Complex structures in the dynamics of Kerr black holes

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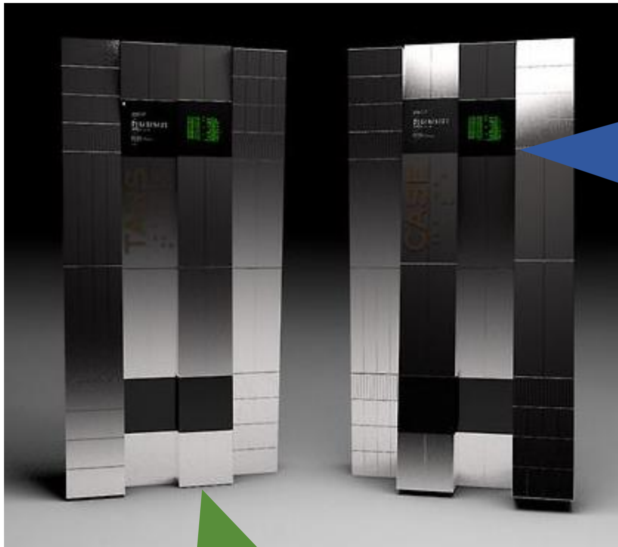
In a certain film…



Help! I need to precisely scatter off a spinning black hole and reach a life supporting planet…

Spinning black holes are very special…maybe some insight will simplify things?

Christopher Nolan and Warner Bros Pictures, 2013

In a certain film…

Okay, you're good with spin; what should I try?

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 6, NUMBER 6 JUNE 1965

Note on the Kerr Spinning-Particle Metric*

E. T. Newman and A. I. Janis

Physics Department, University of Pittsburgh, Pittsburgh, Pennsylvania

(Received 19 June 1964)

It is shown that by means of a complex coordinate transformation performed on the monopole or Schwarzschild metric one obtains a new metric (first discovered by Kerr). It has been suggested that this metric be interpreted as that arising from a spinning particle. We wish to suggest a more

Try using the **Newman-Janis** shift…

$$\mathcal{I}_\ell + i\mathcal{J}_\ell = m\,(ia)^\ell$$

Hansen, 1974

Outline

1. The Newman-Janis shift
2. Kerr black holes as elementary particles?
3. Effective actions and complex worldsheets
4. Spinor equations of motion
5. Discussion

The Newman-Janis shift

Simple statement:

Kerr metric, with spin parameter a, can be obtained **from Schwarzschild** by **complex** transformation $z \to z + ia$

Newman & Janis, 1965

Kerr and Schwarzschild are both **exact** *Kerr-Schild* solutions to GR:

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu}$$

Kerr & Schild, 1965

Background metric

Scalar function

Null vector,

$\eta(k,k) = g(k,k) = 0$

Classical double copy: ∃ gauge theory solution $A_\mu = \phi\, k_\mu$

Monteiro, O'Connell & White, 2014

$\to$ EM analogue of Kerr: $\sqrt{\textbf{Kerr}}$!

The Newman-Janis shift

Metrics: $\phi_{\text{Schwz}}(r) = \dfrac{2GM}{r}$ $\qquad$ $\phi_{\text{Kerr}}(\tilde{r},\theta) = \dfrac{2GM\,\tilde{r}}{\tilde{r}^2 + a^2\cos^2\theta}$

$$\frac{x^2+y^2}{\tilde{r}^2+a^2} + \frac{z^2}{a^2} = 1$$

Under $z \to z + ia$, $r^2 \to (\tilde{r} + ia\cos\theta)^2$.

$$\phi_{\text{Schwz}}(r) \to GM\,\mathrm{Re}\left\{\frac{1}{r}\right\}\Bigg|_{r\mapsto\tilde{r}+ia\cos\theta} = \underbrace{\frac{2GM\tilde{r}}{(\tilde{r}+ia\cos\theta)^2}}$$

$$= \phi_{\text{Kerr}}$$

Why did that just happen???

Scattering amplitudes and spin

No hair theorem → all black hole multipoles specified by mass and spin vector.

$$\mathcal{I}_\ell + i\mathcal{J}_\ell = m\left(\frac{is}{m}\right)^\ell$$

Hansen, 1974

Wigner classification → single particle states specified by mass and spin *s*.

$$\underbrace{\begin{gathered}\mathbb{P}^2|\psi\rangle = m^2|\psi\rangle \\ \mathbb{W}^2|\psi\rangle = -\hbar^2 s(s+1)|\psi\rangle\end{gathered}}_{\text{Little group irreps}}$$

Wigner, 1939

Any state in a little group irrep can be represented using chiral **spinor-helicity** variables:

Distinct spinor reps

$$p^\mu \sigma_\mu^{\alpha\dot{\alpha}} = p^{\alpha\dot{\alpha}} = g^{ij}\lambda_i^\alpha \tilde{\lambda}_j^{\dot{\alpha}}$$

Arkani-Hamed, Huang & Huang, 2017

Little group metric
Massless: $U(1) \rightarrow \delta^{ij}$
Massive: $SU(2) \rightarrow \epsilon^{ji}$

2-spinors

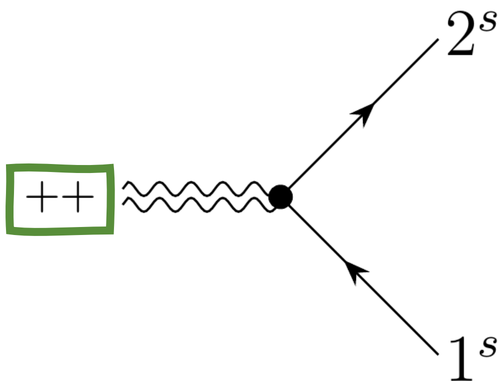
3-point exponentiation

Use to calculate amplitudes for **any** spin *s*:

Chiral

Spin = **parameter**

e.g.

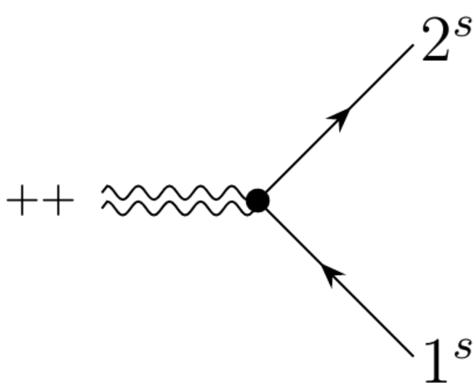


$$= \frac{\kappa}{2}(xm)^2 \frac{\left(\lambda_1^i \cdot \lambda_2^j\right)^{\odot 2s}}{m^{2s}}$$

Derive from coherent states in observables

de la Cruz, **BM**, O'Connell & Ross, 2020

Classical limit: $\hbar \to 0$, $q = \hbar \bar{q}$; $s \to \infty$, $s\hbar \to$ finite.



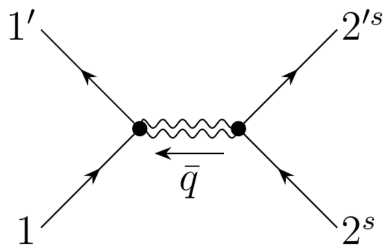
$$= \frac{\kappa}{2}(xm)^2 \lim_{s\to\infty}\left(\mathbb{I} + \frac{\bar{q}\cdot a}{2s}\right)^{2s} \boxed{= \frac{\kappa}{2}(xm)^2 e^{\bar{q}\cdot a}}$$

Arkani-Hamed, Huang & O'Connell, 2019

The 3-point amplitude with spin **exponentiates** in the **classical limit**.

Probe scattering

Scatter a light scalar off of our heavy spinning particle:



$$= \frac{\kappa^2 m_1 m_2}{2\hbar^3 \bar{q}^2}\left(e^{2w} e^{-\bar{q}\cdot a} + e^{-2w} e^{\bar{q}\cdot a}\right)$$

w = rapidity

Arkani-Hamed, Huang & O'Connell, 2019

Now calculate the **impulse** for the probe particle:

via Kosower, **BM** & O'Connell, 2018

$$\Delta p_1^{\mu,(0)} = 4\pi G m_1 m_2 \mathrm{Re}\int \hat{d}^4\bar{q}\,\hat{\delta}(\bar{q}\cdot u_1)\hat{\delta}(\bar{q}\cdot u_2)\frac{\boxed{i\,e^{-i\bar{q}\cdot(b+ia)}}}{\bar{q}^2}$$
$$\times\left(i\cosh 2w\,\bar{q}^\mu + 2\cosh w\,\epsilon^\mu(\bar{q},u_1,u_2)\right)$$

Wait…that's the result for probe scattering off a Kerr black hole

Vines, 2017

NJ shift due to amplitude exponentiation.

Kerr black holes as elementary particles?

One of **many** examples of scattering amplitudes predicting the behaviour of Kerr black holes.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Kerr black holes as elementary particles

Nima Arkani-Hamed,[a] Yu-tin Huang[b,c] and Donal O'Connell[d]

Vaidya, 2014 Guevara, 2017

Guevara, Ochirov & Vines, 2018

Chung, Huang, Kim & Lee, 2018

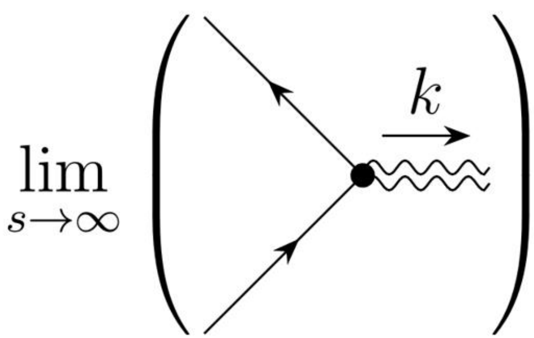
BM, O'Connell & Vines, 2019

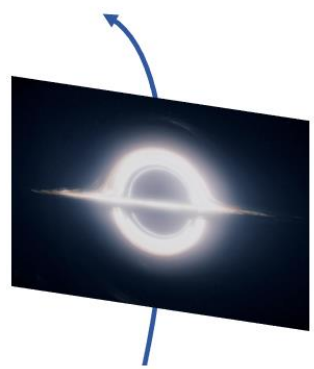
Burger, Emond & Moynihan, 2019

Aoude, Haddad & Helset, 2020

Bern, Luna, Roiban, Shen & Zeng, 2020

"Minimally coupled graviton amplitudes are an on-shell avatar of the no-hair theorem"



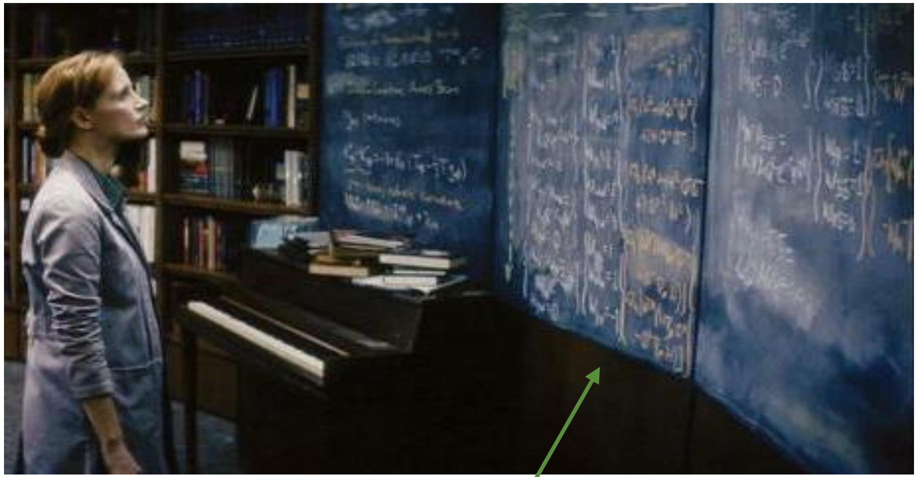


Showed NJ shift for background…but amplitudes calculate **interactions.**

NJ shift for interactions?

Effective actions for spinning particles

To describe the low energy physics of a system we're ignorant about, use an **EFT.**



Effective action for UV theory of quantum gravity

Effective action for dynamics of spinning compact body in classical gravity:

$$S_{\rm pp} = \int d\tau \left\{ -m\sqrt{u^2} \underbrace{- \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu}}_{\text{Spin kinematics}} + \underbrace{L_{\rm SI}[e^A, \bar{g}_{\mu\nu}(r), r(\tau)]}_{\text{Tower of spin-induced UV operators}} \right\}$$

Porto, 2005

Levi & Steinhoff, 2015

Kerr effective action

Guevara, **BM**, Ochirov,
O'Connell & Vines, *in prog.*

For spinning black hole,

Levi & Steinhoff, 2015

$$L_{\rm SI}^{\rm Kerr} = m\sum_{n=1}^{\infty}(-1)^n\left(\frac{(a\cdot\nabla)^{2n-2}}{(2n)!}R_{\mu\alpha\nu\beta} + \frac{(a\cdot\nabla)^{2n-1}}{(2n+1)!}R^*_{\mu\alpha\nu\beta}\right)a^\mu u^\alpha a^\nu u^\beta \boxed{+\cdots}$$

O(R^2) terms

Rewrite:

$$S_{\rm SI}^{\rm Kerr} = -m\sum_{n=1}^{\infty}\mathrm{Re}\int d\tau\,\frac{(ia\cdot\nabla)^{n-1}}{(n+1)!}R^+_{\mu\alpha\nu\beta}a^\mu u^\alpha a^\nu u^\beta + \cdots$$

$$R^+ := R + iR^*$$

$$[\nabla,\omega^{ab}] = -R^{ab}$$

$$\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!} = \frac{e^x-1}{x} = \int_0^1 d\lambda\, e^{\lambda x}$$

$$S_{\rm SI}^{\rm Kerr} = \mathrm{Re}\int d\tau\int_0^1 d\lambda\, i\,\omega_\mu^{+ab}(r+i\lambda a)u^\mu p_a a_b + \cdots$$

2-dimensional sheet living in complex space → **complex worldsheet.**

A worldsheet for Kerr

Guevara, **BM**, Ochirov,
O'Connell & Vines, *in prog.*

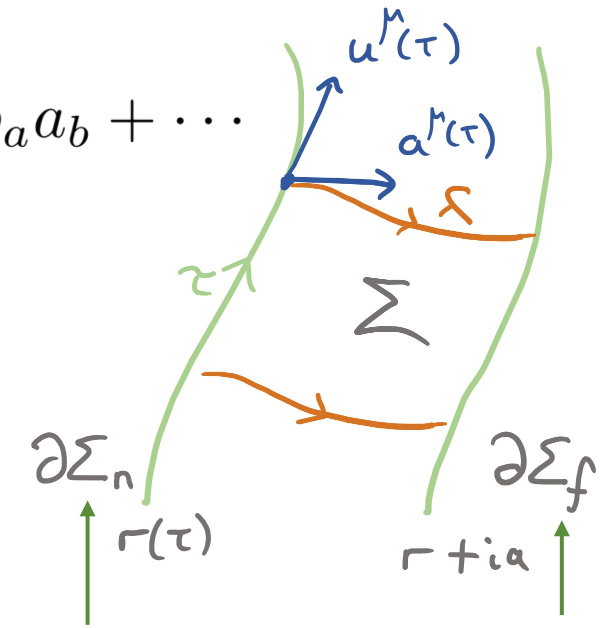
Take sheet coordinates $z(\tau,\lambda) = r(\tau) + i\lambda a(\tau)$:

$$S_{\mathrm{SI}}^{\mathrm{Kerr}} = \mathrm{Re}\int d\tau \int_0^1 d\lambda\, i\, \omega_\mu^{+ab}(r + i\lambda a) u^\mu p_a a_b + \cdots$$

$$= \mathrm{Re}\int_\Sigma \omega_\mu^+(z) u^\mu(z) + \cdots$$

Also find worldsheet for $\sqrt{\text{Kerr}}$ in EM:

$$S_{\mathrm{int}}^{\sqrt{\mathrm{Kerr}}} = -Q\int_{\partial\Sigma_n} A + Q\,\mathrm{Re}\int_\Sigma F^+(z) + \cdots$$



λ=0: "near" bdry → real worldline

λ=1: "far" bdry → NJ shift

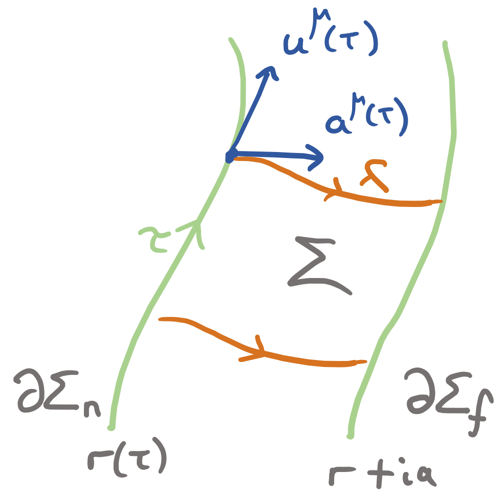
Uh, guys…cute, but is this any use for calculations?

Worldline NJ shift

Guevara, **BM**, Ochirov, O'Connell & Vines, *in prog.*

Worldsheet for $\sqrt{\text{Kerr}}$ in EM:

$$S_{\text{int}}^{\sqrt{\text{Kerr}}} = -Q\int_{\partial\Sigma_n} A + Q\,\text{Re}\int_{\Sigma} F^{+}(z) + \cdots$$



Apply Stokes's theorem in absence of sources:

$$\text{Re}\int_{\Sigma} F^{+}(z) = \text{Re}\left\{\int_{\partial\Sigma_n} A^{+}(z) - \int_{\partial\Sigma_f} A^{+}(z)\right\}$$

$$\boxed{S_{\text{int}}^{\sqrt{\text{Kerr}}} = -Q\,\text{Re}\int d\tau\, A_{\mu}^{+}(r+ia)u^{\mu}}$$

$$A^{+} = A + i\,{}^{\star}\!A$$

Worldsheet action → chiral Newman-Janis shift to **worldline.**

Spinors in classical physics

Exploit chiral splitting: **spinor-helicity**.

An Approach to Gravitational Radiation by a Method of Spin Coefficients*

Ezra Newman
University of Pittsburgh, Pittsburgh, Pennsylvania
and
Roger Penrose†
Syracuse University,‡ Syracuse, New York
(Received September 29, 1961)

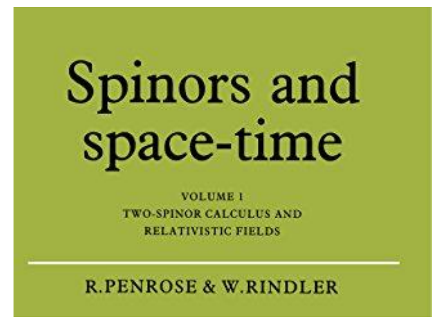
Maxwell:

$$F^{+}_{\mu\nu} = F_{\mu\nu} + i\, F^{\star}_{\mu\nu} = \phi_{\alpha\beta}\, \sigma^{\alpha\beta}_{\mu\nu}\,,$$

$$\text{c.c.} : \tilde{\phi}$$

Weyl:

$$C_{\mu\nu\rho\sigma} + i\, C^{\star}_{\mu\nu\rho\sigma} = \Psi_{\alpha\beta\gamma\delta}\, \sigma^{\alpha\beta}_{\mu\nu} \sigma^{\gamma\delta}_{\rho\sigma}$$



Newman-Janis shifts natural for spinors:

$$\boxed{\begin{aligned} &\phi^{\sqrt{\text{Kerr}}}(z) = \phi^{\text{Coulomb}}(z + ia)\,, \quad \tilde{\phi}^{\sqrt{\text{Kerr}}}(z) = \tilde{\phi}^{\text{Coulomb}}(z - ia) \\ &\Psi^{\text{Kerr}}(z) = \Psi^{\text{Schwz}}(z + ia)\,, \quad \tilde{\Psi}^{\text{Kerr}}(z) = \tilde{\Psi}^{\text{Schwz}}(z - ia) \end{aligned}}$$

Newman & Janis, 1965

From worldsheets to spinors

Guevara, **BM**, Ochirov, O'Connell & Vines, *in prog.*

$$S_{\text{int}}^{\sqrt{\text{Kerr}}} = -Q\,\text{Re}\int d\tau\, A_\mu^+(r+ia)u^\mu$$

$$\frac{dp_\mu}{d\tau} = \frac{Q}{m}\text{Re}\Big\{F_{\mu\nu}^+(r+ia)p^\nu\Big\}$$

$$\frac{da_\mu}{d\tau} = \frac{Q}{m}\text{Re}\Big\{F_{\mu\nu}^+(r+ia)a^\nu\Big\}$$

Common structure:

$$\frac{dV^\mu}{d\tau} = \frac{Q}{2m}\Big(\phi^{\alpha\beta}(r+ia)\sigma^{\mu\nu}_{\alpha\beta} + \tilde{\phi}^{\dot\alpha\dot\beta}(r-ia)\bar{\sigma}^{\mu\nu}_{\dot\alpha\dot\beta}\Big)V_\nu \qquad V^\mu = p^\mu, a^\mu$$

Common basis:

$$V^\mu(\tau) = \frac{1}{2}V^{ij}\,\lambda_i^{\,\alpha}(\tau)\,\tilde{\lambda}_j^{\,\dot\alpha}(\tau)\,\sigma^\mu_{\alpha\dot\alpha}$$

Little group tensor

Dynamical spinors

Arkani-Hamed, Huang & Huang, 2017

Spinor equations

Guevara, **BM**, Ochirov,
O'Connell & Vines, *in prog.*

Particle dynamics encoded by spinors!

$$\boxed{\begin{aligned}\frac{d\lambda_i^{\alpha}}{d\tau} &= \frac{Q}{2m}\phi^{\alpha\beta}\left(r(\tau)+ia(\tau)\right)\lambda_{i\,\beta}(\tau)\\ \frac{d\tilde{\lambda}_i^{\dot\alpha}}{d\tau} &= \frac{Q}{2m}\tilde{\phi}^{\dot\alpha}{}_{\dot\beta}\left(r(\tau)-ia(\tau)\right)\tilde{\lambda}_i^{\dot\beta}(\tau)\end{aligned}}$$

p and a form distinct little group irreps: $p\cdot a = 0$

$$\underbrace{p^{\alpha\dot\alpha}(\tau) = \epsilon^{ij}\lambda_j^{\alpha}(\tau)\tilde{\lambda}_i^{\dot\alpha}(\tau)}_{\text{Singlet state}} \qquad \underbrace{a^{\alpha\dot\alpha}(\tau) = a^{ij}\lambda_i^{\alpha}(\tau)\tilde{\lambda}_j^{\dot\alpha}(\tau)}_{\text{Symmetric}}$$

Gravity

Guevara, **BM**, Ochirov,
O'Connell & Vines, *in prog.*

Kerr worldsheet:

$$S_{\text{int}}^{\text{Kerr}} = \text{Re} \int d\tau \int_0^1 d\lambda \, i \, \omega_\mu^{+ab}(r + i\lambda a) u^\mu p_a a_b + \cdots$$

$\longrightarrow$ spinor equations in gravity are

$$\boxed{\begin{aligned} \frac{d\lambda_i^{\alpha}}{d\tau} &= -\frac{1}{2}\omega_\mu{}^{\alpha\beta}\left(r(\tau) + ia(\tau)\right) u^\mu \lambda_{i\,\beta}(\tau) \\ \frac{d\tilde{\lambda}_i^{\dot\alpha}}{d\tau} &= -\frac{1}{2}\tilde{\omega}_\mu{}^{\dot\alpha}{}_{\dot\beta}\left(r(\tau) - ia(\tau)\right) u^\mu \tilde{\lambda}_i^{\dot\beta}(\tau) \end{aligned}}$$

Connection spinor:

$$\omega_{ab}^{+} = \omega_{\alpha\beta}\, \sigma_{ab}^{\alpha\beta}$$

Worldsheet provides covariant meaning for trajectory's NJ shift.

Kerr-Kerr scattering

Guevara, **BM**, Ochirov,
O'Connell & Vines, *in prog.*

Impulse: $\Delta p_1^{\alpha\dot\alpha} = \epsilon^{ij}\,\mathrm{Re}\,\Delta\lambda_j{}^\alpha\tilde\lambda_i^{\dot\alpha}$ $\qquad$ $\boxed{\Delta a_1^{\alpha\dot\alpha} = a_1^{ij}}\,\mathrm{Re}\,\Delta\lambda_j{}^\alpha\tilde\lambda_i^{\dot\alpha}$

Kerr probe: $\Delta\lambda_i^{\alpha,(0)} = -\frac{1}{2}\int_{-\infty}^{\infty} d\tau\,\omega_\mu{}^{\rho\sigma}(r_1 + ia_1)\,u_1^\mu\sigma_{\rho\sigma}^{\alpha\beta}\lambda_{i\,\beta}$

Worldline NJ shift

Scatter off static Kerr background:

Background NJ shift

$$\Delta a_1^{\mu,(0)} = -4\pi G m_2 \mathrm{Re}\int \hat{d}^4\bar q\,\hat\delta(\bar q\cdot u_1)\hat\delta(\bar q\cdot u_2)\frac{i\,e^{-i\bar q\cdot(b+ia_1+ia_2)}}{\bar q^2}$$
$$\times\Big(i\cosh 2w\,\epsilon^\mu(\bar q, u_1, a_1) - 2\cosh w\,u_1^\mu\epsilon^\mu(\bar q, u_1, u_2, a_1)$$
$$-\,2ia_1\cdot u_2\cosh w\,\bar q^\mu + i\bar q\cdot a_1\,(2\cosh w\,u_2^\mu - u_1^\mu)\Big)$$

Spin-spin scattering as easy as spin-static! Δa^μ **free**.

Also easier than amplitudes:
BM, O'Connell & Vines, 2019

Discussion

Guevara, **BM**, Ochirov,
O'Connell & Vines, *in prog.*

- Newman-Janis shift → important insight for Kerr dynamics.
- Scattering amplitudes explain NJ shift from non-geometric perspective. Kerr worldsheet provides geometric understanding.
- NJ shifts natural for spinors → exploit to provide compact description of dynamics of spinning + charged black holes.
- Simple to extend to Kerr-Taub-NUT.
- Angular impulse as easy as impulse.

Next steps:

- Do worldsheet symmetries constrain tidal $\mathcal{O}(F^2)$ or $\mathcal{O}(R^2)$ terms?
- What are the spinor equations at higher orders?

Results known for Taub-NUT at NLO Kim & Shim, 2020