Classical Yang Mills Observables from Amplitudes

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Based on 2009.03852 with L de la Cruz, B Maybee, D O'Connell

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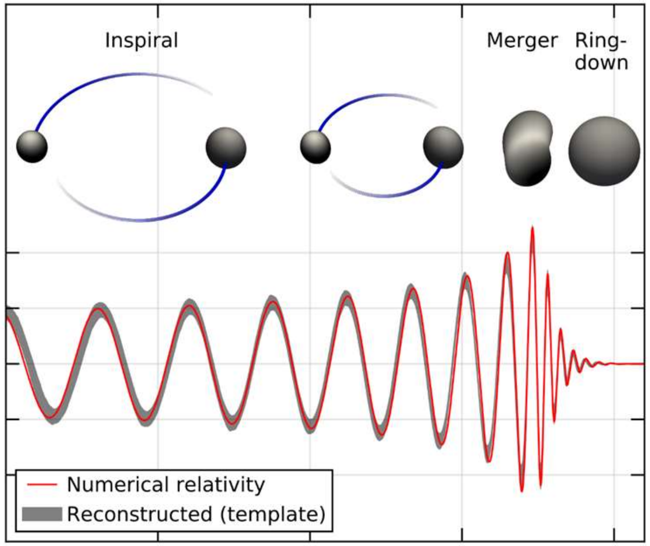
Motivation and Background

Gravity…

- LIGO/Virgo need high precision theoretical computations. This is hard in GR.



Ligo.org

- Tools from amplitudes may prove helpful. In particular the Double Copy (DC)

- DC relates YM to Gravity – roughly $(YM)^2 \sim GR$
  - Perturbative amplitude relation and classical eom's

- YM much simpler to compute than GR. Also have many methods from amplitudes community.

Motivation and Background

Eventually we are interested in scattering/merging of black holes… not quite there yet

We will look at the YM analogue of this – scattering of two colour charged particles, and look in particular at the classical limit.

- Toy model for spin effects (important in real life BH events)
- General interest in YM side of DC, extracting classical observables
- Colour is a unique feature to YM, and plays important role in BCJ DC.

Yang Mills / Wong equations

Classical point particle in YM field

adjoint vector

$$\frac{\mathrm{d}c^a_\alpha}{\mathrm{d}\tau_\alpha} = g f^{abc} v^\mu_\alpha(\tau_\alpha) A^b_\mu(x_\alpha(\tau_\alpha))\, c^c_\alpha(\tau_\alpha)\,,$$

Colour precession

$$\frac{\mathrm{d}p^\mu_\alpha}{\mathrm{d}\tau_\alpha} = g\, c^a_\alpha(\tau_\alpha)\, F^{a\,\mu\nu}(x_\alpha(\tau_\alpha))\, v_{\alpha\,\nu}(\tau_\alpha)\,,$$

'Lorentz force'

$$D^\mu F^a_{\mu\nu}(x) = J^a_\nu(x) = g \sum_{\alpha=1}^{N} \int \mathrm{d}\tau_\alpha\, c^a_\alpha(\tau_\alpha) v^\mu(\tau_\alpha)\, \delta^{(4)}(x - x(\tau_\alpha))$$

'Maxwell' type equations

- Can solve iteratively / perturbatively
- Application in quark gluon plasmas

Quantum Colour

Start with action

$$S = \int \mathrm{d}^4x \left(\sum_{\alpha=1}^{2} \left[(D_\mu \varphi_\alpha^\dagger) D^\mu \varphi_\alpha - \frac{m_\alpha^2}{\hbar^2} \varphi_\alpha^\dagger \varphi_\alpha \right] - \frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} \right)$$

Classical limit $\hbar \to 0$

Associated Noether charge – canonically quantise to get operator

$$\mathbb{C}^a = \hbar \int \mathrm{d}\Phi(p)\; \left(a^\dagger(p)\, T^a_R\, a(p) + b^\dagger(p)\, T^a_{\bar{R}}\, b(p)\right)$$

Dimension of $\hbar$

$$[\mathbb{C}^a, \mathbb{C}^b] = i\hbar f^{abc}$$

Important later for classical limit

Coherent states and Schwinger bosons

General single particle states

$$|\psi\rangle = \sum_i \int d\Phi(p)\, \phi(p) \chi_i \,|p^i\rangle$$

Classical requirements: to match quantities in Wong eqs,

$$\begin{aligned}\langle\psi|\mathbb{C}^a|\psi\rangle &= \text{finite}\,,\\ \langle\psi|\mathbb{C}^a\mathbb{C}^b|\psi\rangle &= \langle\psi|\mathbb{C}^a|\psi\rangle\langle\psi|\mathbb{C}^b|\psi\rangle + \text{negligible}\,.\end{aligned}$$

Use coherent states – nice classical behaviour

Schwinger boson formalism to construct colour (SU(N)) coherent states

Introduced in Sakurai QM book

Coherent states and Schwinger bosons

SU(2) example

Define SU(2) pair of raising / lowering operators $a_\pm, a^\dagger_\pm$

$$J_+ = a^\dagger_+ a_- \qquad J_- = a^\dagger_- a_+ \qquad J_3 = \frac{1}{2}\left(a^\dagger_+ a_+ + a^\dagger_- a_-\right)$$

$$[J_\pm, J_3] = \pm J_\pm$$

Basis of states

$$j = \frac{1}{2}(n_1 + n_2)$$

$$|n_1, n_2\rangle = (a^\dagger_+)^{n_1} (a^\dagger_-)^{n_2} \, |0\rangle$$

$$m = \frac{1}{2}(n_1 - n_2)$$

Build Coherent states

$$|\alpha_1, \alpha_2\rangle = N\, exp(\alpha_1 a^\dagger_+ + \alpha_2 a^\dagger_-)\, |0\rangle$$

Coherent states and Schwinger bosons

Mathur et al. 2000,2010

SU(3) more complex, but same general idea

Build Irreps from $\mathbf{3}, \bar{\mathbf{3}}$

$$\mathbf{b}^\dagger \sim \bar{\mathbf{3}} \qquad \mathbf{a}^\dagger \sim \mathbf{3}$$

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
j_1 & j_2 & \cdots & j_{n_2} & i_1 & i_2 & \cdots & i_{n_1} \\
\hline
 & & & & \multicolumn{4}{c}{} \\
\cline{1-4}
\end{array}$$

Impose symmetry properties of YT

$$\mathcal{P}\left\{\left(b^\dagger_{j_1}\cdots b^\dagger_{j_{n_2}}\right)\left(a^{\dagger i_1}\cdots a^{\dagger i_{n_1}}\right)|0\rangle\right\}$$

Don’t need full coherent states but ‘restricted’ projections, with large rep size (n_1, n_2 large)

$$\boxed{|\xi\,\zeta\rangle_{[n_1,n_2]} \equiv \frac{1}{\sqrt{(n_1!n_2!)}}\left(\zeta\cdot b^\dagger\right)^{n_2}\left(\xi\cdot a^\dagger\right)^{n_1}|0\rangle}$$

$$\xi\cdot\zeta = 0$$

Classical Observables

KMOC formalism

Kosower, Maybee, O'Connell '18

$$\langle \Delta c_1^a \rangle = \langle \Psi | S^\dagger \mathbb{C}^a S | \Psi \rangle - \langle \Psi | \mathbb{C}^a | \Psi \rangle$$

$$= i \langle \Psi | [\mathbb{C}_1^a, T] | \Psi \rangle + \langle \Psi | T^\dagger [\mathbb{C}_1^a, T] | \Psi \rangle$$

$S = 1 + iT$

Algebra gives…

$$\langle \Delta c_1^a \rangle \to \Delta c_1^a = i \left\langle\!\!\left\langle \int \hat{\mathrm{d}}^4 \bar{q}\, \hat{\delta}(2p_1 \cdot \bar{q}) \hat{\delta}(2p_2 \cdot \bar{q}) e^{-ib\cdot\bar{q}}\, \mathcal{G}^a \right\rangle\!\!\right\rangle$$

'Colour Impulse kernel'

$$\mathcal{G}^a \sim Q^a \mathcal{A} + \tilde{Q}^a \int \mathcal{A}^\dagger \mathcal{A}$$

$$Q^a \sim [\mathbb{C}_1^a, \mathcal{C}(\mathcal{A})]$$

Classical Observables

Notation – integration over wavepackets

$$\left\langle\!\!\left\langle f(p_1, p_2, \cdots) \right\rangle\!\!\right\rangle = \int \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_2)|\phi(p_1)|^2|\phi(p_2)|^2 \, \langle \chi_1\, \chi_2 | f(p_1, p_2, \cdots) | \chi_1\, \chi_2 \rangle$$

$$\Longrightarrow \left\{ p_i \to m_i u_i,\, \hat{\mathbb{C}}^a \to c^a \right\}$$

Sharply peaked wavepackets

KMOC procedure

- Massless and loop momenta rescaled $q \to \hbar \bar{q}$
- Expand as series in $\hbar$
- Replace momenta and colours by classical values

$$\mathcal{A} \sim \frac{\mathcal{A}_{-1}}{\hbar} + \mathcal{A}_0 + \ldots$$

$$Q^a \sim Q_0^a + \hbar Q_1^a + \ldots$$

Singular terms will cancel in final result

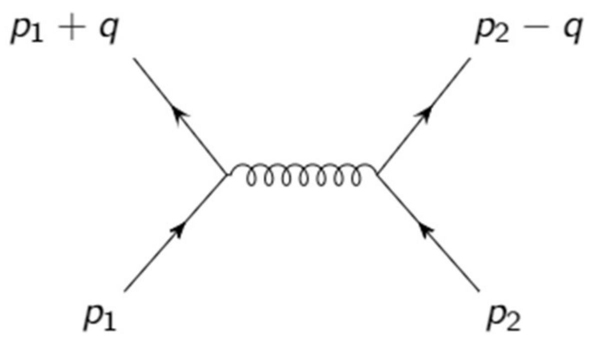
Leading order calculation

Only need linear term in impulse kernel – tree level

$$\mathcal{G}^a_{LO} = \hbar[\mathbb{C}^a_1, \mathcal{C}(tree)]A_{tree}$$

Split colour and kinematics

KMOC procedure



$$A_{tree} = \frac{4p_1 \cdot p_2 + \hbar \bar{q}^2}{\hbar^2 \bar{q}^2}$$

$$\mathcal{C}(tree) = C_1 \cdot C_2$$

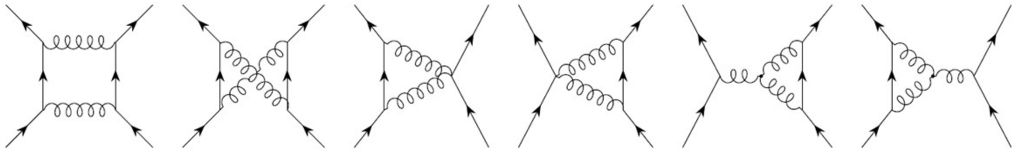
$$\rightarrow [\mathbb{C}^a_1, \mathcal{C}(tree)] = i\hbar f^{abc} C^b_1 C^c_2$$

Classical values for $c^a,\ u_i$

$$\Delta c^a_1 = g^2 f^{abc} c^b_1 c^c_2\, u_1 \cdot u_2 \int \hat{\mathrm{d}}\bar{q}\, \hat{\delta}(u_1 \cdot \bar{q})\hat{\delta}(u_2 \cdot \bar{q}) \frac{e^{-ib\cdot\bar{q}}}{\bar{q}^2}$$

Next to Leading order calculation (brief)

Need one loop linear term and quadratic term with trees.
Focusing on the 1 loop term, colour decomposition yields



$$\mathcal{A}^{(1)} = \mathcal{C}(\text{box})A^{(1)}_{qed} + \hbar^2(\cdots)$$

Contains YM 3point interaction vertex !

Momentum impulse same as QED with replacement $eQ \to gc^a$
Cancellation of kinematic and colour factors – new terms survive

$$\mathcal{G}^a \sim \left(\boxed{\frac{A_{-1}}{\hbar}} + A_0 \cdots\right)(Q_0^a + \boxed{\hbar Q_1^a} + \cdots)$$

Also need cut box term to cancel singular terms

Summary

- Gravitational waves need precision calculations in Black Hole interaction events
- Double Copy and amplitudes may be useful tools - Study scattering in YM side of Double Copy
- Coherent states are needed for colour and kinematic states to give sensible classical behaviour
- Use KMOC formalism to compute classical observables
- Only the QED like amplitude is relevant, YM 3pt vertex is QM correction
- New singular terms in the kinematic amplitude survive - more info in YM than QED
- Can also treat radiation in same way

Thank you!