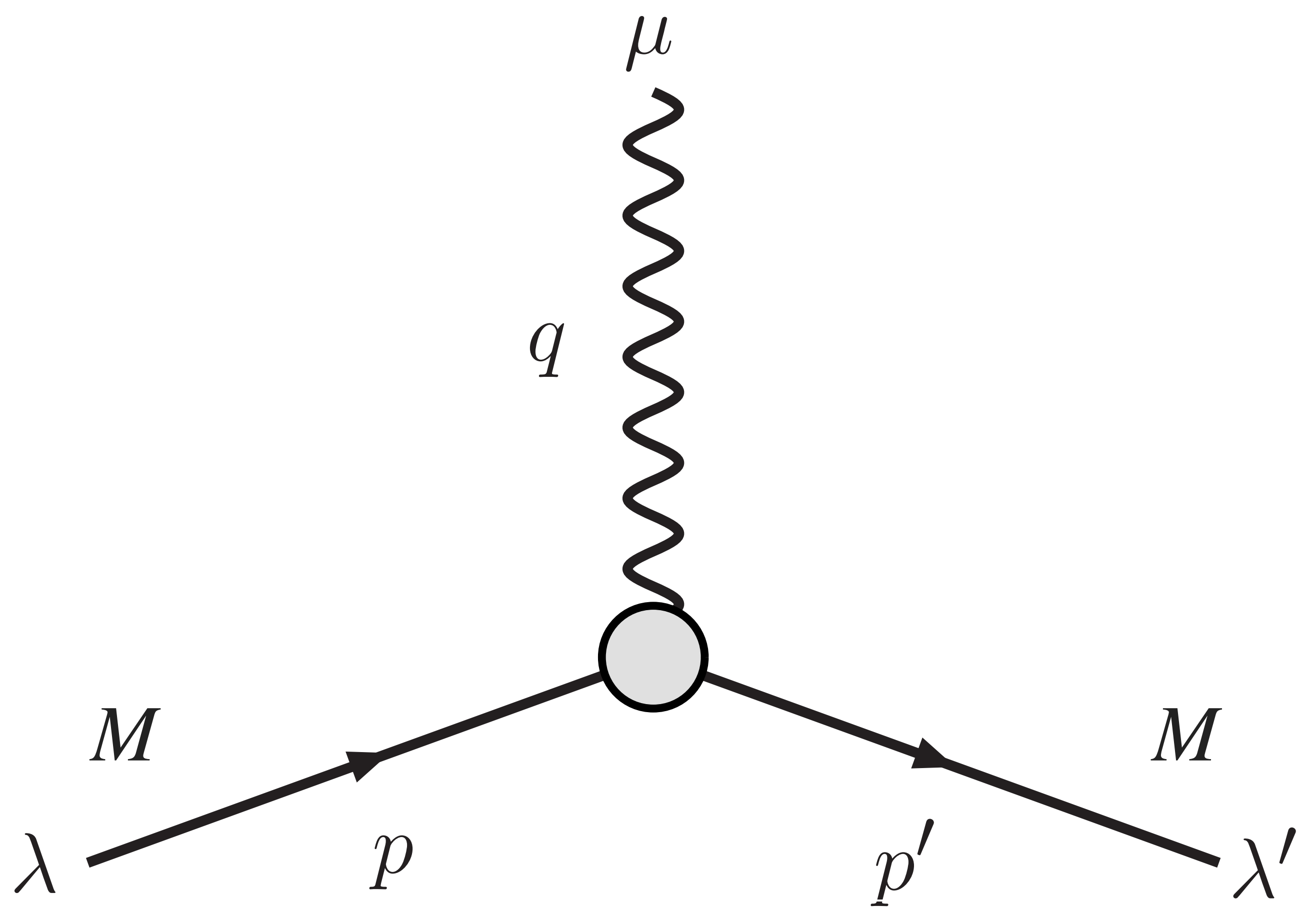
- The half-integer GPTs can be built as

$$u_{\alpha_1\cdots\alpha_n}(p,\lambda)=\sqrt{\frac{j+\lambda}{2j}}u(p,+\tfrac{1}{2})\varepsilon_{\alpha_1\cdots\alpha_n}(p,\lambda-\tfrac{1}{2})+\sqrt{\frac{j-\lambda}{2j}}u(p,-\tfrac{1}{2})\varepsilon_{\alpha_1\cdots\alpha_n}(p,\lambda+\tfrac{1}{2})\,.$$

VECTOR CURRENT

- Let us consider the interaction between a **vector particle and an on-shell particle of arbitrary spin** j **and mass** M through the matrix elements

$$J^\mu \equiv < p', \lambda' \,|\, \hat{J}^\mu_{\mathrm{EM}}(0) \,|\, p, \lambda > .$$



- When $j = n$ is integer:

$$J^{\mu,\alpha'_1\cdots\alpha'_n\alpha_1\cdots\alpha_n}(P,\Delta) = P^\mu \sum_{(k,n)} F^V_{1,k}(t) - (g^{\mu\alpha'_n}\Delta^{\alpha_n} - g^{\mu\alpha_n}\Delta^{\alpha'_n}) \sum_{(k,n-1)} F_{2,k}(t) .$$

- When $j = n + 1/2$ is half-integer:

$$J^{\mu,\alpha'_1\cdots\alpha'_n\alpha_1\cdots\alpha_n}(P,\Delta) = P^\mu \sum_{(k,n)} F^V_{1,k}(t) + \frac{i}{2}\sigma^{\mu\nu}\Delta_\nu \sum_{(k,n)} F^V_{2,k}(t) .$$

- The coefficients $F(t)$ are called form factors (FFs).

$$t = \Delta^2$$

$$\sum_{(k,n)} \equiv \sum_{k=0}^{n} \left[\prod_{i=1}^{k} \left(-\frac{\Delta^{\alpha'_i}\Delta^{\alpha_i}}{2M^2} \right) \prod_{i=k+1}^{n} g^{\alpha'_i\alpha_i} \right]$$

VECTOR CURRENT

- For $j = 0$ we find

$$< p', \lambda' | J^{\mu}(0) | p, \lambda > = P^{\mu} F^{V}_{1,0}(t) \, .$$

- For $j = 1/2$ we find

$$``\bar{u}\gamma^{\mu}u"$$

$$< p', \lambda' | J^{\mu}(0) | p, \lambda > = \bar{u}(p', \lambda') \left[P^{\mu} F^{V}_{1,0}(t) + \frac{i}{2} \sigma^{\mu\nu} \Delta_{\nu} F^{V}_{2,0}(t) \right] u(p, \lambda) \, .$$

- We recover the well established results for $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2.$

  (L. L. Foldy (1952), R. G. Arnold et al. (1980), S. Nozawa et al. (1990), V. Pascalutsa et al. (2007), etc).

TENSOR CURRENT

- Let us study the energy-momentum tensor **(EMT)** $T^{\mu\nu}$.
- The EMT allows us to study a great variety of observables in particle interactions.
- It can be decomposed by using **the metric** $g_{\mu\nu}$, **the Levi-Civita symbol** $\epsilon_{\mu\nu\rho\sigma}$, and the four-vectors of the problem : **the average momentum** P^{μ} and **the momentum transfer** Δ^{μ}, and others.
- On-shell identities reduce the number of independent structures.
- We consider the most general case where the EMT has non conserved terms (for composite particles) and is asymmetric.

TENSOR CURRENT (ASYMMETRIC CONTRIBUTIONS)

- The total angular momentum tensor is given by

$$M^{\mu\nu\rho}(x) = M^{\mu\nu\rho}_{\rm OAM}(x) + M^{\mu\nu\rho}_{\rm spin}(x)\,.$$

where

$$M^{\mu\nu\rho}_{\rm OAM}(x) = x^\nu T^{\mu\rho}(x) - x^\rho T^{\mu\nu}(x),$$

$$M^{\mu\nu\rho}_{\rm spin}(x) = -\,i\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)}(\Sigma^{\nu\rho})^s_r\phi_s(x)\,.$$

Spin	Field transformation
0	$(\Sigma^{\mu\nu})^s_r = 0$
1/2	$(\Sigma^{\mu\nu})^s_r = \frac{1}{2}(\sigma^{\mu\nu})^s_r$
1	$(\Sigma^{\mu\nu})^\beta_\alpha = i(\delta^\mu_\alpha g^{\nu\beta} - \delta^\nu_\alpha g^{\mu\beta})$

TENSOR CURRENT (ASYMMETRIC CONTRIBUTIONS)

- The conservation of the total angular momentum implies

$$\partial_\mu M^{\mu\nu\rho}(x) = 0.$$

- Then, we have

$$T^{\nu\rho}(x) - T^{\rho\nu}(x) = -\,\partial_\mu M^{\mu\nu\rho}_{\text{spin}}(x)\,.$$

Spin	Field transformation
0	$(\Sigma^{\mu\nu})_r^{\ s} = 0$
1/2	$(\Sigma^{\mu\nu})_r^{\ s} = \frac{1}{2}(\sigma^{\mu\nu})_r^{\ s}$
1	$(\Sigma^{\mu\nu})_\alpha^{\ \beta} = i(\delta^\mu_\alpha g^{\nu\beta} - \delta^\nu_\alpha g^{\mu\beta})$

- The spin of the particle introduces an antisymmetric part in the EMT.

TENSOR CURRENT (EXAMPLE)

- Let us parametrise the EMT of a scalar ($j = 0$) particle: $\langle p' | \hat{T}^{\mu\nu}(0) | p \rangle = ?$

  ☑ $j = 0 \Rightarrow$ no antisymmetric part.

- What tensor structures can we have?

  ☑ $P^\mu P^\nu$, $\Delta^\mu \Delta^\nu$, and $g^{\mu\nu}$.

These do not fulfil the conditions!

✗ $P^{\{\mu}\Delta^{\nu\}}$, ✗ $\epsilon^{\mu\nu\rho\sigma} P_\rho P_\sigma$, $\cdots$

- The most general decomposition is given by

$$\langle p' | \hat{T}^{\mu\nu}(0) | p \rangle = A(t) P^\mu P^\nu + B(t)(\Delta^\mu \Delta^\nu - \Delta^2 g^{\mu\nu}) + C(t) g^{\mu\nu} \,.$$

- The decomposition is not unique.

And for arbitrary spin? …

TENSOR CURRENT

- When $j = n$ we find:

$$
\begin{aligned}
T^{\mu\nu,\alpha'_1\cdots\alpha'_n\alpha_1\cdots\alpha_n}(P,\Delta) = 2P^\mu P^\nu \sum_{(k,n)} F^T_{1,k}(t) \\
+2\left(\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2\right)\sum_{(k,n)} F^T_{2,k}(t) \\
+2M^2 g^{\mu\nu}\sum_{(k,n)} F^T_{3,k}(t) \\
-P^{\{\mu}g^{\nu\}[\alpha'_n}\Delta^{\alpha_n]}\sum_{(k,n-1)} F^T_{4,k}(t) \\
-\left(\Delta^{\{\mu}g^{\nu\}\{\alpha'_n}\Delta^{\alpha_n\}} - g^{\mu\nu}\Delta^{\alpha'_n}\Delta^{\alpha_n} - g^{\alpha'_n\{\mu}g^{\nu\}\alpha_n}\Delta^2\right)\sum_{(k,n-1)} F^T_{5,k}(t) \\
+M^2 g^{\alpha'_n\{\mu}g^{\nu\}\alpha_n}\sum_{(k,n-1)} F^T_{6,k}(t) \\
+\Delta^{[\alpha'_n}g^{\alpha_n]\{\mu}g^{\nu\}[\alpha'_{n-1}}\Delta^{\alpha_{n-1}]}\sum_{(k,n-2)} F^T_{7,k}(t) \\
-P^{[\mu}g^{\nu][\alpha'_n}\Delta^{\alpha_n]}\sum_{(k,n-1)} F^T_{8,k}(t) \\
-\Delta^{[\mu}g^{\nu]\{\alpha'_n}\Delta^{\alpha_n\}}\sum_{(k,n-1)} F^T_{9,k}(t),
\end{aligned}
$$

$$a^{\{\mu}b^{\nu\}} = a^\mu b^\nu + a^\nu b^\mu$$

$$a^{[\mu}b^{\nu]} = a^\mu b^\nu - a^\nu b^\mu$$

- When $j = n + 1/2$ we find:

$$
\begin{aligned}
T^{\mu\nu,\alpha'_1\cdots\alpha'_n\alpha_1\cdots\alpha_n}(P,\Delta) = 2P^\mu P^\nu \sum_{(k,n)} F^T_{1,k}(t) \\
+2\left(\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2\right)\sum_{(k,n)} F^T_{2,k}(t) \\
+2M^2 g^{\mu\nu}\sum_{(k,n)} F^T_{3,k}(t) \\
+P^{\{\mu}\frac{i}{2}\sigma^{\nu\}\rho}\Delta_\rho\sum_{(k,n)} F^T_{4,k}(t) \\
-\left(\Delta^{\{\mu}g^{\nu\}\{\alpha'_n}\Delta^{\alpha_n\}} - g^{\mu\nu}\Delta^{\alpha'_n}\Delta^{\alpha_n} - g^{\alpha'_n\{\mu}g^{\nu\}\alpha_n}\Delta^2\right)\sum_{(k,n-1)} F^T_{5,k}(t) \\
+M^2 g^{\alpha'_n\{\mu}g^{\nu\}\alpha_n}\sum_{(k,n-1)} F^T_{6,k}(t) \\
+\Delta^{[\alpha'_n}g^{\alpha_n]\{\mu}g^{\nu\}[\alpha'_{n-1}}\Delta^{\alpha_{n-1}]}\sum_{(k,n-2)} F^T_{7,k}(t) \\
+P^{[\mu}\frac{i}{2}\sigma^{\nu]\rho}\Delta_\rho\sum_{(k,n)} F^T_{8,k}(t) \\
-\Delta^{[\mu}g^{\nu]\{\alpha'_n}\Delta^{\alpha_n\}}\sum_{(k,n-1)} F^T_{9,k}(t)\,.
\end{aligned}
$$

TENSOR CURRENT

- Without operator constraints the number of independent structures grows substantially.
- The same happens if we do not use subsidiary conditions, tensor identities, and on-shell identities.

$$i\epsilon^{\mu\nu\rho\sigma}g^{\tau\lambda} + i\epsilon^{\nu\rho\sigma\tau}g^{\mu\lambda} + i\epsilon^{\rho\sigma\tau\mu}g^{\nu\lambda} + i\epsilon^{\sigma\tau\mu\nu}g^{\rho\lambda} + i\epsilon^{\tau\mu\nu\rho}g^{\sigma\lambda} = 0$$

$$\begin{aligned}
\mathbb{1} &\doteq \frac{\not{P}}{M}, \\
\gamma_5 &\doteq \frac{\not{\Delta}\gamma_5}{2M}, \\
\gamma^\mu &\doteq \frac{P^\mu}{M} + \frac{i\sigma^{\mu\Delta}}{2M}, \\
\gamma^\mu\gamma_5 &\doteq \frac{\Delta^\mu\gamma_5}{2M} + \frac{i\sigma^{\mu P}}{M}, \\
i\sigma^{\mu\nu} &\doteq -\frac{\Delta^{[\mu}\gamma^{\nu]}}{2M} + \frac{i\epsilon^{\mu\nu P\lambda}\gamma_\lambda\gamma_5}{M}, \\
i\sigma^{\mu\nu}\gamma_5 &\doteq -\frac{P^{[\mu}\gamma^{\nu]}\gamma_5}{M} + \frac{i\epsilon^{\mu\nu\Delta\lambda}\gamma_\lambda}{2M},
\end{aligned}$$

$$\begin{aligned}
0 &\doteq \not{\Delta}, \\
0 &\doteq \not{P}\gamma_5, \\
0 &\doteq \frac{\Delta^\mu}{2} + i\sigma^{\mu P}, \\
0 &\doteq P^\mu\gamma_5 + \frac{i\sigma^{\mu\Delta}\gamma_5}{2}, \\
0 &\doteq -P^{[\mu}\gamma^{\nu]} + \frac{i\epsilon^{\mu\nu\Delta\lambda}\gamma_\lambda\gamma_5}{2}, \\
0 &\doteq -\frac{\Delta^{[\mu}\gamma^{\nu]}\gamma_5}{2} + i\epsilon^{\mu\nu P\lambda}\gamma_\lambda,
\end{aligned}$$

$$\begin{aligned}
i\epsilon^{\rho P\alpha\lambda}\gamma_\lambda\gamma_5 &\doteq P^\alpha\gamma^\rho - Mg^{\rho\alpha}, \\
i\epsilon^{\rho\Delta\alpha\lambda}\gamma_\lambda\gamma_5 &\doteq \Delta^\alpha\gamma^\rho, \\
i\epsilon^{P\Delta\alpha\lambda}\gamma_\lambda\gamma_5 &\doteq M\Delta^\alpha, \\
i\epsilon^{\alpha' P\sigma\lambda}\gamma_\lambda\gamma_5 &\doteq P^{\alpha'}\gamma^\sigma - Mg^{\alpha'\sigma}, \\
i\epsilon^{\alpha'\Delta\sigma\lambda}\gamma_\lambda\gamma_5 &\doteq \Delta^{\alpha'}\gamma^\sigma, \\
i\epsilon^{\alpha' P\Delta\lambda}\gamma_\lambda\gamma_5 &\doteq -M\Delta^{\alpha'}, \\
i\epsilon^{\alpha' P\alpha\lambda}\gamma_\lambda\gamma_5 &\doteq -Mg^{\alpha'\alpha}, \\
i\epsilon^{\alpha'\Delta\alpha\lambda}\gamma_\lambda\gamma_5 &\doteq 0,
\end{aligned}$$

$$\begin{aligned}
i\epsilon^{\rho P\alpha\lambda}\gamma_\lambda &\doteq P^\alpha\gamma^\rho\gamma_5, \\
i\epsilon^{\rho\Delta\alpha\lambda}\gamma_\lambda &\doteq \Delta^\alpha\gamma^\rho\gamma_5 - 2Mg^{\rho\alpha}\gamma_5, \\
i\epsilon^{P\Delta\alpha\lambda}\gamma_\lambda &\doteq -2MP^\alpha\gamma_5, \\
i\epsilon^{\alpha' P\sigma\lambda}\gamma_\lambda &\doteq P^{\alpha'}\gamma^\sigma\gamma_5, \\
i\epsilon^{\alpha'\Delta\sigma\lambda}\gamma_\lambda &\doteq \Delta^{\alpha'}\gamma^\sigma\gamma_5 - 2Mg^{\alpha'\sigma}\gamma_5, \\
i\epsilon^{\alpha' P\Delta\lambda}\gamma_\lambda &\doteq 2MP^{\alpha'}\gamma_5, \\
i\epsilon^{\alpha' P\alpha\lambda}\gamma_\lambda &\doteq 0, \\
i\epsilon^{\alpha'\Delta\alpha\lambda}\gamma_\lambda &\doteq -2Mg^{\alpha'\alpha}.
\end{aligned}$$

$$-g^{[\alpha'\{\mu}i\epsilon^{\alpha]\nu\}P\Delta}\gamma_5 \doteq \frac{1}{2M}g^{[\alpha'\{\mu}\left(\Delta^{\alpha]}i\epsilon^{\nu\}P\Delta\lambda} - \Delta^{\nu\}}i\epsilon^{\alpha]P\Delta\lambda} - \Delta^2 i\epsilon^{\alpha]\nu\}P\lambda}\right)\gamma_\lambda\gamma_5 \;\; ? \;\ldots$$

TENSOR CURRENT

- We recover the well established results for $j = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2.$ (I. Kobzarev et al. (1962), H. Pagels (1966), J. Donoghue et al. (1991), S. Taneja (2012), etc.).

- Let us consider $j = 1$:

$$
\begin{aligned}
\langle p',\lambda'|\,\hat{T}^{\mu\nu}(0)\,|p,\lambda\rangle = -\,\varepsilon^{*}_{\alpha'}(p',\lambda')\Bigg[&2P^{\mu}P^{\nu}\left(g^{\alpha'\alpha}\,F^{T}_{1,0}(t)-\frac{\Delta^{\alpha'}\Delta^{\alpha}}{2M^{2}}\,F^{T}_{1,1}(t)\right)\\
&+2\left(\Delta^{\mu}\Delta^{\nu}-g^{\mu\nu}\Delta^{2}\right)\left(g^{\alpha'\alpha}\,F^{T}_{2,0}(t)-\frac{\Delta^{\alpha'}\Delta^{\alpha}}{2M^{2}}\,F^{T}_{2,1}(t)\right)+2M^{2}g^{\mu\nu}\left(g^{\alpha'\alpha}\,F^{T}_{3,0}(t)-\frac{\Delta^{\alpha'}\Delta^{\alpha}}{2M^{2}}\,F^{T}_{3,1}(t)\right)\\
&-P^{\{\mu}g^{\nu\}[\alpha'}\Delta^{\alpha]}F^{T}_{4,0}(t)-\left(\Delta^{\{\mu}g^{\nu\}\{\alpha'}\Delta^{\alpha\}}-g^{\mu\nu}\Delta^{\alpha'}\Delta^{\alpha}-g^{\alpha'\{\mu}g^{\nu\}\alpha}\Delta^{2}\right)F^{T}_{5,0}(t)\\
&+M^{2}g^{\alpha'\{\mu}g^{\nu\}\alpha}F^{T}_{6,0}(t)-P^{[\mu}g^{\nu][\alpha'}\Delta^{\alpha]}F^{T}_{8,0}(t)-\Delta^{[\mu}g^{\nu]\{\alpha'}\Delta^{\alpha\}}F^{T}_{9,0}(t)\Bigg]\varepsilon_{\alpha}(p,\lambda)\,.
\end{aligned}
$$

TENSOR CURRENT

- The FFs describe many interesting quantities.
    - Energy density:

$$U = \frac{1}{2M} < p, \lambda' | T^{00} | p, \lambda > = \left(\frac{1}{2} F^T_{1,0}(0) - \frac{1}{4} F^T_{3,0}(0) \right) M$$

    - Pressure-volume work:

$$W = \frac{\delta_{ij}}{6M} < p, \lambda' | T^{ij}(0) | p, \lambda > = \left(\frac{1}{2} F^T_{3,0}(0) + F^T_{6,0}(0) \right) M$$

- Other observables such as $< r^2 >$, L^i, J^i, I^{ij} can also be expressed as **simple linear combinations** of FFs.

- Additionally, FFs also account for non-perturbative QCD effects.

SUMMARY

- ☑ It is important to have a framework to describe arbitrary spin states within the SM and beyond.

- ☑ We have a general decomposition of (scalar), vector and tensor currents for particles of any spin. It is given in terms of tensor structures, GPTs, and FFs.

- ☑ The FFs of these decompositions give us valuable information about the system.


"That's all Folks!"

THANK YOU FOR YOUR ATTENTION

ANY QUESTIONS?